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A class of Butler groups and their endomorphism rings

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Abstract. We study a class of Butler groups of infinite rank, called Hawaiian groups. They are defined as subgroups of a rational vector space and contain parameters that provide for flexibility but are concrete enough to allow for the computation of certain crucial subgroups and quotient groups, to exhibit endomorphisms and describe the endomorphism rings. Most Hawaiian groups are finitely Butler; under stronger assumptions they are not finitely filtered and hence not B_2-groups.

Key words: torsion-free abelian group of infinite rank, Butler group, finitely Butler, endomorphism ring, free direct summand.

1. Introduction

It is common knowledge that the concept of a Butler group is ambiguous for torsion-free abelian groups that are uncountable. There are competing definitions which we recall. A torsion-free abelian group G is called *finitely Butler* if every finite rank pure subgroup H of G is a Butler group, i.e., a pure subgroup or equivalently a homomorphic image of a completely decomposable group of finite rank. A torsion-free group B is called a B_1-*group* if $\mathrm{Bext}_{\mathbb{Z}}(B, T) = 0$ for all torsion groups T. Here $\mathrm{Bext}_{\mathbb{Z}}(-,-)$ denotes the subfunctor of $\mathrm{Ext}_{\mathbb{Z}}(-,-)$ consisting of all balanced exact extensions. Finally, a torsion-free group G is a B_2-*group* if it has a filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ of pure subgroups G_α such that for every $\alpha < \lambda$, $G_{\alpha+1} = G_\alpha + H_\alpha$ for some Butler group H_α of finite rank.
In [ShSt] groups were constructed that are finitely Butler but not B_2-group and it was shown that in certain models of ZFC some of these groups are B_1-groups.
The purpose of this paper is to investigate further the finitely Butler groups

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constructed in [ShSt] and to determine their endomorphism rings. We generalize the construction from [ShSt] and obtain a large class of torsion-free groups, called Hawaiian groups, that are finitely Butler but not B_2-groups and study their properties as well as characterize the groups that can appear as endomorphism groups. Hawaiian groups may serve as examples or counterexamples for various questions on infinite rank Butler groups. For instance it is not known if Hawaiian groups are pure subgroups of (infinite rank) completely decomposable groups.
Our notation is standard and we write maps on the right. All groups under consideration are abelian and written additively. $\mathbb{P}$ denotes the set of all primes. If we say that a prime p divides an integer m or even a fraction m/n, then this means that p divides m inside $\mathbb{Z}$. If H is a pure subgroup of the abelian group G, then we write $H \subseteq_* G$. Moreover, $H_* \subseteq G$ denotes the purification of the subgroup H of a torsion-free group G. A reasonable knowledge about abelian groups as for instance in [Fu] is assumed. However, the authors have tried to make the paper as accessible as possible.

2. Hawaiian groups

In what follows κ always stands for an infinite cardinal $\leq 2^{\aleph_0}$. We let

$$V := \bigoplus_{n\in\omega} \mathbb{Q}x_n \oplus \bigoplus_{\alpha<\kappa} \mathbb{Q}y_\alpha$$

be the vector space with basis $\{x_n, y_\alpha\}$. Let

$$\mathcal{R} := (R_n \mid n < \omega), \quad \text{and} \quad \mathcal{S} := (S_\alpha \mid \alpha < \kappa)$$

be sequences of *rational groups* by which we mean additive subgroups of $\mathbb{Q}$ that contain $\mathbb{Z}$. Let

$$F := F_\kappa(\mathcal{R}, \mathcal{S}) = \Big(\bigoplus_{n\in\omega} R_n x_n\Big) \oplus \Big(\bigoplus_{\alpha<\kappa} S_\alpha y_\alpha\Big)$$

be the completely decomposable subgroup of V with “decomposition basis” $\{x_n, y_\alpha\}$. We define torsion-free groups sandwiched between F and V as follows.

Definition 2.1 Let $\mathcal{T} := (T_n \mid n < \omega)$ be a sequence of rational groups, let $\mathcal{P} := (p_n\colon n \in \omega)$ be a sequence of (distinct) prime numbers, and finally let $\mathcal{A} = (A_\alpha \colon \alpha < \kappa)$ be a sequence of subsets of ω such that

$$\forall \alpha,\, \beta \colon A_\alpha \cap A_\beta \neq \emptyset, \quad \bigcup_{\alpha<\kappa} A_\alpha = \omega, \quad \text{and} \quad \bigcap_{\alpha<\kappa} A_\alpha = \emptyset.$$

The group

$$B = \langle F_\kappa(\mathcal{R},\, \mathcal{S}),\, p_n^{-1} T_n(y_\alpha - x_n) \colon \alpha < \kappa,\, n \in A_\alpha \rangle \subseteq V$$

is called a *κ-Hawaiian group* or simply a *Hawaiian group*. The κ-Hawaiian group with the specified data we denote by $\mathcal{B}_\kappa(\mathcal{R},\, \mathcal{S},\, \mathcal{T},\, \mathcal{A},\, \mathcal{P})$.

Every element b of $B = \mathcal{B}_\kappa(\mathcal{R},\, \mathcal{S},\, \mathcal{T},\, \mathcal{A},\, \mathcal{P})$ has a representation

$$b = \sum_{n<\omega} r_n x_n + \sum_{\alpha<\kappa} s_\alpha y_\alpha + \sum_{(\alpha,n) \colon \alpha<\kappa, n\in A_\alpha} \frac{t_{\alpha,n}}{p_n}(y_\alpha - x_n) \tag{2.1}$$

$$= \sum_{n<\omega} \Big(r_n - \sum_{\alpha \colon n\in A_\alpha} \frac{t_{\alpha,n}}{p_n} \Big) x_n + \sum_{\alpha<\kappa} \Big(s_\alpha + \sum_{n\in A_\alpha} \frac{t_{\alpha,n}}{p_n} \Big) y_\alpha, \tag{2.2}$$

where $r_n \in R_n$, $s_\alpha \in S_\alpha$ and $t_{\alpha,n} \in T_n$ for $n \in \omega$, $\alpha < \kappa$, and all but a finite number of coefficients are zero.
We will use the following notation where the type of a rational group is its isomorphism class.

Definition 2.2 Let $B = \mathcal{B}_\kappa(\mathcal{R},\, \mathcal{S},\, \mathcal{T},\, \mathcal{A},\, \mathcal{P})$, and $\alpha < \beta < \kappa$. Then set
(i) $\delta := \mathrm{type}(\mathbb{Z})$;
(ii) $P_\alpha := \langle p_n^{-1} \colon n \in A_\alpha \rangle$; $\delta_\alpha := \mathrm{type}(P_\alpha)$;
(iii) $P := \langle p_n^{-1} \colon n \in \omega \rangle = \sum_{\rho<\kappa} P_\rho$; $R := \sum_{n<\omega} R_n$; $S := \sum_{\rho<\kappa} S_\rho$; $T := \sum_{n<\omega} T_n$;
(iv) $P_{\alpha,\beta} := P_\alpha \cap P_\beta = \langle p_n^{-1} \colon n \in A_\alpha \cap A_\beta \rangle$; $\delta_{\alpha,\beta} := \mathrm{type}(P_{\alpha,\beta}) = \delta_\alpha \wedge \delta_\beta$;
(v) $\widehat{T}_\alpha := \sum_{n\in A_\alpha} T_n$; $\tau_\alpha := \mathrm{type}(\widehat{T}_\alpha)$;
(vi) $T_{\alpha,\beta} := \sum_{n\in A_\alpha\cap A_\beta} T_n$; $\tau_{\alpha,\beta} := \mathrm{type}(T_{\alpha,\beta})$;
(vii) $\sigma_\alpha := \mathrm{type}(S_\alpha)$; $S_{\alpha,\beta} = S_\alpha \cap S_\beta$; $\sigma_{\alpha,\beta} = \mathrm{type}(S_{\alpha,\beta}) = \sigma_\alpha \wedge \sigma_\beta$.

We remark that δ_α is represented by the characteristic $[h_1, \ldots, h_i, \ldots]$ where $h_n = 1$ if $n \in A_\alpha$ and $h_n = 0$ otherwise.
As we proceed we will impose one or more conditions on the rational groups involved in the definition of Hawaiian groups. Some such conditions are as follows.

$$P \cap T = \mathbb{Z} \quad \text{and} \quad \forall n < \omega \colon T_n \cap \sum_{i \colon i\in\omega, i\neq n} T_i = \mathbb{Z}. \tag{2.3}$$

$$S \cap T = \mathbb{Z} \quad \text{and} \quad S \cap P = \mathbb{Z}. \tag{2.4}$$

$$R \cap P = \mathbb{Z} \quad \text{and} \quad R \cap T = \mathbb{Z}. \tag{2.5}$$

All these conditions are satisfied if $R = S = T = \mathbb{Z}$, but it is not difficult to exhibit many other examples of rational groups satisfying these conditions. Recall that a rational group is determined by generators of the form p^{-n} ([Ma, Section 1.2]). One only needs to write $\mathbb{P}$ as a countable disjoint union of infinite subsets and pick the generators for the rational groups from different subsets. For example, to satisfy $T_n \cap \sum_{i\in\omega:\, i\neq n} T_i = \mathbb{Z}$ let U be an infinite set of primes and write $U = \bigcup_{i<\omega} U_i$ such that the U_i are pairwise disjoint and infinite. Now, for each i, choose generators p^{-n} of T_i such that $p \in U_i$.

Remark 2.3 It follows from the second assumption in (2.3) that $\widehat{T}_\alpha \cap \widehat{T}_\beta = T_{\alpha,\beta}$.

We will use without explicit mention (see [Ma, Lemma 1.2.13]) the fact that intersection distributes over finite sums in the poset of rational groups. For two rational groups $\tilde{S}$, $\tilde{T}$, let $\tilde{S}\tilde{T} := \{\sum s_i t_i \colon s_i \in \tilde{S},\, t_i \in \tilde{T}\} \cong \tilde{S} \otimes \tilde{T}$ which is again a rational group. The type of a rational group is its isomorphism class and we have

$$\operatorname{type}(\tilde{S} + \tilde{T}) = \operatorname{type}(\tilde{S}) \vee \operatorname{type}(\tilde{T}),$$
$$\operatorname{type}(\tilde{S} \cap \tilde{T}) = \operatorname{type}(\tilde{S}) \wedge \operatorname{type}(\tilde{T}).$$

Before going on we need to record a consequence of the assumption $P \cap T = \mathbb{Z}$.

