Hokkaido Mathematical Journal Vol. 38 (2009) p. 267–281

Rédei's theorem with a factor of order four

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(Received January 21, 2008; Revised September 9, 2008)

Abstract. We will prove that if a finite abelian group is a direct product of its subsets such that one subset has four elements and the others have prime cardinalities, then at least one of the factors must be periodic.

Key words: Factoring abelian groups by subsets, Hajós' theorem, Rédei's theorem, periodic factorizations, normalized factorizations.

1. Introduction

Let G be a finite abelian group and let $A_1, \dots, A_n$ be subsets of G. If the product $A_1 \cdots A_n$ is direct and is equal to G, then we say that the equation $G = A_1 \cdots A_n$ is a factorization of G. The next theorem of L. Rédei is one of the most celebrated factorization results.

Let G be a finite abelian group and let e be the identity element of G. If $G = A_1 \cdots A_n$ is a factorization of G, $|A_i|$ is a prime and $e \in A_i$ for each i, $1 \leq i \leq n$, then at least one of the factors must be a subgroup of G.

We say that a subset A of G is periodic with period g if $g \neq e$ and $Ag = A$. Rédei's theorem can be reformulated using periodicity in the following way. If $G = A_1 \cdots A_n$ is a factorization of the finite abelian group G and each $|A_i|$ is a prime, then at least one of the factors must be periodic. Examples show that the condition that each $|A_i|$ is a prime cannot be dropped from the theorem. However, for 2-groups K. Amin, K. Corrádi and A. D. Sands ([1, Theorem 15]) proved a slightly more general version. Namely, if $G = BA_1 \cdots A_n$ is a factorization of the finite abelian 2-group G such that $|B| = 4$ and $|A_1| = \cdots = |A_n| = 2$, then at least one of the factors is periodic.

It is Problem 1 in [7] that if a finite abelian group is factorized as a product of factors each of which has prime order or order four must one of the factor be periodic.

We will extend the above result of K. Amin, K. Corrádi and A. D. Sands proving that if $G = BA_1 \cdots A_n$ is a factorization of the finite abelian group

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2000 Mathematics Subject Classification : 20K01, 20D60, 20K25.

G such that $|B| = 4$ and each $|A_i|$ is a prime, then at least one of the factors must be periodic. This provides a partial answer for Sands' problem.

2. Preliminaries

If χ is a character of G, then $\chi(A)$ will denote

$$\sum_{a \in A} \chi(a).$$

Lemma 1 *Let A, B be subsets of a finite abelian group G. If $\chi(A) = \chi(B)$ for each character χ of G, then $A = B$.*

Proof. Let $g_1, \ldots, g_n$ be all the elements of G. Let $\chi_1, \ldots, \chi_n$ be all the characters of G. As $\chi_i(A) = \chi_i(B)$ for each i, $1 \le i \le n$ we get the following system of linear equations.

$$\begin{array}{ccc} \chi_1(g_1)(x_1 - y_1) + & \cdots & + \chi_1(g_n)(x_n - y_n) = 0 \\ \vdots & \ddots & \vdots \\ \chi_n(g_1)(x_1 - y_1) + & \cdots & + \chi_n(g_n)(x_n - y_n) = 0 \end{array}$$

Here $x_i = 1$ if $g_i \in A$ and $x_i = 0$ if $g_i \notin A$. Similarly, $y_i = 1$ if $g_i \in B$ and $y_i = 0$ if $g_i \notin B$. By the standard orthogonality relations the matrix $[\chi_i(g_j)]$ is orthogonal. In particular the columns are linearly independent. It follows that the system of equations has only the trivial solution

$$x_1 - y_1 = \cdots = x_n - y_n = 0$$

and so $A = B$. □

Corollary 1 *If A is a subset of a finite abelian group G and $\chi(A) = 0$ for each nonprincipal character χ of G, then $A = G$.*

