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On an F-algebra of holomorphic functions on the upper half plane

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Abstract. In this paper, we shall consider the class $N^p(D)$ $(p > 1)$ of holomorphic functions on the upper half plane $D := \{z \in \mathbf{C} \mid \mathrm{Im}\, z > 0\}$ satisfying $\sup_{y>0} \int_{\mathbf{R}} \log(1 + |f(x + iy)|)^{\,p}\, dx < \infty$. We shall prove that $N^p(D)$ is an F-algebra with respect to a natural metric on $N^p(D)$. Moreover, a canonical factorization theorem for $N^p(D)$ will be given.

Key words: Nevanlinna-type spaces, Nevanlinna class, Smirnov class, N^p, Hardy spaces.

0. Introduction

Let U and T denote the unit disk and the unit circle in $\mathbf{C}$, respectively. For $p > 1$, we denote by $N^p(U)$ the class of functions f holomorphic on U and satisfying

$$\sup_{0<r<1} \int_T \bigl(\log(1+|f(r\zeta)|)\bigr)^p \, d\sigma(\zeta) < +\infty,$$

where $d\sigma$ denotes normalized Lebesgue measure on T. Letting $p = 1$, we have the Nevanlinna class $N(U)$. It is well-known that each function f in $N(U)$ has the nontangential limit $f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$ (a.e. $\zeta \in T$) and that $\log(1 + |f|)$ (and hence, $(\log(1 + |f|))^p$ for $p > 1$) is subharmonic if f is holomorphic.

We denote the Smirnov class by $N_*(U)$, which consists of all holomorphic functions f on U such that $\log(1 + |f(z)|) \leqq Q[\phi](z)$ $(z \in U)$ for some $\phi \in L^1(T)$, $\phi \geqq 0$, where the right side denotes the Poisson integral of ϕ on U.

It is well-known that $H^q(U) \subset N^p(U) \subset N_*(U) \subset N(U)$ $(0 < q \leqq \infty$, $p > 1)$, where $H^q(U)$ denotes the Hardy space on U. These inclusion relations are proper. Stoll [11] introduced the class $N^p(U)$. This was further studied by several authors (see [1] and [2]). The spaces $N(U)$, $N_*(U)$ and

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$N^p(U)$ are called *Nevanlinna-type spaces.*

Mochizuki [7] introduced the Nevanlinna class $N_0(D)$ and the Smirnov class $N_*(D)$ on the upper half plane $D := \{z \in \mathbf{C} \mid \mathrm{Im}\, z > 0\}$: the class $N_0(D)$ is the set of all holomorphic functions f on D satisfying

$$d(f,0) := \sup_{y>0} \int_{\mathbf{R}} \log\bigl(1+|f(x+iy)|\bigr)\, dx < +\infty$$

and $N_*(D)$ the set of all holomorphic functions f on D satisfying $\log(1+|f(z)|) \leqq P[\phi](z)$ $(z \in D)$ for some $\phi \in L^1(\mathbf{R})$, $\phi \geqq 0$, where the right side denotes the Poisson integral of ϕ on D.

In this paper, we shall define a new class $N^p(D)$, analogous to $N^p(U)$; i.e., we denote by $N^p(D)$ $(p > 1)$ the set of all holomorphic functions f on D such that

$$[d_p(f,0)]^p := \sup_{y>0} \int_{\mathbf{R}} \bigl(\log(1+|f(x+iy)|)\bigr)^p\, dx < +\infty.$$

Hardy spaces on D, $H^q(D)$ $(0 < q < \infty)$, are defined by $L^q(dx)$-boundedness of holomorphic functions $f(x+iy)$. Although $N_*(D) \subset N_0(D)$, we have, in contrast to the open unit disc U, that $H^p(D) \not\subset N_0(D)$ and $N^p(D) \not\subset N_0(D)$ $(p > 1)$. In fact, if $p^{-1} < \alpha < 1$, then $(z+i)^{-\alpha} \in H^p(D)$ and $(z+i)^{-\alpha} \in N^p(D)$ but $(z+i)^{-\alpha} \notin N_0(D)$ (see [7, Remark]).

First we obtain a factorization theorem for the class $N^p(D)$, as Mochizuki [7] does for the class $N_0(D)$. Moreover, we show that $N^p(D)$ becomes an F-algebra, in the sense that $N^p(D)$ is a complete linear metric space with multiplication continuous.