Lemma 2.4 *Suppose that $P \cap T = \mathbb{Z}$. Then the following statements are true.*
(i) $\sum_{n\in A_\alpha} (1/p_n) T_n = P_\alpha + \widehat{T}_\alpha = P_\alpha \widehat{T}_\alpha$.
(ii) $\delta_\alpha \vee \tau_\beta \geq \delta_{\alpha'} \vee \tau_{\beta'}$ *if and only if* $\delta_\alpha \geq \delta_{\alpha'}$ *and* $\tau_\beta \geq \tau_{\beta'}$.

Proof. (i) Let $n \in A_\alpha$ and $t_n \in T_n$. Then $t_n = (1/p_n)(t_n p_n) \in (1/p_n)T_n$, so $\widehat{T}_\alpha \subseteq \sum_{n\in A_\alpha}(1/p_n)T_n$. Further, $P_\alpha = \sum_{n\in A_\alpha}(1/p_n)\mathbb{Z} \subseteq \sum_{n\in A_\alpha}(1/p_n)T_n$. Hence $P_\alpha + \widehat{T}_\alpha \subseteq \sum_{n\in A_\alpha}(1/p_n)T_n$.
Conversely, let $n \in A_\alpha$ and $0 \neq t_n/p_n \in (1/p_n)T_n$. Write t_n as a reduced fraction $t_n = r/s$. Then the assumption $P \cap T = \mathbb{Z}$ implies that $\gcd(p_n, s) = 1$. In fact, if $s = p_n s'$, then $r/p_n = (r/s)s' \in P \cap T = \mathbb{Z}$, a contradiction.

We have a Bezout equation $up_n + vs = 1$ and

$$\frac{t_n}{p_n} = \frac{r}{sp_n} = \frac{r(up_n + vs)}{sp_n} = \frac{ru}{s} + \frac{rv}{p_n} \in T_n + P_\alpha.$$

This shows that $\sum_{n\in A_\alpha}(1/p_n)T_n \subseteq P_\alpha + \widehat{T}_\alpha$ and the desired equality is established.

To show that $P_\alpha + \widehat{T}_\alpha = P_\alpha \widehat{T}_\alpha$ we first note that trivially $P_\alpha + \widehat{T}_\alpha \subseteq P_\alpha \widehat{T}_\alpha$. The reverse containment follows from the partial fraction decomposition of an element $r \in P_\alpha$:

$$r = \frac{r_1}{p_{n_1}} + \frac{r_2}{p_{n_2}} + \cdots, \quad r_i \in \mathbb{Z},\ n_i \in A_\alpha.$$

(ii) Suppose that $\delta_\alpha \vee \tau_\beta \geq \delta_{\alpha'} \vee \tau_{\beta'}$. The lattice of types is distributive and the hypothesis $P \cap T = \mathbb{Z}$ implies that always $\delta_\rho \wedge \tau_\sigma = \delta = \operatorname{type}(\mathbb{Z})$. Hence $\delta_{\alpha'} \wedge \delta_\alpha = \delta_{\alpha'} \wedge (\delta_\alpha \vee \tau_\beta) \geq \delta_{\alpha'} \wedge (\delta_{\alpha'} \vee \tau_{\beta'}) = \delta_{\alpha'}$ and $\delta_\alpha \wedge \delta_{\alpha'} \geq \delta_{\alpha'}$ implies that $\delta_\alpha \geq \delta_{\alpha'}$. The inequality $\tau_\beta \geq \tau_{\beta'}$ is obtained analogously. □

To establish properties of Hawaiian groups we need a preparatory numerical lemma.

Lemma 2.5 *Assume* (2.3). *Then a finite sum* $\sum_{n\in\omega} t_n/p_n$ *with* $t_n \in T_n$ *is in* $\mathbb{Z}$ *if and only if* $t_n/p_n \in \mathbb{Z}$ *for all* $n \in \omega$.

Proof. Let F be a finite subset of ω and set $m := \sum_{i\in F} t_i/p_i \in \mathbb{Z}$ with $0 \neq t_i \in T_i$. Let $n \in F$ and write t_n as a reduced fraction $t_n = r/s$. Then the first equation of (2.3) implies that $\gcd(s, p_n) = 1$. We have $r/p_n = s(t_n/p_n)$, so

$$\frac{r}{p_n}\Big(\prod_{n\neq i\in F} p_i\Big) = s\Big(\prod_{n\neq i\in F} p_i\Big)\Big(m - \sum_{n\neq i\in F} \frac{t_i}{p_i}\Big) \in P \cap T = \mathbb{Z}.$$

Hence p_n divides r. Now $m - \sum_{n\neq i\in F} t_i/p_i = t_n/p_n \in T_n \cap \sum_{n\neq i\in F} T_i = \mathbb{Z}$. □

We introduce important subgroups of B.

Definition 2.6 Let $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ be a Hawaiian group. For each $\alpha \leq \kappa$ define

$$F_\alpha(\mathcal{R}, \mathcal{S}) := \Big(\bigoplus_{n\in\omega} R_n x_n\Big) \oplus \Big(\bigoplus_{\rho<\alpha} S_\rho y_\rho\Big)$$

and

$$B_\alpha := \langle F_\alpha(\mathcal{R}, \mathcal{S}),\, p_n^{-1}T_n(y_\rho - x_n)\colon \rho < \alpha,\, n \in A_\rho \rangle.$$

In particular, $B_0 = \bigoplus_{n\in\omega} R_n x_n$ and $B_\kappa = B$.

Lemma 2.7 *Let B be a Hawaiian group and let B_α $(\alpha < \kappa)$ be the ascending chain of subgroups defined in Definition 2.6. So $B = \bigcup_{\alpha<\kappa} B_\alpha$. Assume* (2.3) *and* (2.4). *Then the following statements hold.*

(i) *B_α is pure in B for every $\alpha < \kappa$.*

(ii) *$B/B_0 \cong \bigoplus_{\rho<\kappa}(S_\rho + \sum_{n\in A_\rho}(1/p_n)T_n)y_\rho$, the isomorphism being induced by the projection*

$$\pi\colon V \to \bigoplus_{\rho<\kappa} \mathbb{Q}y_\rho \quad \textit{along} \quad \mathbb{Q}B_0.$$

(iii) *For $\beta < \alpha$, $B_\alpha/B_\beta \cong \bigoplus_{\beta\le\rho<\alpha}(S_\rho + \sum_{n\in A_\rho}(1/p_n)T_n)y_\rho$, the isomorphism being induced by the projection*

$$\pi\colon \mathbb{Q}B_0 \oplus \Big(\bigoplus_{\rho<\alpha} \mathbb{Q}y_\rho\Big) \to \bigoplus_{\beta\le\rho<\alpha} \mathbb{Q}y_\rho$$
$$\textit{along} \quad \mathbb{Q}B_\beta = \mathbb{Q}B_0 \oplus \Big(\bigoplus_{\rho<\beta} \mathbb{Q}y_\rho\Big).$$

Proof. (i) Let $b \in B$ be as in (2.1) and assume that $mb \in B_\alpha$ for some nonzero integer m. By (2.2)

$$\forall \sigma \ge \alpha\colon s_\sigma + \sum_{n\in A_\sigma} \frac{t_{\sigma,n}}{p_n} = 0. \tag{2.6}$$

Hence $\forall \sigma \ge \alpha\colon s_\sigma = -\sum_{n\in A_\sigma}(t_{\sigma,n}/p_n) \in S_\sigma \cap P_\sigma \widehat{T}_\sigma \overset{\textit{Lemma } 2.4}{=} S_\sigma \cap (P_\sigma + \widehat{T}_\sigma) \subseteq S \cap (P + T) = (S \cap P) + (S \cap T) = \mathbb{Z}$ and by Lemma 2.5 we have that

$$\forall \sigma \ge \alpha,\, \forall n \in A_\sigma\colon \frac{t_{\sigma,n}}{p_n} \in \mathbb{Z}. \tag{2.7}$$

Therefore

$$\begin{aligned} b &= \sum_{n\in\omega} r_n x_n + \sum_{\rho<\kappa} s_\rho y_\rho + \sum_{(\rho,n)\colon\, \rho<\kappa, n\in A_\rho} \frac{t_{\rho,n}}{p_n}(y_\rho - x_n) \\ &= \sum_{n\in\omega}\Big(r_n - \sum_{\rho\colon\, n\in A_\rho} \frac{t_{\rho,n}}{p_n}\Big)x_n + \sum_{\rho<\kappa}\Big(s_\rho + \sum_{n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)y_\rho \end{aligned}$$

$$\begin{aligned}
&\overset{(2.6)}{=} \sum_{n\in\omega}\Big(r_n - \sum_{\rho\colon n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)x_n + \sum_{\rho<\alpha}\Big(s_\rho + \sum_{n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)y_\rho \\
&= \sum_{n\in\omega} r_n x_n - \sum_{n\in\omega}\Big(\sum_{\rho\colon n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)x_n + \sum_{\rho<\alpha} s_\rho y_\rho + \sum_{(\rho,n)\colon \rho<\alpha, n\in A_\rho}\frac{t_{\rho,n}}{p_n}y_\rho \\
&= \sum_{n\in\omega} r_n x_n - \sum_{n\in\omega}\Big(\sum_{\rho\colon \rho\geq\alpha, n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)x_n + \sum_{\rho<\alpha} s_\rho y_\rho \\
&\quad + \sum_{(\rho,n)\colon \rho<\alpha, n\in A_\rho}\frac{t_{\rho,n}}{p_n}(y_\rho - x_n) \overset{(2.7)}{\in} B_\alpha.
\end{aligned}$$

(iii) Let

$$\pi\colon \mathbb{Q}B_0 \oplus \Big(\bigoplus_{\rho<\alpha}\mathbb{Q}y_\rho\Big) \to \bigoplus_{\beta\leq\rho<\alpha}\mathbb{Q}y_\rho$$

be the projection along $\mathbb{Q}B_\beta$. We claim that $\pi\restriction_{B_\alpha}$ has image

$$\bigoplus_{\beta\leq\rho<\alpha}\Big(S_\rho + \sum_{n\in A_\rho}\frac{1}{p_n}T_n\Big)y_\rho$$

and kernel B_β.
By (i) B_β is pure in B_α and hence $B_\alpha \cap \mathbb{Q}B_\beta = B_\beta$. Therefore $\mathrm{Ker}(\pi\restriction_{B_\alpha}) = B_\beta$. Let $b \in B_\alpha$ be given in the representation (2.2). Then

$$b\pi = \sum_{\beta\leq\rho<\alpha}\Big(s_\rho + \sum_{n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)y_\rho \in \bigoplus_{\beta\leq\rho<\alpha}\Big(S_\rho + \sum_{n\in A_\rho}\frac{1}{p_n}T_n\Big)y_\rho.$$

Now suppose that $y \in \bigoplus_{\beta\leq\rho<\alpha}(S_\rho + \sum_{n\in A_\rho}(1/p_n)T_n)y_\rho$. Then y has the form

$$y = \sum_{\beta\leq\rho<\alpha}\Big(s_\rho + \sum_{n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)y_\rho \quad \text{with} \quad s_\rho \in S_\rho \text{ and } t_{\rho,n} \in T_n.$$

Then

$$b := \sum_{\beta\leq\rho<\alpha} s_\rho y_\rho + \sum_{(\rho,n)\colon \beta\leq\rho<\alpha, n\in A_\rho}\frac{t_{\rho,n}}{p_n}(y_\rho - x_n) \in B_\alpha$$

and clearly $b\pi = y$. This proves (iii).
Finally, (ii) is the special case $\alpha = \kappa$ and $\beta = 0$ of (iii). □

Using Lemma 2.4 we can reformulate results in Lemma 2.7.