Proof. Let $|A| = r$, $|G| = n$. The multiset nA contains the elements of A with multiplicity n. Similarly, the multiset rG contains the elements of G with multiplicity r. Note that $\chi(nA) = \chi(rG)$ holds for each character χ of G. Indeed, if χ is not the principal character of G, then $\chi(nA) = n\chi(A) = 0$, $\chi(rG) = r\chi(G) = 0$. If χ is the principal character of G, then $\chi(nA) = \chi(rG)$ is equivalent to $n|A| = r|G|$. Using the argument we have seen in the proof of Lemma 1 we get that the multisets nA and rG are equal. It follows that $A = G$. □

Let A, A' be subsets of the finite abelian group G. We say that A is replaceable by A' if $G = A'B$ is a factorization of G whenever $G = AB$ is a factorization of G. If $\chi(A) = 0$, then we say χ annihilates A. The set of all characters of G that annihilate A is denoted by $\mathrm{Ann}(A)$. The connection between annihilators and replacement we will need is the following. If $|A| = |A'|$ and $\mathrm{Ann}(A) \subset \mathrm{Ann}(A')$, then A can be replaced by A'. The periodicity of a subset A of G can be tested by means of its translates. Namely, if $g \neq e$ and

$$g \in \bigcap_{a \in A} a^{-1}A,$$

then A is periodic with period g. (A proof can be found for example in [8].) We will need the following lemmas.

Lemma 2 *If $\rho_1, \ldots, \rho_4$ are complex numbers such that $|\rho_1| = \cdots = |\rho_4| = 1$ and $\rho_1 + \cdots + \rho_4 = 0$, then there is a permutation $\sigma_1, \ldots, \sigma_4$ of $\rho_1, \ldots, \rho_4$ such that $\sigma_1 + \sigma_2 = 0$ and $\sigma_3 + \sigma_4 = 0$.*

Proof. Draw a closed polygon on the complex plane with vertices $P_1, \ldots, P_4$ such that

$$\overrightarrow{P_1P_2} = \rho_1, \quad \overrightarrow{P_2P_3} = \rho_2, \quad \overrightarrow{P_3P_4} = \rho_3, \quad \overrightarrow{P_4P_1} = \rho_4.$$

In the "triangles" $P_1P_2P_4$ and $P_3P_2P_4$ the "side" P_2P_4 is common. If $P_2 = P_4$ then

$$\overrightarrow{P_1P_2} + \overrightarrow{P_4P_1} = \rho_1 + \rho_4 = 0,$$

and

$$\overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} = \rho_2 + \rho_3 = 0,$$

as required. Thus for the remaining part of the proof we may assume that $P_2 \neq P_4$.

In the "triangles" $P_4P_1P_3$ and $P_2P_1P_3$ the "side" P_1P_3 is common. If $P_1 = P_3$ then

$$\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} = \rho_1 + \rho_2 = 0,$$

and

$$\overrightarrow{P_3P_4} + \overrightarrow{P_4P_1} = \rho_3 + \rho_4 = 0,$$

as required. Therefore for the remaining part of the proof we may assume that $P_1 \neq P_3$.

Since $P_2 \neq P_4$ and $P_1 \neq P_3$, the polygon is a rhombus with diagonals P_2P_4 and P_1P_3. In this case

$$\overrightarrow{P_1P_2} + \overrightarrow{P_3P_4} = \rho_1 + \rho_3 = 0,$$

and

$$\overrightarrow{P_2P_3} + \overrightarrow{P_4P_1} = \rho_2 + \rho_4 = 0.$$

This completes the proof. □

Lemma 3 *Let $B = \{e, ax, by, cz\}$ be a subset of a finite abelian group G such that a, b, c are 2-elements and x, y, z are $2'$-elements. Then B can be replaced by*

$$B' = \{e, ax^s, by^s, cz^s\}$$

for each integer s.