1. Preliminaries

Let ν be a real measure on T. Set $\Psi(z) = (z-i)/(z+i)$ $(z \in \overline{D})$. Then there corresponds a finite real measure μ on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} h(t)\, d\mu(t) = \int_{T^*} (h \circ \Psi^{-1})(\eta)\, d\nu(\eta) \qquad (h \in C_c(\mathbf{R})),$$

where $T^* = T \setminus \{1\}$. Let $H(w,\eta) = (\eta+w)/(\eta-w)$ $((w,\eta) \in U \times T)$. There holds

$$\frac{1}{i} \int_{\mathbf{R}} \frac{1+tz}{t-z}\, d\mu(t) = \int_{T^*} H(\Psi(z),\eta)\, d\nu(\eta) \tag{1}$$

$$= \int_T H(\Psi(z), \eta)\, d\nu(\eta) - i\alpha z \qquad (z \in D),$$

where $\alpha = -\nu(\{1\})$. We write the Poisson integrals of measures μ on $\mathbf{R}$ and ν on T as follows:

$$P[\mu](z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{y}{(x-t)^2 + y^2}\, d\mu(t) \qquad (z = x + iy \in D),$$

$$Q[\nu](w) = \int_T \frac{1 - |w|^2}{|\eta - w|^2}\, d\nu(\eta) \qquad (w \in U).$$

Taking the real parts in (1), we have

$$P[\pi(1+t^2)\, d\mu(t)](z) = Q[\nu](\Psi(z)) + \alpha \cdot \mathtt{Im}\, z \qquad (z \in D). \tag{2}$$

When $f \in L^1(\mathbf{R}, (1+t^2)^{-1} dt)$ and $g \in L^1(T)$, we write $P[f]$ and $Q[g]$ instead of $P[f(t)\, dt]$ and $Q[g\sigma]$, respectively. If $g \in L^1(T)$, then we have $g \circ \Psi \in L^1(\mathbf{R}, (1+t^2)^{-1} dt)$ and

$$P[g \circ \Psi](z) = Q[g](\Psi(z)). \tag{2$'$}$$

2. Some properties on $N^p(U)$

In this section, we shall summarize some properties on $N^p(U)$ $(p > 1)$. For the following results, the reader refers to [1], [2] and [11].

Proposition 2.1 *Let $f \in N^p(U)$ $(p > 1)$, $f \neq 0$. Then,* $\log|f^*| \in L^1(T)$ *and* $\log(1 + |f^*|) \in L^p(T)$. *Moreover, f can be uniquely factored as follows,*

$$f(z) = aB(z)F(z)S(z),$$

where $a \in T$ is a constant, $B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$ $(z \in U)$ is a Blaschke product with respect to the zeros of f, $F(z) = \exp\left(\int_T \frac{\zeta+z}{\zeta-z} \log|f^*(\zeta)|\, d\sigma(\zeta) \right)$ and $S(z) = \exp\left(- \int_T \frac{\zeta + z}{\zeta - z}\, d\nu(\zeta) \right)$, where ν is a positive singular measure.

Proposition 2.2 *Let $f \in N(U)$ and $p > 1$. Then $f \in N^p(U)$ if and only if* $(\log(1 + |f|))^p$ *has a harmonic majorant.*

Proposition 2.3 *Let $f \in N^p(U)$, $p > 1$. Then* $(\log(1 + |f|))^p$ *has the least harmonic majorant* $Q[(\log(1 + |f^*|))^p]$.

3. A factorization theorem for the class $N^p(D)$

Let $f \in N^p(D)$, $p > 1$. Then we easily have the following proposition by [9, Chapter II, Theorem 4.6].

Proposition 3.1 *Let $p > 1$ and $f \in N^p(D)$. Then,*
(i) $(\log(1+|f|))^p$ *has the least harmonic majorant $P[g]$, where $g \in L^p(dx)$.*
(ii) $\|g\|_p \leqq d_p(f, 0)$.
(iii) *Let $D_\delta = \{z \in \mathbf{C} \mid \mathtt{Im}\, z > \delta\}$. Then $\log(1 + |f(z)|) \to 0$ as $|z| \to +\infty$ provided $z \in D_\delta$, for each $\delta > 0$.*

Using the above proposition, we have the following characterization of function f in $N^p(D)$.