Corollary 2.8 *Let B be a Hawaiian group and assume* (2.3) *and* (2.4)*. Then*

$$B/B_0 \cong \bigoplus_{\rho<\kappa} (P_\rho + S_\rho + \widehat{T}_\rho) y_\rho.$$

The groups $Y_{\alpha,\beta} := \langle y_\alpha - y_\beta \rangle_*^B \subseteq B$ turn out to be very important in studying the endomorphism ring of a Hawaiian group. We describe next the structure of $Y_{\alpha,\beta}$.

Lemma 2.9 *Let $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ be a Hawaiian group. Assume* (2.3), (2.4)*, and* (2.5)*. Then*

$$\forall \alpha < \beta < \kappa \colon Y_{\alpha,\beta} = (P_{\alpha,\beta} + S_{\alpha,\beta} + T_{\alpha,\beta})(y_\alpha - y_\beta).$$

In particular,

$$\forall \alpha < \beta < \kappa \colon \operatorname{type}(Y_{\alpha,\beta}) = \delta_{\alpha,\beta} \vee \tau_{\alpha,\beta} \vee \sigma_{\alpha,\beta}.$$

Proof. Suppose that $\alpha < \beta < \kappa$.
Let $s \in S_{\alpha,\beta} = S_\alpha \cap S_\beta$. Then $sy_\alpha \in B$ and $sy_\beta \in B$, so $s(y_\alpha - y_\beta) \in B$. Hence

$$S_{\alpha,\beta}(y_\alpha - y_\beta) \subseteq B.$$

The assumption $A_\alpha \cap A_\beta \neq \emptyset$ assures that there is $m \in A_\alpha \cap A_\beta$. Let m be such an integer.

Let $t \in T_m$. Then $t(y_\alpha - x_m) \in B$ and $t(y_\beta - x_m) \in B$. Hence

$$t(y_\alpha - y_\beta) = t(y_\alpha - x_m) - t(y_\beta - x_m) \in B.$$

This shows that $T_m(y_\alpha - y_\beta) \subseteq B$ and therefore, m having been any element of $A_\alpha \cap A_\beta$,

$$T_{\alpha,\beta}(y_\alpha - y_\beta) \subseteq B.$$

Finally, let $p_m^{-1} \in P_{\alpha,\beta}$. Then $p_m^{-1}(y_\alpha - x_m) \in B$ and $p_m^{-1}(y_\beta - x_m) \in B$, hence also $p_m^{-1}(y_\alpha - y_\beta) \in B$ which shows that

$$P_{\alpha,\beta}(y_\alpha - y_\beta) \in B.$$

Altogether we have

$$(S_{\alpha,\beta} + T_{\alpha,\beta} + P_{\alpha,\beta})(y_\alpha - y_\beta) \subseteq Y_{\alpha,\beta}.$$

For the reverse containment we make use of the projection π in Lemma 2.7(ii). We have $Y_{\alpha,\beta} = U(y_\alpha - y_\beta)$ for some rational group U containing $S_{\alpha,\beta} + T_{\alpha,\beta} + P_{\alpha,\beta}$ and, by Corollary 2.8,

$$\begin{aligned}(Y_{\alpha,\beta})\pi &= U(y_\alpha - y_\beta)\pi \subseteq Uy_\alpha \oplus Uy_\beta\\ &\subseteq (S_\alpha + P_\alpha + \widehat{T}_\alpha)y_\alpha \oplus (S_\beta + P_\beta + \widehat{T}_\beta)y_\beta.\end{aligned}$$

Hence by our assumptions ([Ma, Lemma 1.2.13] for distributivity)

$$\begin{aligned}U &\subseteq (S_\alpha + P_\alpha + \widehat{T}_\alpha) \cap (S_\beta + P_\beta + \widehat{T}_\beta)\\ &= (S_\alpha \cap S_\beta) + (S_\alpha \cap P_\beta) + (S_\alpha \cap \widehat{T}_\beta) + (P_\alpha \cap S_\beta) + (P_\alpha \cap P_\beta)\\ &\quad +(P_\alpha \cap \widehat{T}_\beta) + (\widehat{T}_\alpha \cap S_\beta) + (\widehat{T}_\alpha \cap P_\beta) + (\widehat{T}_\alpha \cap \widehat{T}_\beta)\\ &\overset{2.4(\mathrm{ii})}{=} (S_\alpha \cap S_\beta) + \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + (P_\alpha \cap P_\beta) + \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + (\widehat{T}_\alpha \cap \widehat{T}_\beta)\\ &\overset{2.3}{=} S_{\alpha,\beta} + P_{\alpha,\beta} + T_{\alpha,\beta}.\end{aligned}$$

We have established that $U = S_{\alpha,\beta} + P_{\alpha,\beta} + T_{\alpha,\beta}$. □

Lemma 2.10 *Assume* (2.3), (2.4), (2.5) *and* $R \cap S = \mathbb{Z}$. *Then*

$$\forall \alpha < \kappa,\ \forall n \in A_\alpha \colon \langle y_\alpha - x_n\rangle_*^B = \frac{1}{p_n} T_n(y_\alpha - x_n).$$

Proof. As it is clear that $\langle y_\alpha - x_n\rangle_*^B \supset (1/p_n)T_n(y_\alpha - x_n)$, we suppose that

$$b = \sum_{i\in\omega}\Big(r_i - \sum_{\rho\colon i\in A_\rho}\frac{t_{\rho,i}}{p_i}\Big)x_i + \sum_{\rho<\kappa}\Big(s_\rho + \sum_{i\in A_\rho}\frac{t_{\rho,i}}{p_i}\Big)y_\rho \in \langle y_\alpha - x_n\rangle_*^B.$$

Then there is a positive integer m such that $mb = q(y_\alpha - x_n)$ for some rational number q. It follows that

$$\forall i \neq n\colon r_i - \sum_{\rho\colon i\in A_\rho}\frac{t_{\rho,i}}{p_i} = 0, \quad \text{and} \quad \forall \rho \neq \alpha\colon s_\rho + \sum_{i\in A_\rho}\frac{t_{\rho,i}}{p_i} = 0.$$

Hence

$$b = \Big(r_n - \sum_{\rho\colon n\in A_\rho}\frac{t_{\rho,n}}{p_n}\Big)x_n + \Big(s_\alpha + \sum_{i\in A_\alpha}\frac{t_{\alpha,i}}{p_i}\Big)y_\alpha,$$

and

$$m\Big(r_n - \frac{1}{p_n}\sum_{\rho\colon n\in A_\rho} t_{\rho,n}\Big)x_n + m\Big(s_\alpha + \sum_{i\in A_\alpha}\frac{t_{\alpha,i}}{p_i}\Big)y_\alpha = q(y_\alpha - x_n).$$

So

$$m\Big(r_n - \frac{1}{p_n}\sum_{\rho\colon n\in A_\rho} t_{\rho,n}\Big) = -q \quad\text{and}\quad m\Big(s_\alpha + \sum_{i\in A_\alpha}\frac{t_{\alpha,i}}{p_i}\Big) = q.$$

Adding the two equations and canceling m we obtain

$$r_n - \frac{1}{p_n}\sum_{\rho\colon n\in A_\rho} t_{\rho,n} + s_\alpha + \sum_{i\in A_\alpha}\frac{t_{\alpha,i}}{p_i} = 0.$$

Hence

$$\begin{aligned} r_n + s_\alpha &= \frac{1}{p_n}\sum_{\rho\colon n\in A_\rho} t_{\rho,n} - \sum_{i\in A_\alpha}\frac{t_{\alpha,i}}{p_i} \\ &\in (R+S)\cap PT = (R+S)\cap(P+T) = \mathbb{Z}. \end{aligned}$$

Our assumption $R\cap S = \mathbb{Z}$ further implies that

$$r_n \in \mathbb{Z},\ \ s_\alpha \in \mathbb{Z},\ \ \frac{1}{p_n}\sum_{\rho\colon n\in A_\rho} t_{\rho,n} - \sum_{i\in A_\alpha}\frac{t_{\alpha,i}}{p_i} \in \mathbb{Z}.$$

In the last expression the term $t_{\alpha,n}/p_n$ cancels and we are left with

$$\frac{1}{p_n}\sum_{\rho\colon \rho\neq\alpha, n\in A_\rho} t_{\rho,n} - \sum_{i\colon i\neq n, i\in A_\alpha}\frac{t_{\alpha,i}}{p_i} \in \mathbb{Z}.$$

By Lemma 2.5

$$-\frac{1}{p_n}\sum_{\rho\colon \rho\neq\alpha, n\in A_\rho} t_{\rho,n} \in \mathbb{Z} \quad\text{and}\quad \forall i\neq n,\ i\in A_\alpha\colon \frac{t_{\alpha,i}}{p_i}\in\mathbb{Z}.$$

Hence $b = zx_n + sy_\alpha + (1/p_n)t_{\alpha,n}(y_\alpha - x_n)$ for integers z, s. Using again that $mb = q(y_\alpha - x_n)$ we find that $z = -s$, and finally that

$$\begin{aligned} b &= \Big(s + \frac{1}{p_n}t_{\alpha,n}\Big)(y_\alpha - x_n) \\ &= \frac{1}{p_n}(sp_n + t_{\alpha,n})(y_\alpha - x_n) \in \frac{1}{p_n}T_n(y_\alpha - x_n). \end{aligned}$$

□

Before determining the endomorphism ring of certain Hawaiian groups we show that all Hawaiian groups are finitely Butler.