Proof. It is enough to show that $\chi(B) = 0$ implies $\chi(B') = 0$ for each character χ of G.

$$0 = \chi(B) = \chi(e) + \chi(ax) + \chi(by) + \chi(cz)$$

can hold only in the following three ways

$$\begin{aligned} 1 + \chi(ax) = 0, &\quad \chi(by) + \chi(cz) = 0, \\ 1 + \chi(by) = 0, &\quad \chi(ax) + \chi(cz) = 0, \\ 1 + \chi(cz) = 0, &\quad \chi(ax) + \chi(by) = 0. \end{aligned}$$

Let us deal with the first possibility. (The others can be handled in a similar way.) It follows that

$$(-1)\chi(a) = \chi(x^{-1}), \quad (-1)\chi(bc^{-1}) = \chi(zy^{-1}).$$

The left sides have 2 power orders and the right sides have odd orders. Therefore

$$-\chi(a) = 1, \quad \chi(x^{-1}) = 1, \quad -\chi(bc^{-1}) = 1, \quad \chi(zy^{-1}) = 1,$$

that is,

$$\chi(a) = -1, \quad \chi(x) = 1, \quad \chi(c) = -\chi(b), \quad \chi(y) = \chi(z).$$

Computing $\chi(B')$ shows that $\chi(B') = 0$.

This completes the proof. □

3. The result

We are ready to prove the main result of the paper.

Theorem 1 *Let $G = BA_1 \cdots A_n$ be a factorization of the finite abelian group G such that $|A_1|, \dots, |A_n|$ are primes and $|B| = 4$. Then at least one of the factors $B, A_1, \dots, A_n$ is periodic.*

Proof. We may assume that $|A_1| = \cdots = |A_m| = 2$ and $|A_{m+1}|, \dots, |A_n|$ are odd primes. In order to prove the theorem we assume the contrary that there is a counterexample. In the $n = 0$ case the theorem holds. So in a counterexample $n \geq 1$.

Let us consider an A_i and assume that $|A_i| = p$ is a prime. If each $a \in A_i \setminus \{e\}$ has order p, then do nothing with A_i. If there is an element $a \in A_i \setminus \{e\}$ with $p^2 \mid |a|$, then replace A_i by $A_i' = \{e, a, a^2, \dots, a^{p-1}\}$. Clearly A_i' is not periodic. If there is an element $a \in A_i$ whose order is not a prime power then there is an integer s such that $A_i^s = \{a^s : a \in A_i\}$ contains only (p, q)-elements, where q is a prime distinct from p. In this case replace A_i by A_i^s. By Proposition 3 of [6], this can be done. In short we may assume that there is a counterexample in which each A_i contains only (p, q)-elements.

If each of $B, A_1, \dots, A_m$ contains only 2-elements, then the product $BA_1 \cdots A_m$ forms a factorization of the 2-component of G and so by the Amin-Corrádi-Sands result at least one of the factors $B, A_1, \dots, A_m$ is periodic. We choose a counterexample with minimal n and among these we choose one for which the quantity

$$h = \prod_{b \in B} |b_{|2'}| \prod_{i=1}^{m} \prod_{a \in A_i} |a_{|2'}|$$

is minimal. We may assume that one of $B, A_1, \ldots, A_m$ contains not only 2-elements.

Let us consider B. Clearly $B = \{e, ax, by, cz\}$, where the orders of a, b, c are 2 powers and the orders of x, y, z are odd. Assume first that each of x, y, z is equal to e. This case can be settled by the method of K. Amin, K. Corrádi and A. D. Sands. In this case $B = \{e, a, b, c\}$. If one of a, b, c has order 2, say $|a| = 2$, then the squares of e and a are equal. From the factorization $G = B(A_1 \cdots A_n)$ by Lemma 1 of [5], it follows that B or $A_1 \cdots A_n$ is periodic. As B is not periodic, $A_1 \cdots A_n$ is periodic. In this case the conditions of the Theorem 2 of [3] are satisfied and so by this theorem, one of the factors $A_1, \ldots, A_n$ is periodic. This is a contradiction. We may assume that none of a, b, c has order 2. Write c in the form $c = abd$, where d is an element of G.

$$B = \{e, a, b, abd\}$$

If $d = e$, then $B = \{e, a\}\{e, b\}$. Here $\{e, a\}$, $\{e, b\}$ are not subgroups of G as $|a| \neq 2$, $|b| \neq 2$. From the factorization $G = \{e, a\}\{e, b\}A_1 \cdots A_n$, by Rédei's theorem, it follows that at least one of the factors is periodic. This is not possible. Thus we may assume that $d \neq e$. If there is no character χ of G for which $\chi(B) = 0$, then $0 = \chi(G) = \chi(A_1 \cdots A_n)$ for each nonprincipal character χ of G. By Corollary 1, we get the contradiction $G = A_1 \cdots A_n$. Consider a character χ of G for which $\chi(B) = 0$.