Theorem 3.2 *Let $p > 1$. A function $f \in N^p(D)$ has the following properties:*
(i) $f \circ \Psi^{-1} \in N^p(U)$.
(ii) *The nontangential limit $f^*(x)$ exists a.e. for $x \in \mathbf{R}$.*
(iii)
$$\sup_{y>0} \int_{\mathbf{R}} \big(\log(1 + |f(x + iy)|)\big)^p\, dx = \lim_{y \to 0^+} \int_{\mathbf{R}} \big(\log(1 + |f(x + iy)|)\big)^p\, dx = \int_{\mathbf{R}} \big(\log(1 + |f^*(x)|)\big)^p\, dx$$

Proof. Suppose $f \in N^p(D)$. Then it is seen that $f \circ \Psi^{-1} \in N^p(U)$ by Proposition 2.2 and part (i) in Proposition 3.1. Hence, (i) and (ii) hold. We have

$$\sup_{y>0} \int_{\mathbf{R}} \big(\log(1+|f(x+iy)|)\big)^p\, dx = \lim_{y \to 0^+} \int_{\mathbf{R}} \big(\log(1+|f(x+iy)|)\big)^p\, dx$$

by part (iii) in Proposition 3.1 and [3, Theorem 1]. By Proposition 3.1, the least harmonic majorant of $(\log(1+|f|))^p$ is the form $P[g]$, where $g \in L^p(\mathbf{R})$; and it follows that

$$\begin{aligned} \|g\|_p^p &\leqq \sup_{y>0} \int_{\mathbf{R}} \big(\log(1 + |f(x + iy)|)\big)^p dx \\ &\leqq \sup_{y>0} \int_{\mathbf{R}} \big(P[g](x + iy)\big)^p dx \leqq \|g\|_p^p. \end{aligned}$$

Here, the last inequality holds by [4, inequality (3.5)]. Since $f \circ \Psi^{-1} \in N^p(U)$, the least harmonic majorant $P[g]$ of $(\log(1 + |f|))^p$ is also given by $P[(\log(1 + |f^*|))^p]$ by Proposition 2.3. Therefore, $g = \log(1 + |f^*|))^p$. This shows part (iii). □

Theorem 3.3 *Let $p > 1$. $f \in N^p(D)$, $f \neq 0$, is factorized in the form*

$$f(z) = ae^{i\alpha z}b(z)d(z)g(z) \qquad (z \in D), \tag{3}$$

where the factors above have the following properties:

(i) $a \in T$, $\alpha \geqq 0$.

(ii) $b(z)$ *is the Blaschke product with respect to the zeros of* f.

(iii) $d(z) = \exp\left(\dfrac{1}{\pi i}\displaystyle\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log h(t)\,dt\right)$,
where $h(t) \geqq 0$, $\log h \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$ *and* $\log(1+h) \in L^p(\mathbf{R})$.

(iv) $g(z) = \exp\left(-\dfrac{1}{i}\displaystyle\int_{\mathbf{R}} \frac{1+tz}{t-z}\,d\mu(t)\right)$, *where μ is a finite nonnegative measure on $\mathbf{R}$, singular with respect to Lebesgue measure, and such that* $\displaystyle\int_{\mathbf{R}}(1+t^2)\,d\mu(t) < +\infty$.

If f is expressed in the form (3), *then* $f \in N^p(D)$.

The proof of this theorem needs the following:

Lemma 3.4 *Let $\mathfrak{N}^p$ $(p > 1)$ be the class of all holomorphic functions on D which satisfy*

$$\sup_{y>0}\int_{\mathbf{R}} \left(\log^+|f(x+iy)|\right)^p dx < +\infty$$

(*Letting $p = 1$, we have the Nevanlinna class $\mathfrak{N}$ introduced by Krylov* [6]). *Then the following including relations hold,*

$$N^p(D) \subset \mathfrak{N}^p \subset \mathfrak{N}.$$

Proof. It is easy to see that the first containment holds. The fact that $\mathfrak{N}^p \subset \mathfrak{N}$ for $p > 1$ is a consequence of [5, Remark]. □

Proof of Theorem 3.3. Let $f \in N^p(D)$, $f \neq 0$. Then

$$(f \circ \Psi^{-1})(w) = aB(w)F(w)S(w) \qquad (w \in U)$$

by Theorem 3.2 (i) and Proposition 2.1. Now, in the factorization $f(z) = aB(\Psi(z))F(\Psi(z))S(\Psi(z))$ $(z \in D)$, $b(z) := B(\Psi(z))$ is the Blaschke product formed from the zeros of f, and by changing the variables $\eta = \Psi(t)$

($t \in \mathbf{R}$), we shall show that $d(z) := F(\Psi(z))$ is of the form (iii). Since $\log|(f \circ \Psi^{-1})^*| \in L^1(T)$, we have $\log|f^*| \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$ by $(2)'$. Theorem 3.2 shows $\log(1+|f^*|) \in L^p(\mathbf{R})$. Setting $\alpha = \nu(\{1\})$, we have $S(\Psi(z)) := S_1(\Psi(z)) = g(z)e^{i\alpha z}$, where g is of the form (iv). Moreover, it follows from (2) that $\int_{\mathbf{R}} (1+t^2)\, d\mu(t) < \infty$.