Theorem 2.11 *Let* $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ *and assume* (2.3)*,* (2.4)*, and* (2.5)*. Then* B *is finitely Butler.*

Proof. Let $H \subseteq_* B$ be a pure subgroup of B of finite rank. Then there is a finite set $E \subseteq \kappa$ and a finite subset $W \subseteq \omega$ such that

$$H \subseteq_* L_* \text{ with } L := \Big\langle F_H,\, \frac{1}{p_n} T_n(y_\alpha - x_n) \colon \alpha \in E,\, n \in A_\alpha \cap W \Big\rangle$$

where $F_H = (\bigoplus_{m \in W} R_m x_m) \oplus (\bigoplus_{\rho \in E} S_\rho y_\rho)$. Without loss of generality we may assume that $A_\alpha \cap A_\beta \cap W \neq \emptyset$ for all $\alpha, \beta \in E$ which assures that $Y_{\alpha,\beta} \subseteq L_*$.

We claim that $L_* \subseteq B$ is a Butler group. Since the class of Butler groups is closed under pure subgroups, also H is then a Butler group of finite rank. We first define a completely decomposable group of finite rank as an external direct sum

$$C = F_H \oplus \bigoplus_{(\alpha,n) \colon \alpha \in E, n \in A_\alpha \cap W} \frac{1}{p_n} T_n(y_\alpha - x_n) \oplus \bigoplus_{\alpha,\beta \in E, \alpha < \beta} Y_{\alpha,\beta}.$$

Since all summands of C are subgroups of L_* we obtain a homomorphism

$$\varphi \colon C \to L_*$$

induced by the inclusion of the summands of C. It remains to prove that φ is surjective. Let $b \in B$ and assume that $mb \in L$ for some $0 \neq m \in \mathbb{N}$, i.e., $b \in L_*$. Then

$$mb = \sum_{n \in W} r'_n x_n + \sum_{\rho \in E} s'_\rho y_\rho + \sum_{(\rho,n) \colon \rho \in E, n \in A_\rho \cap W} \frac{t'_{\rho,n}}{p_n}(y_\rho - x_n) \tag{2.8}$$

for some $r'_n \in R_n$, $s'_\rho \in S_\rho$ and $t'_{\rho,n} \in T_n$ with $n \in W$, $\rho \in E$. Let b be given in the form of (2.1). Then

$$mb = \sum_{n \in \omega} m r_n x_n + \sum_{\rho < \kappa} m s_\rho y_\rho + \sum_{(\rho,n) \colon \rho < \kappa, n \in A_\rho} \frac{m t_{\rho,n}}{p_n}(y_\rho - x_n)$$

for some $r_n \in R_n$, $s_\rho \in S_\rho$ and $t_{\rho,n} \in T_n$ where $n \in \omega$, $\rho < \kappa$. Equating coefficients of y_α in V, we obtain that

$$\forall \alpha \notin E \colon m\Big(s_\alpha + \sum_{n \in A_\alpha} \frac{t_{\alpha,n}}{p_n}\Big) = 0.$$

By Lemma 2.5 it follows that

$$\forall \alpha \notin E,\, \forall n \in A_\alpha \colon \frac{t_{\alpha,n}}{p_n} \in \mathbb{Z}.$$

Note that $S \cap P = S \cap T = \mathbb{Z}$ and hence $s_\alpha = -\sum_{n \in A_\alpha} t_{\alpha,n}/p_n \in S \cap (P + T) = \mathbb{Z}$. Hence, combining and renaming coefficients we may assume without loss of generality that

$$\begin{aligned} b &= \sum_{n\in\omega} r_n x_n + \sum_{\rho\in E} s_\rho y_\rho + \sum_{(\rho,n)\colon\, \rho\in E, n\in A_\rho} \frac{t_{\rho,n}}{p_n}(y_\rho - x_n) \\ &= \Big(\sum_{n\in W} r_n x_n + \sum_{\rho\in E} s_\rho y_\rho + \sum_{(\rho,n)\colon\, \rho\in E, n\in A_\rho\cap W} \frac{t_{\rho,n}}{p_n}(y_\rho - x_n) \Big) \\ &\quad + \sum_{n\notin W} r_n x_n + \sum_{(\rho,n)\colon\, \rho\in E, n\in A_\rho\setminus W} \frac{t_{\rho,n}}{p_n}(y_\rho - x_n). \end{aligned}$$

Obviously, $h := \sum_{n\in W} r_n x_n + \sum_{\rho\in E} s_\rho y_\rho + \sum_{(\rho,n)\colon\, \rho\in E, n\in A_\rho\cap W} (t_{\rho,n}/p_n)(y_\rho - x_n)$ is in the image of φ, so it remains to prove that

$$\sum_{n\notin W} r_n x_n + \sum_{(\rho,n)\colon\, \rho\in E, n\in A_\rho\setminus W} \frac{t_{\rho,n}}{p_n}(y_\rho - x_n) \in C\varphi.$$

Fix $k \notin W$. For our current b, the multiple mb must have the form (2.8). Equating coefficients of x_k yields that

$$m\Big(r_k - \sum_{\rho\colon\, \rho\in E, k\in A_\rho\setminus W} \frac{t_{\rho,k}}{p_k} \Big) = 0.$$

If there is no $\rho \in E$ such that $k \in A_\rho$, then $r_k = 0$. So assume that there is $\rho \in E$ with $k \in A_\rho$. Then

$$r_k = \sum_{\rho\colon\, \rho\in E, k\in A_\rho\setminus W} \frac{t_{\rho,k}}{p_k} \in R_k \cap \frac{1}{p_k} T_k = \mathbb{Z}.$$

Hence

$$\frac{1}{p_k} \sum_{\rho\colon\, \rho\in E, k\in A_\rho\setminus W} t_{\rho,k} \in \mathbb{Z}. \tag{2.9}$$

Let $\{\alpha_1, \ldots, \alpha_n\} = \{\alpha \in E\colon k \in A_\alpha \setminus W\}$. Then

$$\begin{aligned} & r_k x_k + \frac{1}{p_k} \sum_{\rho\colon \rho \in E, k \in A_\rho \setminus W} t_{\rho,k}(y_\rho - x_k) \\ &= r_k x_k + \frac{1}{p_k} \sum_{\rho \in \{\alpha_1, \ldots, \alpha_n\}} t_{\rho,k}(y_\rho - x_k) \\ &= r_k x_k + \frac{1}{p_k} t_{\alpha_1,k}(y_{\alpha_1} - y_{\alpha_2}) + \frac{1}{p_k}(t_{\alpha_1,k} + t_{\alpha_2,k})(y_{\alpha_2} - y_{\alpha_3}) + \cdots \\ &\quad + \frac{1}{p_k}(t_{\alpha_1,k} + \cdots + t_{\alpha_{n-1},k})(y_{\alpha_{n-1}} - y_{\alpha_n}) \\ &\quad + \frac{1}{p_k}(t_{\alpha_1,k} + \cdots + t_{\alpha_n,k})(y_{\alpha_n} - x_k), \end{aligned}$$

which is an element of $\operatorname{Im}\varphi$ since $(1/p_k)(t_{\alpha_1,k} + \cdots + t_{\alpha_n,k}) \in \mathbb{Z}$ by equation (2.9) and $(1/p_k)(t_{\alpha_1,k} + \cdots + t_{\alpha_i,k}) \in (1/p_k)T_k$ for all i. This finishes the proof. $\square$

3. Strong Hawaiian groups

We now need to make stronger assumptions on our Hawaiian groups by placing further restrictions on the defining sequences $\mathcal{A}$.

Definition 3.1 Let $\aleph$ be an infinite cardinal. A set of types $\{\delta_\alpha \mid \alpha < \aleph\}$ is called a *strong anti-chain of size* $\aleph$ if it satisfies the following condition. If $\alpha < \aleph$ and $E \subseteq \aleph$ is a finite subset of $\aleph$ such that $\alpha \notin E$, then δ_α and $\bigwedge_{\rho \in E} \delta_\rho$ are incomparable.

Strong anti-chains satisfy stronger conditions than those postulated by the definition.

Lemma 3.2 *Let $\Delta = \{\delta_\alpha \mid \alpha < \aleph\}$ be a strong anti-chain of size $\aleph$. If E and E' are finite subsets of $\aleph$ such that $E \not\subseteq E'$ and $E' \not\subseteq E$, then $\bigwedge_{\beta \in E} \delta_\beta$ and $\bigwedge_{\beta \in E'} \delta_\beta$ are incomparable.*

Proof. By way of contradiction assume that $\bigwedge_{\beta \in E} \delta_\beta$ and $\bigwedge_{\beta \in E'} \delta_\beta$ are comparable. Without loss of generality we may assume that $\bigwedge_{\beta \in E} \delta_\beta \leq \bigwedge_{\beta \in E'} \delta_\beta$. Choosing $\alpha \in E' \setminus E$ which exists by assumption, we obtain that $\bigwedge_{\beta \in E} \delta_\beta \leq \delta_\alpha$ contradicting the fact that Δ is a strong anti-chain. $\square$

We prove next that strong anti-chains exist in ZFC.

Lemma 3.3 *There exist strong anti-chains of size* $\aleph_0$.

Proof. We define subsets A_m of ω using the binary expansion of integers. For $m \in \omega$ let

$$A_0 = \{0\} \cup \Big\{ \sum_{i \in \omega} a_i 2^i \;\Big|\; a_0 = 1 \Big\}$$

$$\text{and for } m \geq 1 \quad A_m := \Big\{ \sum_{i \in \omega} a_i 2^i \;\Big|\; a_m = 1 \Big\}.$$

Then the following statements are evident.

- A_m is infinite;
- $\bigcup_{m<\omega} A_m = \omega$;
- $\bigcap_{m<\omega} A_m = \emptyset$;
- For disjoint finite subsets E and E' of ω the intersection

$$\Big(\bigcap_{n \in E} A_n \Big) \cap \Big(\bigcap_{n \in E'} \complement A_n \Big) \text{ is infinite.}$$

Here $\complement$ stands for complementation in ω. Let $\mathcal{P} = \{p_0, p_1, \ldots\}$ be an infinite set of primes and define $\delta_m := \operatorname{type}\langle p_i^{-1} \mid i \in A_m \rangle$. Then it is easy to see that $\{\delta_m \mid m \in \omega\}$ is a strong anti-chain of size $\aleph_0$. $\square$

We can also show that there exist strong anti-chains of size $2^{\aleph_0}$.

Theorem 3.4 *There exists a strong anti-chain of cardinality* $2^{\aleph_0}$.