$$0 = \chi(B) = \chi(e) + \chi(a) + \chi(b) + \chi(abd)$$

The sum of 4 complex numbers whose length are all 1 is 0. This can happen only in the following 3 ways

$$\chi(e) + \chi(a) = 0, \quad \chi(b) + \chi(abd) = 0,$$

$$\chi(e) + \chi(b) = 0, \quad \chi(a) + \chi(abd) = 0,$$

$$\chi(e) + \chi(abd) = 0, \quad \chi(a) + \chi(b) = 0.$$

From $\chi(a) = -1$, $\chi(b) = -\chi(abd)$ it follows that $\chi(d) = 1$. From $\chi(b) = -1$,

$\chi(a) = -\chi(abd)$ it follows that $\chi(d) = 1$. In other words $\chi(B) = 0$, $\chi(d) \neq 1$ implies $\chi(abd) = -1$.

Set $C = \{e, abd\}A_1 \cdots A_n$. Notice that the product is direct. We claim that $Cd = C$. We verify the claim showing that $\chi(Cd) = \chi(C)$ for each character χ of G. It is clear when $\chi(d) = 1$ or $\chi(C) = 0$. So suppose that $\chi(d) \neq 1$, $\chi(C) \neq 0$. Now χ is not the principal character of G. From the factorization $G = BA_1 \cdots A_n$ it follows that $\chi(B) = 0$ or $\chi(A_i) = 0$ for some i, $1 \leq i \leq n$. As $\chi(C) \neq 0$ we get $\chi(B) = 0$. But $\chi(B) = 0$, $\chi(d) \neq 1$ implies $\chi(\{e, abd\}) = 0$ and $\chi(C) = 0$. Therefore $\chi(Cd) = \chi(C)$. Thus C is periodic with period d. Theorem 2 of [3] gives that one of the factors $A_1, \ldots, A_n$, $\{e, abd\}$ is periodic. This is a contradiction.

Turn to the cases when not all of x, y, z is equal to e. By Lemma 3, B can be replaced by $B' = \{e, a, b, c\}$. From the factorization $G = B'A_1 \cdots A_n$ by the minimality of the counterexample, it follows that one of the factors B', $A_1, \ldots, A_n$, is periodic. This leads to a contradiction unless B' is periodic. By relabeling a, b, c we may assume that $B' = \{e, a\}\{e, b\}$, where $|a| = 2$. Thus $B = \{e, ax, by, abz\}$, that is, $c = ab$. We distinguish 5 cases.

Case 1. Two of x, y, z are equal to e.

Case 2. One of x, y, z is equal to e.

Case 3. None of x, y, z is equal to e and $|x||y||z|$ is not a power of a prime.

Case 4. x, y, z are nonidentity p-elements and two of them generate the same subgroup.

Case 5. x, y, z are of order p and $z = x^k y$.

Turn to case 1. Suppose that exactly one of x, y, z is not equal to e. Now factor B is in one of the following forms

$$B = \{e, ax, b, ab\}, \quad x \neq e,$$

$$B = \{e, a, by, ab\}, \quad y \neq e,$$

$$B = \{e, a, b, abz\}, \quad z \neq e.$$

In the last two cases B contains an element of order 2. This leads to the contradiction that one of the factors $B, A_1, \ldots, A_n$ is periodic. We may assume that the first possibility occurs. Consider a character χ of G with $\chi(B) = 0$.

$$0 = \chi(B) = \chi(e) + \chi(ax) + \chi(b) + \chi(ab)$$

It can happen only in one of the following three ways

$$\begin{aligned} 1+\chi(ax)=0, &\quad \chi(b)+\chi(ab)=0,\\ 1+\chi(b)=0, &\quad \chi(ax)+\chi(ab)=0,\\ 1+\chi(ab)=0, &\quad \chi(ax)+\chi(b)=0. \end{aligned}$$