Conversely, suppose that f is of the form (3). Then

$$\begin{aligned} |f(z)| &= |e^{i\alpha z}|\,|b(z)| \exp\bigl(P[\log h - \pi(1+t^2)\, d\mu(t)](z)\bigr) \\ &\leqq \exp(P[\log h](z)). \end{aligned}$$

Since $\log^+|(f \circ \Psi^{-1})(w)| \leqq Q[\log^+(h \circ \Psi^{-1})](w)$, we have $f \circ \Psi^{-1} \in N^p(U)$. Letting $y \to 0^+$ in $|f(x+iy)|$, we have $|f^*(x)| = h(x)$ for a.e. $x \in \mathbf{R}$.

Furthermore, $(\log(1+|f \circ \Psi^{-1}|))^p$ has the least harmonic majorant $v' = Q[(\log(1+|(f\circ\Psi^{-1})^*|))^p]$ by Proposition 2.3, $v := v' \circ \Psi$ is the least harmonic majorant of $(\log(1+|f|))^p$: i.e., $(\log(1+|f(z)|))^p \leqq P[(\log(1+|f^*|))^p](z)$. Integrating the both sides, we have $f \in N^p(D)$. □

4. The class $N^p(D)$ as an F-algebra

For f, $g \in N^p(D)$, $p > 1$, let $d_p(f,g) = d_p(f-g, 0)$. By Theorem 3.2,

$$d_p(f,g) = \left\{ \int_{\mathbf{R}} \bigl(\log(1+|f^*(x)-g^*(x)|)\bigr)^p\, dx \right\}^{1/p}.$$

The above definition of d_p has been motivated by the metric on $N_0(D)$, which was introduced by Mochizuki [7].

We see that d_p defines a translation invariant metric on $N^p(D)$. In fact, we obtain the following theorem.

Theorem 4.1 *Let $p > 1$. The space $(N^p(D), d_p)$ is an F-algebra, that is, a complete linear metric space with multiplication continuous.*

Proof. We shall prove the theorem using the idea due to Stoll [10; 11, Theorem 4.2]. The inequalities

$$\begin{aligned} &\log(1+|x+y|) \leqq \log(1+|x|) + \log(1+|y|), \\ &\log(1+|xy|) \leqq \log(1+|x|) + \log(1+|y|) \quad \text{and} \\ &\log(1+|cx|) \leqq \max(1,|c|) \log(1+|x|) \end{aligned}$$

imply

$$\begin{aligned} &d_p(f+g,0) \leqq d_p(f,0)+d_p(g,0), \\ &d_p(fg,0) \leqq d_p(f,0)+d_p(g,0) \quad \text{and} \\ &d_p(cf,0) \leqq \max(1,|c|)\, d_p(f,0) \qquad (c \in \mathbf{C}). \end{aligned}$$

Hence, $N^p(D)$ forms an algebra.

Next, we show that multiplication is continuous. Let c_n, $c \in \mathbf{C}$ and f_n, g_n, f, $g \in N^p(D)$. Suppose $|c_n - c| \to 0$, $d_p(f_n, f) \to 0$ and $d_p(g_n, g) \to 0$. Obviously, $d_p((f_n + g_n) - (f + g), 0) \to 0$ and $d_p(cf_n - cf, 0) \to 0$. In order to see $d_p(c_n f - cf, 0) \to 0$, we may assume $|c_n - c| \leqq 1$. Then, $\log(1 + |c_n f - cf|) \leqq \log(1 + |f|)$. Since $\log(1 + |f|) \in L^p(\mathbf{R})$, the Lebesgue dominated convergence theorem yields $d_p(c_n f - cf, 0) \to 0$. Since

$$f_n g_n - fg = (f_n - f)(g_n - g) + (fg_n - fg) + (gf_n - gf),$$

we have

$$d_p(f_n g_n, fg) \leqq d_p(f_n, f) + d_p(g_n, g) + d_p(fg_n, fg) + d_p(gf_n, gf).$$

Therefore, it is suffice to see that, for all $g \in N^p(D)$, $gf_n \to gf$ if $f_n \to f$. Fix $g \in N^p(D)$ and let $\alpha = \limsup_{n\to\infty} d_p(gf_n, gf)$. We only have to show that $\alpha = 0$. Replacing $\{f_n\}$ by a subsequence, if necessary, we may assume that $d_p(gf_n, gf) \to \alpha$. It is easy to see the following weak-type inequality

$$(\log(1+\varepsilon))^p \int_{\{x;\,|f| \geqq \varepsilon\}} dx \leqq \int_{\mathbf{R}} \bigl(\log(1+|f|)\bigr)^p\, dx = d_p(f, 0).$$