Proof. The result follows from [Je, Lemma 24.8] but for the convenience of the reader we recall it. Let λ be an infinite cardinal. A family $\mathfrak{C}$ of subsets of a cardinal λ is called *uniformly independent* if for any distinct sets $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ in $\mathfrak{C}$ the intersection $X_1 \cap \cdots \cap X_n \cap \complement Y_1 \cap \cdots \cap \complement Y_m$ has cardinality λ. Here $\complement$ denotes complementation in λ. Obviously, for $\lambda = \omega$, the existence of a uniformly independent family $\{A_\alpha \colon \alpha < 2^{\aleph_0}\}$ of subsets of ω yields the desired strong anti-chain by choosing an infinite set of primes $\mathcal{P} = \{p_0, p_1, \ldots\}$ putting $R_\alpha = \langle p_i^{-1} \colon i \in A_\alpha \rangle$, and $\delta_\alpha = \operatorname{type} R_\alpha$ for $\alpha < 2^{\aleph_0}$.

[Je, Lemma 24.8] states that for every infinite cardinal λ there is a uniformly independent family of size 2^λ. A proof is as follows.

Let P be the collection of all pairs $(F, \mathfrak{F})$ where F is a finite subset of λ and $\mathfrak{F}$ is a finite set of finite subsets of λ. Then $|P| = \lambda$ and it suffices to find a uniformly independent family of subsets of P of size 2^λ. For $u \subseteq \lambda$

let $X_u = \{(F, \mathfrak{F}) \in P\colon F \cap u \in \mathfrak{F}\}$ and let $\mathfrak{C} = \{X_u\colon u \subseteq \lambda\}$. It is easy to check that all the X_u's are distinct and hence $|\mathfrak{C}| = 2^\lambda$. To show that $\mathfrak{C}$ is uniformly independent let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be distinct subsets of λ. Choose $\alpha_{i,j} \in u_i \backslash v_j$ or $\alpha_{i,j} \in v_j \backslash u_i$ for $i \leq n$ and $j \leq m$. For any finite subset F of λ containing all the $\alpha_{i,j}$'s we let $\mathfrak{F} = \{F \cap u_i\colon i \leq n\}$. Then $(F, \mathfrak{F}) \in X_{u_i}$ for all $i \leq n$ and $(F, \mathfrak{F}) \notin X_{v_j}$ for all $j \leq m$. Thus the intersection $X_{u_1} \cap \cdots \cap X_{u_n} \cap \complement X_{v_1} \cap \cdots \cap \complement X_{v_m}$ has cardinality λ. $\square$

Recall that in the context of Hawaiian groups we have $P_\alpha := \langle p_n^{-1}\colon n \in A_\alpha\rangle$ and $\delta_\alpha := \mathrm{type}(P_\alpha)$.

Definition 3.5 A Hawaiian group $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ is called a *strong Hawaiian group* if $\forall\, \alpha < \kappa\colon S_\alpha = \mathbb{Z}$ and $\{\delta_\alpha \mid \alpha < \kappa\}$ is a strong anti-chain of size κ. We denote the strong Hawaiian group with the specified data by $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$.

Note that for $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ the condition (2.4) is satisfied due to the special choice of $\mathcal{S}$. For the next theorem recall that a torsion-free group G is called *finitely filtered* if G is the union of an ascending continuous sequence $\{G_\alpha\colon \alpha < \lambda\}$ of pure subgroups G_α such that for every $\alpha < \lambda$ there is a finite rank subgroup H_α of G with $G_{\alpha+1} = G_\alpha + H_\alpha$.

Theorem 3.6 *Let* $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ *and assume that* $\mathrm{cf}(\kappa) > \aleph_0$. *Then B is not a finitely filtered group. In particular, B is not a B_2-group.*

Proof. The proof is essentially contained in [ShSt, Theorem 5.1] but for the convenience of the reader we shall recall the main steps. By way of contradiction assume that B is finitely filtered. Hence (the B_α in this proof have no relation to the B_α in Definition 2.6)

$$B = \bigcup_{\alpha<\kappa} B_\alpha$$

where $B_{\alpha+1} = B_\alpha + H_\alpha$ for some finite rank subgroup H_α. It is now straight forward to check that the set $C = \{\delta < \kappa \mid B_\delta = \langle x_n, y_\beta\colon n < \omega, \beta < \delta\rangle_*\}$ is a closed unbounded subset (cub) of κ since κ has uncountable cofinality (see [ShSt, Lemma 5.2]). Now let $\delta \in C$ be such that $\delta > \aleph_0$ and w.l.o.g. let δ be a limit ordinal. This is possible since C is a cub. Note that $y_\delta \notin B_\delta$. However, as in [ShSt, Lemma 5.3] one proves that there exist $n^* < \omega$ and a sequence of ordinals $\delta \leq \alpha_1 \leq \alpha_2 \cdots \leq \alpha_{n^*} < \kappa$ such that

$$\langle B_\delta + \mathbb{Z}y_\delta \rangle_* \subseteq \sum_{m \leq n^*} H_{\alpha_m} + B_\delta.$$

For every $m \leq n^*$ we choose a finite set $W_m \subseteq \kappa$ and an integer $n_m < \omega$ such that

$$H_{\alpha_m} \subseteq \Big\langle \sum_{\gamma \in W_m} \mathbb{Z}y_\gamma + \sum_{i \leq n_m} R_i x_i \Big\rangle_*.$$

Collecting all these generators and letting $W = \bigcup_{m \leq n^*} W_m$, $k = \max\{n_m : m \leq n^*\}$ and $H = \langle \sum_{\gamma \in W} \mathbb{Z}y_\gamma + \sum_{i \leq k} R_i x_i \rangle_*$ we obtain

$$\langle B_\delta + \mathbb{Z}y_\delta \rangle_* \subseteq H + B_\delta. \tag{3.1}$$

Now choose $\beta < \delta \backslash W$ and let $n \geq k$ be such that

$$n \in (A_\beta \cap A_\delta) \backslash \bigcup_{\gamma \in W, \gamma \neq \delta} A_\gamma.$$

Note that this choice is possible since B is a strong Hawaiian group and hence the types δ_α $(\alpha < \kappa)$ form a strong anti-chain. It is now straightforward to see that $p_n^{-1}(y_\delta - y_\beta)$ is an element of $\langle B_\delta + \mathbb{Z}y_\delta \rangle_*$ but it is not an element of $H + B_\delta$ contradicting equation (3.1) (see also [ShSt, Lemma 5.3]). $\square$

Note that every $\aleph_0$-Hawaiian group is a B_2-group since it is finitely Butler. Finally we show that a strong Hawaiian group possesses many fully invariant subgroups.

Lemma 3.7 *Let* $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ *and assume* (2.3) *and* (2.5). *Then the following hold for* $\alpha < \beta < \kappa$.
(i) $B(\text{type}(Y_{\alpha,\beta})) = Y_{\alpha,\beta}$;
(ii) $Y_{\alpha,\beta}$ *is a pure fully invariant subgroup of* B.

Proof. Put $\mu_{\alpha,\beta} = \text{type}(Y_{\alpha,\beta}) = \delta_{\alpha,\beta} \vee \tau_{\alpha,\beta}$. The inclusion $Y_{\alpha,\beta} \subseteq B(\mu_{\alpha,\beta})$ is obvious. Conversely, let $b \in B(\mu_{\alpha,\beta})$. Then $\text{type}^B(b) \geq \mu_{\alpha,\beta}$. Hence $\text{type}^{B/B_0}(b + B_0) \geq \mu_{\alpha,\beta}$. Since B is strong Hawaiian, the types δ_γ, $\gamma < \kappa$, form a strong anti-chain and hence $(B/B_0)(\mu_{\alpha,\beta}) \cong (P_\alpha + \widehat{T}_\alpha)y_\alpha \oplus (P_\beta + \widehat{T}_\beta)y_\beta$ by Lemma 3.2 and Lemma 2.7(ii). By Lemma 2.7(ii) the types $\delta_\gamma \vee \tau_\gamma$, $\gamma < \kappa$, form a strong anti-chain. Using the isomorphism π from Lemma 2.7(ii) we obtain that

$$b = r_\alpha y_\alpha + r_\beta y_\beta + \sum_{n\in\omega} r_n x_n \\ + \sum_{n\in A_\alpha} \frac{t_{\alpha,n}}{p_n}(y_\alpha - x_n) + \sum_{n\in A_\beta} \frac{t_{\beta,n}}{p_n}(y_\beta - x_n)$$

for some $r_\alpha \in P_\alpha$, $r_\beta \in P_\beta$, $t_{\alpha,n}, t_{\beta,n} \in T_n$, and $r_n \in R_n$ for $n \in \omega$. Hence multiplying by a suitable product K of prime numbers and renaming coefficients we may assume without loss of generality that

$$Kb = r_\alpha(y_\alpha - y_\beta) + (r_\beta + r_\alpha)y_\beta + \sum_{n\in\omega} r_n x_n + \sum_{n\in A_\alpha} t_{\alpha,n}(y_\alpha - y_\beta) \\ + \sum_{n\in A_\alpha} t_{\alpha,n}(y_\beta - x_n) + \sum_{n\in A_\beta} t_{\beta,n}(y_\beta - x_n)$$

with $t_{\alpha,n}, t_{\beta,n} \in \mathbb{Z}$. Thus

$$Kb - r_\alpha(y_\alpha - y_\beta) - \sum_{n\in A_\alpha} t_{\alpha,n}(y_\alpha - y_\beta) \\ = (r_\beta + r_\alpha)y_\beta + \sum_{n\in\omega} r_n x_n + \sum_{n\in A_\alpha} t_{\alpha,n}(y_\beta - x_n) \\ + \sum_{n\in A_\beta} t_{\beta,n}(y_\beta - x_n) \in B(\mu_{\alpha,\beta}).$$

Using that B is a strong Hawaiian group (hence the types δ_α where $\alpha < \kappa$, form a strong anti-chain) it is now easy to check that this can only happen if $Kb - r_\alpha(y_\alpha - y_\beta) - \sum_{n\in A_\alpha} t_{\alpha,n}(y_\alpha - y_\beta) = 0$ and hence $Kb \in Y_{\alpha,\beta}$. By purity it follows that $b \in Y_{\alpha,\beta}$. This shows that $Y_{\alpha,\beta} = B(\mu_{\alpha,\beta})$ and hence (i) and (ii) hold. □

4. Endomorphism rings of Hawaiian groups

We begin with an explanation on notations used for classes of Hawaiian groups. These are as follows.

- $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ is most general Hawaiian group where $\mathcal{R} = (R_n \mid n < \omega)$, $\mathcal{S} = (S_\alpha \mid \alpha < \kappa)$, $\mathcal{T} = (T_n \mid n < \omega)$, are sequences of arbitrary rational groups, $\mathcal{A} = (A_\alpha \mid \alpha < \kappa)$ is a family of subsets of ω such that $A_\alpha \cap A_\beta \neq \emptyset$, $\bigcup_{\alpha<\kappa} A_\alpha = \omega$, and $\bigcap_{\alpha<\kappa} A_\alpha = \emptyset$, and $\mathcal{P} = (p_n : n \in \omega)$ is a sequence of primes.
- $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ is the group $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ with $\mathcal{S} = (\mathbb{Z}, \mathbb{Z}, \ldots)$.