From $\chi(ax)=-1$, $\chi(b)=-\chi(ab)$ it follows that $\chi(x)=1$. From $\chi(b)=-1$, $\chi(ax)=-\chi(ab)$ it follows that $\chi(x)=1$. From $\chi(ab)=-1$, $\chi(ax)=-\chi(b)$ it follows that $\chi(x)=1$. In short $\chi(B)=0$ implies $\chi(x)=1$. Thus B can be replaced by

$$B'=\{e,ax,bx,abx^2\}=\{e,ax\}\{e,bx\}.$$

From the factorization $G=\{e,ax\}\{e,bx\}A_1\cdots A_n$ by Rédei's theorem it follows that one of the factors is periodic. This is a contradiction.

Turn to case 2 when exactly 2 of x, y, z are not equal to e. Now B is in one of the following forms

$$\begin{aligned} B&=\{e,ax,by,ab\}, &\quad x\neq e,\ y\neq e,\\ B&=\{e,ax,b,abz\}, &\quad x\neq e,\ z\neq e,\\ B&=\{e,a,by,abz\}, &\quad y\neq e,\ z\neq e. \end{aligned}$$

In the third case B contains an element of order 2. This leads to the contradiction that one of the factors is periodic. We may assume that the first two possibilities occur. Suppose $B=\{e,ax,by,ab\}$. Consider a character χ of G for which $\chi(B)=0$.

$$0=\chi(B)=\chi(e)+\chi(ax)+\chi(by)+\chi(ab)$$

can hold only in the following ways

$$\begin{aligned} 1+\chi(ax)=0, &\quad \chi(by)+\chi(ab)=0,\\ 1+\chi(by)=0, &\quad \chi(ax)+\chi(ab)=0,\\ 1+\chi(ab)=0, &\quad \chi(ax)+\chi(by)=0. \end{aligned}$$

From $\chi(ax)=-1$, $\chi(by)=-\chi(ab)$ it follows that $\chi(x)=1$, $\chi(y)=1$. From

$\chi(by) = -1$, $\chi(ax) = -\chi(ab)$ it follows that $\chi(x) = 1$, $\chi(y) = 1$. From $\chi(ab) = -1$, $\chi(ax) = -\chi(by)$ it follows that $\chi(x) = \chi(y)$. In short $\chi(B) = 0$ implies $\chi(x) = \chi(y)$.

If $x = y$, then

$$B = \{e, ax, by, ab\} = \{e, ax\}\{e, ab\}.$$

In the $|ab| = 2$ case B is periodic which is a contradiction. We may assume that $|ab| \neq 2$. From the factorization $G = \{e, ax\}\{e, ab\}A_1 \cdots A_n$ by Rédei's theorem, it follows that one of the factors is periodic. So we may assume that $x \neq y$. Set $v = x^{-1}y$. Note that $v \neq e$ and $\chi(B) = 0$ implies $\chi(v) = 1$. This gives that B can be replaced by

$$B' = \{e, av, bv, abv^2\} = \{e, av\}\{e, bv\}.$$

The factorization $G = \{e, av\}\{e, bv\}A_1 \cdots A_n$ contradicts Rédei's theorem.

Suppose $B = \{e, ax, b, abz\}$. This case can be settled in a similar manner.

Turn to case 3 when none of x, y, z is equal to e and there are distinct primes p, q such that both p and q divides $|x||y||z|$. Now $B = \{e, ax, by, abz\}$. Assume first that $|b| \neq 2$. Replace B by $B' = \{e, ax^p, by^p, abz^p\}$ in the factorization $G = BA_1 \cdots A_n$. In the factorization $G = B'A_1 \cdots A_n$ the value of h decreased. By the minimality of the counterexample, B' is periodic. This gives that $x^p = e$, that is, $|x| \bigm| p$. Replacing B by $\{e, ax^q, by^q, abz^q\}$ gives $x^q = e$, that is, $|x| \bigm| q$. Therefore $x = e$. This is a contradiction. So we may assume that $|b| = 2$. Then $|ab| = 2$. In this situation the roles of a, b, ab are symmetric and we write B in the form $B = \{e, ax, by, cz\}$. Replacing B by $B' = \{e, ax^p, by^p, cz^p\}$ gives that B' is periodic. By symmetry we may assume that $x^p = e$ and $y^p = z^p$. Replacing B by $B'' = \{e, ax^q, by^q, cz^q\}$ gives that B'' is periodic. We face the following possibilities