Since $d_p(f_n - f, 0) \to 0$, we see that f_n coverges to f in measure. Hence, there exists a subsequence $\{f^*_{n_k}\}$ of $\{f^*_n\}$ such that $f^*_{n_k} \to f^*$ a.e. on $\mathbf{R}$. Thus, $\log(1 + |g^* f^*_{n_k} - g^* f^*|) \to 0$ a.e. on $\mathbf{R}$. It follows that

$$\begin{aligned} &\bigl\{\log(1 + |g^* f^*_{n_k} - g^* f^*|)\bigr\}^p \\ &\quad \leqq \bigl\{\log(1 + |g^*|) + \log(1 + |f^*_{n_k} - f^*|)\bigr\}^p \\ &\quad \leqq 2^p \bigl\{(\log(1 + |g^*|))^p + (\log(1 + |f^*_{n_k} - f^*|))^p\bigr\}. \end{aligned}$$

Note that the term in the right of the above inequality converges a.e. to $2^p(\log(1 + |g^*|))^p$. Then

$$\alpha = \lim_{n\to\infty} d_p(gf_n, gf) = \lim_{k\to\infty} \left\{ \int_{\mathbf{R}} \bigl(\log(1 + |g^* f^*_{n_k} - g^* f^*|)\bigr)^p\, dx \right\}^{1/p}$$

$$= \left\{ \int_{\mathbf{R}} \lim_{k\to\infty} \bigl(\log(1 + |g^* f^*_{n_k} - g^* f^*|)\bigr)^p \, dx \right\}^{1/p} = 0,$$

where we use a generalization of Lebesgue's dominated convergence theorem [8, p.270]. Therefore we have $\lim_{n\to\infty} d_p(g_n f_n, gf) = 0$, which proves the multiplication continuous.

Next we show the completeness. Suppose $\{f_n\}$ is a Cauchy sequence in $N^p(D)$. Since the function $(\log(1 + |f_m - f_n|))^p$ is subharmonic, we have, by [4, p.39],

$$\begin{aligned} &\bigl(\log(1 + |f_m(x+iy) - f_n(x+iy)|)\bigr)^p \\ &\quad \leqq \frac{2}{\pi y} \sup_{\eta>0} \int_{\mathbf{R}} \bigl(\log(1 + |f_m(\xi + i\eta) - f_n(\xi + i\eta)|)\bigr)^p \, d\xi \\ &\qquad\qquad\qquad\qquad\qquad (z = x+iy,\ y>0). \end{aligned}$$

Then, for $z = x + iy \in \overline{D_\delta}$,

$$\log(1+|f_m(z)-f_n(z)|) \leqq \left(\frac{2}{\pi\delta}\right)^{1/p} d_p(f_m - f_n, 0).$$

The right side of the above inequality tends to zero as m, $n \to \infty$, so $f_n(z)$ converges uniformly on every compact subset on D to a holomorphic function $f(z)$. Since $\{f_n\}$ is a Cauchy sequence in $N^p(D)$, we have $d_p(f_n, 0) \leqq C$, where C is a positive constant. Therefore,

$$\begin{aligned} \int_I \bigl(\log(1 + |f(x+iy)|)\bigr)^p \, dx &= \lim_{n\to\infty} \int_I \bigl(\log(1 + |f_n(x+iy)|)\bigr)^p \, dx \\ &\leqq C^p \qquad (y > 0) \end{aligned}$$

for each finite interval I on $\mathbf{R}$. This shows that $f \in N^p(D)$.

It remains to be shown that $d_p(f_n, f) \to 0$. We obtain

$$\begin{aligned} &\int_I \bigl(\log(1 + |f_n(x+iy) - f(x+iy)|)\bigr)^p \, dx \\ &\quad \leqq \lim_{m\to\infty} \int_{\mathbf{R}} \bigl(\log(1 + |f_n(x+iy) - f_m(x+iy)|)\bigr)^p \, dx \\ &\quad \leqq \lim_{m\to\infty} [d_p(f_m, f_n)]^p \qquad (y > 0). \end{aligned}$$

Therefore we have $d_p(f_n, f) \leqq \lim_{m\to\infty} d_p(f_m, f_n)$, which shows $d_p(f_n, f) \to 0$. This finishes the proof. □

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