- $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ is the group $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}, \mathcal{P})$ where $\mathcal{A}$ is such that the types δ_α form a strong anti-chain.
- $\mathcal{B}_\kappa(\mathcal{R}^*, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ is the group $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ where $\mathcal{R}$ is such that the types $\operatorname{type}(R_n)$ are pairwise incomparable.
- Finally, $\mathcal{B}_\kappa(\mathcal{A}^*, \mathcal{P})$ (respectively $\mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$) is the group $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ (respectively $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}, \mathcal{P})$) where $\mathcal{R}$ and $\mathcal{T}$ are all sequences of the integers $\mathbb{Z}$.

The scheme is to drop sequences of $\mathbb{Z}$ from the listing, and to place a $*$ if a special condition is imposed on the sequence.

In this section we consider Hawaiian groups $\mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$. Note that (2.3), (2.4), and (2.5) are satisfied for these special groups. We will show that these Hawaiian groups have many endomorphisms. Recall that all Hawaiian groups $\mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$ are of the form

$$B = \langle F,\, p_n^{-1}(y_\alpha - x_n) \colon n \in A_\alpha,\, \alpha < \kappa\rangle \quad \text{where} \quad F = \bigoplus_{n\in\omega} \mathbb{Z}x_n \oplus \bigoplus_{\alpha<\kappa} \mathbb{Z}y_\alpha.$$

Definition 4.1 Let $B = \mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$. Recall that $\mathcal{A} = (A_\alpha \colon \alpha < \kappa)$. Let
(i) $A'_\alpha \subseteq A_\alpha$ for $\alpha < \kappa$;
(ii) $\mathcal{A}' = (A'_\alpha : \alpha < \kappa)$;
(iii) $N = \bigcup_{\alpha<\kappa} A'_\alpha$ and $\delta \colon N \to \kappa$ be a function such that $(n)\delta \in \{\alpha \colon \alpha < \kappa,\, n \in A'_\alpha\}$ for every $n \in N$;
(iv) $\bar{b} = (b,\, b_n \colon n \in \omega)$ with $b,\, b_n \in B$ for all $n \in \omega$;
(v) $m \in \mathbb{Z}$.

Then a linear transformation $\phi = \phi_{\mathcal{A}',\delta,\bar{b},m} \in \operatorname{End}_{\mathbb{Q}}(V)$ is defined as follows:
(i) $(y_\alpha)\phi = my_\alpha + b$ for $\alpha < \kappa$;
(ii) $(x_n)\phi = my_{(n)\delta} + b - p_n b_n$ if $n \in N$;
(iii) $(x_n)\phi = mx_n + b - p_n b_n$ if $n \notin N$.

A linear transformation ϕ of V is called a *Hawaiian transformation* if there exist $\mathcal{A}'$, δ, $\bar{b}$, m as above such that $\phi = \phi_{\mathcal{A}',\delta,\bar{b},m}$.

Lemma 4.2 *Let $B = \mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$. Then every Hawaiian transformation of V is an endomorphism of B.*

Proof. Let $\phi = \phi_{\mathcal{A}',\delta,\bar{b},m}$ be a Hawaiian transformation of V. By definition ϕ satisfies $(F)\phi \subseteq B$. Now, let $\alpha < \kappa$ and $n \in A_\alpha$. If $n \notin N$, then

$$(y_\alpha - x_n)\phi = my_\alpha + b - (mx_n + b - p_n b_n) = m(y_\alpha - x_n) + p_n b_n$$

and hence $(y_\alpha - x_n)\phi$ is p_n-divisible in B which shows that $(1/p_n)(y_\alpha - x_n)\phi \in B$. If $n \in N$, then

$$\begin{aligned}(y_\alpha - x_n)\phi &= my_\alpha + b - (my_{\delta(n)} + b - p_n b_n)\\ &= m(y_\alpha - y_{\delta(n)}) + p_n b_n\\ &= m(y_\alpha - x_n) - m(y_{\delta(n)} - x_n) + p_n b_n\end{aligned}$$

which is p_n-divisible in B since $n \in A_\alpha$ and $n \in A_{\delta(n)}$, hence p_n divides $y_\alpha - y_{\delta(n)}$ in B. This finishes the proof. □

To illustrate the Hawaiian transformations that induce endomorphisms of Hawaiian groups let us give three examples.

Example 4.3 Let $B = \mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$. If $b \in B$ and $m \in \mathbb{Z}$, then the linear transformation of V defined by $y_\alpha \mapsto my_\alpha + b$ and $x_n \mapsto mx_n + b$ for $\alpha < \kappa$, and $n \in \omega$ is an endomorphism of B.

Proof. Follows from the Lemma 4.2 choosing $N = \emptyset = A'_\alpha$ for all $\alpha < \kappa$, and $b_n = 0$ for all $n \in \omega$. □

Example 4.4 Let $B = \mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$. For (fixed) $\alpha < \kappa$ and $m \in \mathbb{Z}$, the linear transformation of V defined by $y_\beta \mapsto m(y_\beta - y_\alpha)$ for all $\beta < \kappa$, $x_n \mapsto m(x_n - y_\alpha)$ for $n \notin A_\alpha$ and $x_n \mapsto 0$ for $n \in A_\alpha$, restricts to an endomorphism of B.

Proof. This follows from Lemma 4.2 choosing $b = -my_\alpha$, $b_n = 0$ for every $n \in \omega$, $N = A_\alpha = A'_\alpha$ and $A'_\beta = \emptyset$ for $\beta \neq \alpha$ as well as $\delta(n) = \alpha$ for $n \in N$. □

Example 4.5 Let $B = \mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$. For $m \in \mathbb{Z}$, the linear transformation φ of V defined by $y_\beta \mapsto my_\beta$ and $x_n \mapsto mx_n - p_n x_n$ for every $n \in \omega$ restricts to an endomorphism of B. Moreover, B_0 is invariant under φ and the induced homomorphism $\bar{\varphi}\colon B/B_0 \to B/B_0$ is multiplication by m.

Proof. This follows from Lemma 4.2 choosing $b = 0$, $b_n = x_n$ for every $n \in \omega$ and $N = \emptyset$. □

Note that the linear transformations from Example 4.5 are in fact endomorphisms of any Hawaiian group $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{P})$.
The following lemma shows that a Hawaiian group $\mathcal{B}_\kappa(\mathcal{A}^*, \mathcal{P})$ usually has many free summands.

Lemma 4.6 *Let $B = \mathcal{B}_\kappa(\mathcal{A}, \mathcal{P})$ and $\alpha < \kappa$. Then $\langle y_\alpha, (1/p_n)(y_\alpha - x_n) \colon n \in A_\alpha\rangle$ is a free summand of B.*

Proof. Choose $\varphi \in \mathrm{End}_{\mathbb{Z}}(B)$ as in Example 4.4 with $m = 1$. Thus
(i) $(y_\beta)\varphi = y_\beta - y_\alpha$ for all $\beta < \kappa$;
(ii) $(x_n)\varphi = x_n - y_\alpha$ for $n \notin A_\alpha$;
(iii) $(x_n)\varphi = 0$ for $n \in A_\alpha$.
Since $(y_\alpha)\varphi = 0$ it is easily checked that φ is a projection, i.e., $\varphi^2 = \varphi$. Consequently, $B = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$. We claim that $\mathrm{Ker}(\varphi) = \langle y_\alpha, (1/p_n)(y_\alpha - x_n) \colon n \in A_\alpha\rangle$. By definition of φ we have $\langle y_\alpha, (1/p_n)(y_\alpha - x_n) \colon n \in A_\alpha\rangle \subseteq \mathrm{Ker}(\varphi)$. Therefore, let $b \in B$ such that $\varphi(b) = 0$. Assume that b is represented as in equation (2.1), i.e.,

$$b = \sum_{n\in\omega} r_n x_n + \sum_{\beta<\kappa} s_\beta y_\beta + \sum_{(\beta,n)\colon\, \beta<\kappa, n\in A_\beta} \frac{t_{\beta,n}}{p_n}(y_\beta - x_n) \tag{4.1}$$

for some $r_n, s_\beta, t_{\beta,n} \in \mathbb{Z}$. Then

$$\begin{aligned}(b)\varphi = &\sum_{n\notin A_\alpha} r_n(x_n - y_\alpha) + \sum_{\beta<\kappa} s_\beta(y_\beta - y_\alpha)\\ &+ \sum_{(\beta,n)\colon\, \beta<\kappa, n\in A_\beta\setminus A_\alpha} \frac{t_{\beta,n}}{p_n}(y_\beta - x_n)\\ &+ \sum_{(\beta,n)\colon\, \beta<\kappa, n\in A_\beta\cap A_\alpha} \frac{t_{\beta,n}}{p_n}(y_\beta - y_\alpha) = 0.\end{aligned}$$

Now choose $\alpha \neq \beta < \kappa$ and equate coefficients for y_β inside V to get

$$s_\beta + \sum_{n\in A_\beta} \frac{t_{\beta,n}}{p_n} = 0$$

which (Lemma 2.5) implies that p_n must divide $t_{\beta,n}$ for all $n \in A_\beta$. Hence, combining and renaming coefficients in (4.1) shows that b is of the form

$$b = \sum_{n\in\omega} r_n x_n + s_\alpha y_\alpha + \sum_{n\in A_\alpha} \frac{t_{\alpha,n}}{p_n}(y_\alpha - x_n).$$

Now, $(b)\varphi = 0$ reads as $(b)\varphi = \sum_{n\notin A_\alpha} r_n(x_n - y_\alpha) = 0$ and hence $r_n = 0$ for all $n \notin A_\alpha$. Thus $b \in \langle y_\alpha, (1/p_n)(y_\alpha - x_n) \colon n \in A_\alpha\rangle$ which proves that $\langle y_\alpha, (1/p_n)(y_\alpha - x_n) \colon n \in A_\alpha\rangle = \mathrm{Ker}(\varphi)$.
Finally, we have to show that $\langle y_\alpha, (1/p_n)(y_\alpha - x_n) \colon n \in A_\alpha\rangle$ is free. Fix

$n' \in A_\alpha$. We claim that

$$\Big\langle y_\alpha,\ \frac{1}{p_n}(y_\alpha - x_n)\colon n \in A_\alpha \Big\rangle = \bigoplus_{n\in A_\alpha} \frac{1}{p_n}\mathbb{Z}(y_\alpha - x_n) \oplus \mathbb{Z}x_{n'}.$$

The equality of the two sides is obvious, hence it suffices to prove that $\bigoplus_{n\in A_\alpha}(1/p_n)\mathbb{Z}(y_\alpha - x_n) \oplus \mathbb{Z}x_{n'}$ is indeed a direct sum. Therefore, assume that

$$\sum_{n\in A_\alpha} \frac{t_n}{p_n}(y_\alpha - x_n) + t'x_{n'} = 0 \tag{4.2}$$

for some $t_n,\ t' \in \mathbb{Z}$. Equating coefficients for y_α we obtain $\sum_{n\in A_\alpha} t_n/p_n = 0$ and thus (Lemma 2.5) p_n must divide t_n for all $n \in A_\alpha$. Moreover, for $n \neq n'$ we get $t_n/p_n = 0$ by looking at the coefficient of x_n in (4.2) and thus taken together $t_{n'}/p_{n'} = 0$. Finally, for n' we have $t' = t_{n'}/p_{n'} = 0$. Hence the sum is direct. □

Corollary 4.7 *Let $B = \mathcal{B}_\kappa(\mathcal{A},\ \mathcal{P})$. Then* $|\operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z})| \geq 2^{\aleph_0}$.