$$x^q = e, \quad y^q = z^q,$$

$$y^q = e, \quad x^q = z^q,$$

$$z^q = e, \quad x^q = y^q.$$

In the first case $x^p = e$, $x^q = e$ imply the contradiction $x = e$. In the second case from $x^p = e$, $y^q = e$ it follows that $|x| = p$, $|y| = q$. From

$z^{pq} = (z^p)^q = (y^p)^q = (y^q)^p = e$ it follows that $|z| \mid pq$. Set $z = \alpha\beta$ where $|\alpha|$ is a p power and $|\beta|$ is a q power. From $y^p = z^p = (\alpha\beta)^p = \beta^p$ it follows that $y = \beta$. From $x^q = z^q = (\alpha\beta)^q = \alpha^q$ it follows that $x = \alpha$. Hence $z = xy$ and

$$B = \{e, ax, by, abz\} = \{e, ax\}\{e, by\}.$$

By Rédei's theorem one of the factors in the factorization

$$G = \{e, ax\}\{e, by\}A_1 \cdots A_n$$

is periodic. This is a contradiction. The third possibility can be settled in a similar way.

Turn to case 4 when x, y z are p-elements and two of x, y, z generate the same subgroup. This can happen in the following ways

$$\langle x\rangle = \langle y\rangle, \quad y = x^s,\ p \not| s,$$
$$\langle x\rangle = \langle z\rangle, \quad z = x^s,\ p \not| s,$$
$$\langle y\rangle = \langle z\rangle, \quad z = y^s,\ p \not| s,$$

In the first case $B = \{e, ax, bx^s, abz\}$. Consider a character χ of G with $\chi(B) = 0$.

$$0 = \chi(B) = \chi(e) + \chi(ax) + \chi(bx^s) + \chi(abz)$$

can hold only in the following ways

$$1 + \chi(ax) = 0, \quad \chi(bx^s) + \chi(abz) = 0,$$
$$1 + \chi(bx^s) = 0, \quad \chi(ax) + \chi(abz) = 0,$$
$$1 + \chi(abz) = 0, \quad \chi(ax) + \chi(bx^s) = 0.$$

From $\chi(a) = -1$, $\chi(x) = 1$ we get $\chi(x^s) = \chi(z)$ then $\chi(z) = 1$. From $\chi(b) = -1$, $\chi(x^s) = 1$ we get $\chi(x) = \chi(z)$ then $\chi(z) = 1$. From $\chi(ab) = -1$, $\chi(z) = 1$ we get $\chi(z) = 1$. In short $\chi(B) = 0$ implies $\chi(z) = 1$. So B can be replaced by

$$B' = \{e, az, bz, abz^2\} = \{e, az\}\{e, bz\}.$$

From the factorization $G = \{e, az\}\{e, bz\}A_1 \cdots A_n$ by Rédei's theorem we get the contradiction that one of the factors is periodic. The second and third possibilities can be settled in similar ways.

Turn to case 5 when $|x| = |y| = |z| = p$ and $z = x^k y$. Now $B = \{e, ax, by, abx^k y\}$. If $k = 0$, then $\langle y\rangle = \langle z\rangle$ and by case 4, we are done. If $k = 1$, then

$$B = \{e, ax, by, abxy\} = \{e, ax\}\{e, by\}$$

and by Rédei's theorem we are done. If $k = -1$, then

$$B = \{e, ax, by, abx^{-1}y\} = \{e, ax\}\{e, abx^{-1}y\}$$

and by Rédei's theorem we are done. We may assume that $2 \leq k \leq p-2$. Consider a character χ of G with $\chi(B) = 0$.

$$0 = \chi(B) = \chi(e) + \chi(ax) + \chi(by) + \chi(abx^k y)$$

This can hold only in the following ways

$$\begin{aligned} 1 + \chi(ax) = 0, &\quad \chi(by) + \chi(abx^k y) = 0, \\ 1 + \chi(by) = 0, &\quad \chi(ax) + \chi(abx^k y) = 0, \\ 1 + \chi(abx^k y) = 0, &\quad \chi(ax) + \chi(by) = 0. \end{aligned}$$

If $\chi(a) = -1$, $\chi(x) = 1$ we get $\chi(x) = 1$. If $\chi(b) = -1$, $\chi(y) = 1$ we get $\chi(x^{k-1}) = 1$. If $\chi(ab) = -1$, $\chi(x^k y) = 1$ we get $\chi(x^{k+1}) = 1$. In short $\chi(B) = 0$ implies $\chi(x) = 1$ and so B can be replaced by

$$B' = \{e, ax, bx, abx^2\} = \{e, ax\}\{e, bx\}.$$

From the factorization $G = \{e, ax\}\{e, bx\}A_1 \cdots A_n$ by Rédei's theorem we get the contradiction that one of the factors is periodic.

In the factorization $G = BA_1 \cdots A_n$ replace B by $B' = \{e, a\}\{e, b\}$. Set $H_0 = \{e, a\} = \langle a\rangle$, $A_0 = \{e, b\}$. We get the factorization $G = H_0 A_0 A_1 \cdots A_n$ and then the factorization

$$G/H_0 = (A_0 H_0)/H_0 \cdots (A_n H_0)/H_0$$

of the factor group G/H_0. Here $(A_iH_0)/H_0$ stands for $\{aH_0 : a \in A_i\}$. By Rédei's theorem, one of the factors, say $(A_0H_0)/H_0$, is a subgroup of G/H_0. It means that $A_0H_0 = H_1$ is a subgroup of G. Then, by relabelling the factors if necessary, we get the factorization

$$G/H_1 = (A_1H_1)/H_1 \cdots (A_nH_1)/H_1$$

of the factor group G/H_1. Continuing in this way we get that there is a maximal subgroup M of G such that M contains all but one of the factors $H_0, A_0, \ldots, A_n$. We may assume that

(i) $A_0 \not\subset M$ or
(ii) one of $A_1, \ldots, A_m$ is not contained by M, say $A_m \not\subset M$ or
(iii) one of $A_{m+1}, \ldots, A_n$ is not contained by M, say $A_n \not\subset M$.

If $A_0 \not\subset M$, then $|G : M| = 2$ and each $2'$-element of G is in M. In particular $x \in M$. So $\{e, ax\} \subset M$. Therefore $M = \{e, ax\}A_1 \cdots A_n$ is a factorization of M. By Rédei's theorem one of the factors is a subgroup of M. This is a contradiction.

If $A_m \not\subset M$, then $|G : M| = 2$ and each $2'$-element of G is in M. In particular $x, y, z \in M$. Hence $B = \{e, ax, by, cz\} \subset M$ and so

$$M = BA_1 \cdots A_{m-1}A_{m+1} \cdots A_n$$

is a factorization of M. By the minimality of the counterexample one of the factors is periodic. This is a contradiction. We may assume that $A_n \not\subset M$. This means $|G : M| = p$ is an odd prime. If $\{x, y, z\} \subset M$, then $B \subset M$ and so $M = BA_1 \cdots A_{n-1}$ is a factorization of M. By the minimality of the counterexample one of the factors is periodic. This is a contradiction. We may assume that $\{x, y, z\} \not\subset M$. By Proposition 3 of [6], A_n can be replaced by A_n' such that A_n' contains only p-elements. Choose an element $c \in A_n' \setminus \{e\}$ and replace A_n' by $C = \{e, c, c^2, \ldots, c^{p-1}\}$ to get the factorization $G = BA_1 \cdots A_{n-1}C$. By Lemma 3 of [9], this replacement is possible. As $G = M(Cd^{-1})$ is a factorization of G for each $d \in C$, Cd^{-1} is a complete set of representatives modulo M. There are elements $u_d, v_d, w_d \in Cd^{-1}$ such that