Proof. This follows easily from Lemma 4.6 and the fact that $\operatorname{Hom}_{\mathbb{Z}}(\bigoplus_\omega \mathbb{Z},\ \mathbb{Z}) \cong \prod_\omega \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\ \mathbb{Z})$. □

The final result of this section shows that the lower bound in Corollary 4.7 is best possible if we deal with strong Hawaiian groups.

Corollary 4.8 *Let $B = \mathcal{B}_\kappa(\mathcal{A}^*,\ \mathcal{P})$ and assume that A_α is infinite for every α. Then*

$$\operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z}) \cong \prod_{n\in\omega} \mathbb{Z} \quad \textit{and} \quad |\operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z})| = 2^{\aleph_0}.$$

Proof. Let B be a strong Hawaiian group and $f \in \operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z})$. Since B is strong each type $\delta_{\alpha,\beta}$ is bigger than $\mathbb{Z}$ for every $\alpha < \beta < \kappa$. Thus $(y_\alpha - y_\beta)f = 0$ $(\alpha < \beta < \kappa)$ which implies that $(y_\alpha)f = (y_0)f$ for all $\alpha < \kappa$. We put $(y_0)f = z^f \in \mathbb{Z}$. Now, let $n \in \omega$ and choose $\alpha < \kappa$ such that $n \in A_\alpha$. Then $(y_\alpha - x_n)f = p_n\big((1/p_n)(y_\alpha - x_n)\big)f = p_n z_n^f$ for some $z_n^f \in \mathbb{Z}$. Therefore, $(x_n)f = z^f - p_n z_n^f$. We now define a mapping $\Psi\colon \operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z}) \to \mathbb{Z}\times\prod_{n\in\omega}\mathbb{Z}$ by sending $f \in \operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z})$ to the sequence $(z^f,\ z_n^f\colon n \in \omega)$. It is immediately verified that Ψ is a group isomorphism and hence $\operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z}) \cong \prod_{n\in\omega}\mathbb{Z}$ which shows $|\operatorname{Hom}_{\mathbb{Z}}(B,\ \mathbb{Z})| = 2^{\aleph_0}$. □

The next result shows that the Hawaiian transformations from Definition 4.1

form a subgroup of the endomorphism ring of Hawaiian groups.

Theorem 4.9 *Let $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}, \mathcal{P})$. Then there is an injective homomorphism of additive groups*

$$\Phi\colon \mathbb{Z} \oplus B \oplus \prod_{n\in\omega} p_n B \to \mathrm{End}_{\mathbb{Z}}(B).$$

Proof. We first define Φ. Let $\bar{b} = (z, b, p_n b_n : n \in \omega) \in \mathbb{Z} \times B \times \prod_{n\in\omega} p_n B$. We define $f_{\bar{b}} \in \mathrm{End}_{\mathbb{Z}}(B)$ to be the endomorphism of B induced by the Hawaiian transformation defined by $y_\alpha \mapsto zy_\alpha + b$ and $x_n \mapsto zx_n + b - p_n b_n$ for all $\alpha < \kappa$ and $n \in \omega$. By Lemma 4.2 $f_{\bar{b}}$ is a well-defined endomorphism of B and it is easy to see that $\Phi\colon \mathbb{Z} \oplus B \oplus \prod_{n\in\omega} p_n B \to \mathrm{End}_{\mathbb{Z}}(B)$, $\bar{b} \mapsto f_{\bar{b}}$, is a well-defined injective homomorphism of additive groups. $\square$

5. Endomorphism rings of strong Hawaiian groups

In this section we will study the endomorphism rings of general strong Hawaiian groups, i.e., groups $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$. It turns out that these groups have endomorphism rings that are "rather small" which means "close to $\mathbb{Z}$". Recall that the *nucleus* $\mathrm{nuc}(P)$ of a rational group $P \subseteq \mathbb{Q}$ is defined to be the largest subring contained in P. Equivalently, if we consider $\mathrm{End}_{\mathbb{Z}}(P)$ as a subring of $\mathbb{Q}$, then $\mathrm{nuc}(P) = \mathrm{End}_{\mathbb{Z}}(P)$.

Lemma 5.1 *Let $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ and $f \in \mathrm{End}_{\mathbb{Z}}(B)$. Assume* (2.3), *and* (2.5). *Then there exists an integer $r \in \mathbb{Z}$ such that $f\restriction_{Y_{\alpha,\beta}}$ is multiplication by r for every $\alpha < \beta < \kappa$.*

Proof. Let $\alpha < \beta < \kappa$ and $f \in \mathrm{End}_{\mathbb{Z}}(B)$ as stated. By Lemma 3.7 we know that $Y_{\alpha,\beta}$ is a pure fully invariant subgroup of B, hence $(y_\alpha - y_\beta)f \in Y_{\alpha,\beta}$. Since (Lemma 2.9) $Y_{\alpha,\beta} \cong P_{\alpha,\beta} + T_{\alpha,\beta}$ we obtain that there is an element $r_{\alpha,\beta} \in \mathrm{nuc}(T_{\alpha,\beta})$ such that $f\restriction_{Y_{\alpha,\beta}}$ is multiplication by $r_{\alpha,\beta}$. It remains to prove that $r_{\alpha,\beta}$ is independent of α, β and an integer. Therefore, let $\alpha < \beta < \gamma < \kappa$. Applying f to the equation

$$(y_\alpha - y_\beta) + (y_\beta - y_\gamma) = y_\alpha - y_\gamma$$

implies that

$$r_{\alpha,\beta}(y_\alpha - y_\beta) + r_{\beta,\gamma}(y_\beta - y_\gamma) = r_{\alpha,\gamma}(y_\alpha - y_\gamma).$$

Equating coefficients inside V it follows that

$$r_{\alpha,\beta} = r_{\alpha,\gamma} = r_{\beta,\gamma}.$$

Since α, β, γ were arbitrary we have that $r := r_{0,1} = r_{\alpha,\beta} \in \bigcap_{\alpha<\beta<\kappa} \mathrm{nuc}(T_{\alpha,\beta})$ for all $\alpha < \beta < \kappa$. By (2.3) it follows that $r \in \mathbb{Z}$ and therefore $f \restriction_{Y_{\alpha,\beta}}$ is multiplication by $r \in \mathbb{Z}$ for every $\alpha < \beta < \kappa$. □

The next result shows that the Hawaiian transformations from Definition 4.1 determine completely the endomorphism ring of strong Hawaiian groups.

Theorem 5.2 *Let $B = \mathcal{B}_\kappa(\mathcal{A}^*, \mathcal{P})$ be a strong Hawaiian group. Then there is a bijective homomorphism of additive groups*

$$\Phi\colon \mathbb{Z} \oplus B \oplus \prod_{n\in\omega} p_n B \to \mathrm{End}_{\mathbb{Z}}(B).$$

Proof. By Theorem 4.9 we only need to show that Φ is surjective. Therefore, let $f \in \mathrm{End}_{\mathbb{Z}}(B)$. By Lemma 5.1 there is an integer $z \in \mathbb{Z}$ such that $(y_\alpha - y_\beta)f = z(y_\alpha - y_\beta)$ for all $\alpha < \beta < \kappa$. Putting $b = (y_0)f - zy_0$ this implies that $(y_\alpha)f = zy_\alpha + b$ for all $\alpha < \kappa$. Moreover, if $n \in \omega$, then there is $\alpha < \kappa$ such that $x_n \in A_\alpha$ and hence $(y_\alpha - x_n)f = p_n b_n'$ for some $b_n' \in B$. Thus $(x_n)f = zy_\alpha + b - p_n b_n'$. Since $p_n^{-1}(y_\alpha - x_n) \in B$ we let $b_n = b_n' - p_n^{-1}z(y_\alpha - x_n)$ and obtain $(x_n)f = zx_n + b - p_n b_n$ for all $n \in \omega$. Thus $f = f_{\bar{b}}$ where $\bar{b} = (z, b, p_n b_n \colon n \in \omega)$ and so Φ is surjective. □

The next theorem deals with possible direct decompositions of strong Hawaiian groups.

Theorem 5.3 *Let $B = \mathcal{B}_\kappa(\mathcal{A}^*, \mathcal{P})$. If $f \in \mathrm{End}_{\mathbb{Z}}(B)$ is idempotent, then there are $b, b_n \in B$ $(n \in \omega)$ such that either*

$$\mathrm{Ker}(f) = \langle b, b_n \colon n \in \omega \rangle_* \quad \textit{or} \quad \mathrm{Im}(f) = \langle b, b_n \colon n \in \omega \rangle_*.$$

Hence, $B = \langle b, b_n \colon n \in \omega\rangle_* \oplus C$ for some $C \subseteq B$. In particular, if $B = A \oplus C$, then A or C must be countable.