$$(ax)^{-1} \in u_dM, \quad (by)^{-1} \in v_dM, \quad (abz)^{-1} \in w_dM.$$

Hence $axu_d, byv_d, abzw_d \in M$. Set

$$D_d = \{e, (ax)u_d, (by)v_d, (abz)w_d\}.$$

Note that $M = D_d A_1 \cdots A_{n-1}$ is a factorization of M. Indeed, product coming from $D_d A_1 \cdots A_{n-1}$ are among the products coming from $BA_1 \cdots A_{n-1}(Cd^{-1})$. By the minimality of the counterexample D_d is periodic. If $|b| \neq 2$, then the periodicity of D_d gives that $xu_d = e$ and $yv_d = zw_d$ for each $d \in C$. We get

$$x^{-1} \in \bigcap_{d \in C} Cd^{-1}.$$

By Lemma 6 of [8], C is periodic. As $|C| = p$ we get $C = \langle c \rangle$, that is, $|c| = p$. Choose d to be e and consider $xu_e = e$, $yv_e = zw_e$. As $u_e, v_e, w_e \in \langle c \rangle$, we get that there is an integer k such that $z = x^k y$. Now $B = \{e, ax, by, abx^k y\}$ and by case 5 we are done. We may assume that $|b| = 2$. In this situation $|ab| = 2$ and the roles of a, b, ab are symmetric. We write B back in the form $B = \{e, ax, by, cz\}$, where $c = ab$. By the periodicity of D_d, one of the following holds

$$xu_d = e, \quad yv_d = zw_d,$$
$$yv_d = e, \quad xu_d = zw_d,$$
$$zw_d = e, \quad xu_d = yv_d.$$

Suppose there are $g, h \in C$ such that $g \neq h$ and

$$xu_g = e, \quad yv_g = zw_g,$$
$$yv_h = e, \quad xu_h = zw_h.$$

Then $x = c^i$, $y = c^j$ for some i, j, $-p+1 \leq i, j \leq p-1$. Here $i \neq 0$, $j \neq 0$ as $x \neq e$, $y \neq e$. If $i = j$, then $x = y$ and by case 4, we are done. Therefore $i \neq j$ and $y = x^s$, where $p \not| s$. Again by case 4, we are done. We may assume that $xu_d = e$ for each $d \in C$. It means that

$$x^{-1} \in \bigcap_{d \in C} Cd^{-1}$$

and by Lemma 6 of [8], C is periodic. Since $|C| = p$, we get that C is a subgroup of G and so $|c| = p$. Using the fact that D_d is periodic for $d = e$ by symmetry we may assume that $xu_e = e$, $yv_e = zw_e$. As $u_e, v_e, w_e \in \langle c \rangle$, it follows that there is an integer k for which $z = x^k y$. Now by case 5, we are done.

This completes the proof. $\square$

4. An open problem

We describe a construction of N. G. De Bruijn [2] explicitly. Let p, q be primes with $p \geq 5$ and let G be a group with basis elements x, y, z, $|x| = |y| = p$, $|z| = q$. Set

$$A_1 = \{e, xy, (xy)^2, \ldots, (xy)^{p-3}, x^{p-2}y^{p-1}, x^{p-1}y^{p-2}\},$$
$$B = \langle x \rangle \cup z\langle x \rangle \cup \cdots \cup z^{q-2}\langle x \rangle \cup z^{q-1}\langle y \rangle.$$

A routine computation shows that $G = BA_1$ is a factorization and none of the factors is periodic. Clearly $|A_1| = p$, $|B| = pq$.

Let $G = BA_1 \cdots A_n$ be a factorization of G such that $|A_1|, \ldots, |A_n|$ are primes and $|B|$ is a product of two primes, say $|B| = pq$. Does it follow that at least one of the factors is always periodic? The example above shows that the answer is "no" if $p \geq 5$. We can hope an affirmative answer only in the $p \leq 3$ case. The $p = q = 2$ case is settled by this paper. The answer is not known when $p = q = 3$ or $p = 3$, $q = 2$. This is Problem 2. of [7] which reads as follows.

Let $G = BA_1 \cdots A_n$ be a factorization of a finite abelian group G such that $|A_1|, \ldots, |A_n|$ are primes and $|B|$ is 6 or 9. Does it follow that one of the factors is periodic?

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