Proof. Let $f \in \mathrm{End}_{\mathbb{Z}}(B)$ and assume that f is idempotent, i.e., $f^2 = f$. By Theorem 4.9 there are elements $b, b_n \in B$ and $m \in \mathbb{Z}$ such that f is induced by the Hawaiian transformation $y_\alpha \mapsto my_\alpha + b$ and $x_n \mapsto mx_n + b - p_n b_n$ for all $\alpha < \kappa$ and $n \in \omega$. Therefore we obtain

$$my_\alpha + b = (y_\alpha)f = (y_\alpha)f^2 = m(y_\alpha)f + (b)f = m^2 y_\alpha + mb + (b)f.$$

If we pick $\alpha < \kappa$ large enough (so that y_α does not appear in the representations of b and $(b)f$), then equating coefficients implies that $m = m^2$, hence $m = 0$ or $m = 1$. Let us first assume that $m = 0$. Then $(y_\alpha)f = b$ for all $\alpha < \kappa$. Thus $b = (y_\alpha)f = (y_\alpha)f^2 = (b)f$. Moreover, for $n \in \omega$ we have

$$b - p_n b_n = (x_n)f = (x_n)f^2 = (b)f - p_n(b_n)f = b - p_n(b_n)f,$$

hence $b_n = (b_n)f$ for all $n \in \omega$. It now follows easily that $\mathrm{Im}(f) = \langle b, b_n : n \in \omega \rangle_* \subseteq B$.
On the other hand, if $m = 1$, then we consider $g = 1 - f$ instead of f. Then g is again idempotent and $\mathrm{Im}(g) = \mathrm{Ker}(f)$. Moreover, in the case of g we have $m = 0$, hence the above shows that $\mathrm{Im}(g) = \langle b, b_n : n \in \omega \rangle_* = \mathrm{Ker}(f)$. The other claims follow immediately. □

The next lemma shows that the endomorphism ring of a strong Hawaiian group is very special if the group has many invariant subgroups.

Lemma 5.4 *Let $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ and assume* (2.3), *and* (2.5). *Assume further that for all $\alpha < \kappa$ and $n \in \omega$, the map $f \in \mathrm{End}_{\mathbb{Z}}(B)$ is such that $(y_\alpha - x_n)f = r_{\alpha,n}(y_\alpha - x_n)$ for some $r_{\alpha,n} \in \mathbb{Q}$. Then there is $r \in \mathbb{Z}$ such that $r_{\alpha,n} = r$. Moreover, f is induced by the linear transformation $y_\alpha \mapsto r y_\alpha + b$, $x_n \mapsto r x_n + (y_0 f - r y_0)$ for $\alpha < \kappa$ and $n \in \omega$.*

Proof. Since $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$, Lemma 5.1 applies and there is $r \in \mathbb{Z}$ such that $(y_\alpha - y_\beta)f = r(y_\alpha - y_\beta)$ for all $\alpha < \beta < \kappa$. Fix $\alpha < \kappa$ and $n \in \omega$. Choose any $\beta > \alpha$. Then

$$y_\alpha - y_\beta = (y_\alpha - x_n) - (y_\beta - x_n)$$

and hence

$$r(y_\alpha - y_\beta) = r_{\alpha,n}(y_\alpha - x_n) - r_{\beta,n}(y_\beta - x_n)$$

and equating coefficients we see that $r = r_{\alpha,n} = r_{\beta,n} \in \mathbb{Z}$.
Now, $(y_0 - x_n)f = r(y_0 - x_n)$ implies that

$$x_n f = y_0 f - r y_0 + r x_n = r x_n + (y_0 f - r y_0).$$

Moreover, $(y_0 - y_\beta)f = r(y_0 - y_\beta)$ says that

$$y_\beta f = y_0 f - r y_0 + r y_\beta = r y_\beta + (y_0 f - r y_0)$$

for all $\beta < \kappa$ and the claim is established. □

Finally, we study some special strong Hawaiian groups with especially nice properties.

Lemma 5.5 *Let $B = \mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ and assume* (2.3)*, and* (2.4)*. Assume further that $f \in \operatorname{End}_{\mathbb{Z}}(B)$ and that $x_n f = r_n x_n$ for some $r_n \in \mathbb{Q}$ and all $n < \omega$. Then there is $r \in \mathbb{Z}$ and $\epsilon_n \in \operatorname{nuc}(R_n)$ such that f is induced by the linear transformation $y_\alpha \mapsto r y_\alpha$, $x_n \mapsto r x_n + \epsilon_n p_n x_n$ where $\alpha < \kappa$ and $n \in \omega$.*

Proof. By Lemma 5.1 we obtain $r \in \mathbb{Z}$ such that $(y_\alpha - y_\beta)f = r(y_\alpha - y_\beta)$ for all $\alpha < \beta < \kappa$. Then, choosing $\alpha = 0$,

$$y_\beta f = r y_\beta + (y_0 f - r y_0)$$

for every $\beta < \kappa$ and hence

$$\begin{aligned}(y_\beta - x_n)f &= r y_\beta + (y_0 f - r y_0) - r_n x_n \\ &= r(y_\beta - x_n) + (r - r_n)x_n + (y_0 f - r y_0)\end{aligned}$$

for all $\beta < \kappa$ and $n \in \omega$. Thus, for $\beta < \kappa$ and $n \in A_\beta$ we get that

$$p_n \text{ divides } (r - r_n)x_n + (y_0 f - r y_0) \text{ in } B. \tag{5.1}$$

Passing to the quotient B/B_0 this implies that $(y_0 f - r y_0) + B_0$ is divisible by p_n whenever $n \in A_\beta$. As $\bigcup_{\alpha<\kappa} A_\alpha = \omega$ it follows that $(y_0 f - r y_0) + B_0$ is divisible by every $p \in \mathcal{P}$. Using Lemma 2.7(ii) it follows that $y_0 f - r y_0 \in B_0$ and hence $y_0 f - r y_0 = \sum_{m\in\omega} r'_m x_m$ for some $r'_m \in R_m$. Therefore, by (5.1),

$$(r - r_n)x_n + \sum_{m\in\omega} r'_m x_m \in p_n B,$$

and by purity of B_0 and its summands $R_m x_m$,

$$(r - r_n) + r'_n \in p_n R_n, \quad \text{and} \quad \forall n \neq m \colon r'_m \in p_n R_m. \tag{5.2}$$

We will show that this is impossible except when $r'_m = 0$ for all $m \in \omega$. Suppose to the contrary that there is an m such that $r'_m \neq 0$. Write r'_m as a canceled fraction $r'_m = a/b$. Since n was arbitrary in (5.2) there is some p_n that does not divide a. Now $a/b = r'_m \in p_n R_m$, so $a/p_n = b(1/p_n) r'_m \in R_m$ and since R_m contains 1, also $1/p_n \in R_m$. But then $1/p_n \in R \cap P = \mathbb{Z}$, a contradiction. We have established that $r'_m = 0$ for all m. Thus $y_0 f = r y_0$ and p_n divides $r - r_n$ inside R_n for every $n \in \omega$. Let $r_n - r = p_n \epsilon_n$ for some $\epsilon_n \in R_n$. Then $r_n = p_n \epsilon_n + r$ and hence $x_n f = r_n x_n = r x_n + p_n \epsilon_n x_n$ as claimed. Since f is a homomorphism and $R_n x_n$ is fully invariant, we must have $r_n \in \mathrm{nuc}(R_n)$. Furthermore $r \in \mathbb{Z} \subseteq \mathrm{nuc}(R_n)$ and $\mathrm{nuc}(R_n)$ is pure, therefore also $\epsilon_n \in \mathrm{nuc}(R_n)$ for all $n \in \omega$. $\square$

We specialize further to groups $\mathcal{B}_\kappa(\mathcal{R}, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ with the additional property that $q_0 R = R$ for some prime number $q_0 \notin \mathcal{P}$. The following result completely describes the (small) endomorphism ring of such a group $\mathcal{B}_\kappa(\mathcal{R}^*, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$.

Corollary 5.6 *Let $B = \mathcal{B}_\kappa(\mathcal{R}^*, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ satisfying the additional property that $q_0 R = R$ for some prime number $q_0 \notin \mathcal{P}$ and $\{\mathrm{type}(R_n) \colon n \in \omega\}$ is an anti-chain. Assume* (2.3), *and* (2.5). *Then $f \in \mathrm{End}_{\mathbb{Z}}(B)$ if and only if f is induced by a linear transformation $y_\alpha \mapsto r y_\alpha$, $x_n \mapsto r x_n + \epsilon_n p_n x_n$ where $r \in \mathbb{Z}$ and $\epsilon_n \in \mathrm{nuc}(R_n)$.*

Proof. By hypothesis $q_0 R = R$ and $\{\mathrm{type}(R_n) \colon n \in \omega\}$ is an anti-chain, hence $q_0 B_0 = B_0$ and therefore B_0 is fully invariant in B. In fact, every $R_n x_n$ is a fully invariant subgroup of B for $n \in \omega$. Thus, any $f \in \mathrm{End}_{\mathbb{Z}}(B)$ satisfies $x_n f = r_n x_n$ for some $r_n \in \mathbb{Q}$ and all $n \in \omega$. By Lemma 5.5 it follows that f is of the desired form. Conversely, every linear transformation of the described form is an endomorphism of B (see also Example 4.5). $\square$

We end with a second corollary.

Corollary 5.7 *Let $B = \mathcal{B}_\kappa(\mathcal{R}^*, \mathcal{T}, \mathcal{A}^*, \mathcal{P})$ satisfying the additional property that $q_0 R = R$ for some prime number $q_0 \notin \mathcal{P}$ and $\{\mathrm{type}(R_n) \colon n \in \omega\}$ is an anti-chain. Assume* (2.3), *and* (2.5) *and furthermore that $\forall n < \omega \colon T_n \not\subseteq R_n$. Then* $\mathrm{End}_{\mathbb{Z}}(B) = \mathbb{Z}$.

Proof. Let B be given and let $f \in \mathrm{End}_{\mathbb{Z}}(B)$. By Corollary 5.6 it follows that f is induced by the linear transformation $y_\alpha \mapsto r y_\alpha$, $x_n \mapsto r x_n + \epsilon_n p_n x_n$ for $\alpha < \kappa$, $n \in \omega$ for some $r \in \mathbb{Z}$, $\epsilon_n \in \mathrm{nuc}(R_n)$. Let $n \in \omega$ and choose $\alpha < \kappa$ such that $n \in A_\alpha$. Then $\{t \in \mathbb{Q} \mid t(y_\alpha - x_n) \in B\} \supseteq T_n$. Hence also $\{t \in \mathbb{Q} \mid t\big((y_\alpha - x_n)f - r(y_\alpha - x_n)\big) \in B\} \supseteq T_n$. But $(y_\alpha - x_n)f - r(y_\alpha - x_n) = \epsilon_n x_n$. By way of contradiction assume that $\epsilon_n \neq 0$. Then $R_n = \{t \in \mathbb{Q} \mid t\epsilon_n x_n \in B\} \supseteq T_n$ contradicting our assumptions. Therefore $\epsilon_n = 0$, f is multiplication by $r \in \mathbb{Z}$, and $\mathrm{End}_{\mathbb{Z}}(B) = \mathbb{Z}$. $\square$

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