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Group Actions on Stacks and Applications

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The motivation at the origin of this article is to investigate some ways of constructing moduli for curves, and covers above them, using tools from stack theory. This idea arose from reading Bertin–Mézard [BeM, esp. Sec. 5] and Abramovich–Corti–Vistoli [ACV]. Our approach is in the spirit of most recent works, where one uses the flexibility of the language of algebraic stacks. This language has two (twin) aspects: category-theoretic on one side and geometric on the other. Some of our arguments, especially in Section 8, are formal arguments involving general constructions concerning group actions on algebraic stacks (this is more on the categoric side). They are, intrinsically, natural enough to preserve the "modular" aspect. In trying to isolate these arguments, we were led to write results of independent interest. It seemed therefore more adequate to present them in a separate, self-contained part. Thus the article is split into two parts of comparable size.

More specifically, groups are ubiquitous in algebraic geometry (when one focuses on curves and maps between them, examples include the automorphism group, fundamental group, monodromy group, permutation group of the ramification points, ...). It is natural to ask whether we can handle group actions on stacks in the same fashion as we do on schemes. For example, we expect: that the quotient of the stack of curves with ordered marked points $\mathcal{M}_{g,n}$ by the symmetric group should classify curves with unordered marked points; that if G acts on a scheme X then the fixed points of the stack $\mathcal{P}ic(X)$ under G should be related to G-linearized line bundles on X; and that the quotient of the modular stack curve $\mathcal{X}_1(N)$ by $(\mathbb{Z}/N\mathbb{Z})^\times$ should be $\mathcal{X}_0(N)$ (the notation is, we hope, well known to the reader). Other important examples appear in the literature: action of tori on stacks of stable maps in Gromov–Witten theory [Ko; GrPa], and action of the symmetric group $\mathfrak{S}_d$ on a stack of multisections in [L-MB, (6.6)]. Our aim is to provide the material necessary to handle the questions raised here and then answer them, as well as to give other applications.

Let us now explain in more detail the structure and results of this paper.

In Part A we discuss the notion of a group action on a stack. We are mainly interested in giving general conditions under which the fixed points and the quotient of an algebraic stack are algebraic. In Sections 1 and 2 we give definitions and basics on actions. For simplicity let us now consider a flat group scheme G and an algebraic stack $\mathcal{M}$, both of finite presentation (abbreviated fp) over some

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base scheme. Slightly sharper assumptions are in the text. In Sections 3 and 4 we establish the following.

- Assume that the structure sheaf $\mathcal{O}_G$ is locally free over the base. If, moreover, G is proper or if $\mathcal{M}$ is a Deligne–Mumford stack, then there is an algebraic stack of fixed points $\mathcal{M}^G$. The map $\mathcal{M}^G \to \mathcal{M}$ is representable and separated, with even better properties when G is finite (Theorem 3.3, Proposition 3.7).
- If G is separated then there is a quotient algebraic stack $\mathcal{M}/G$. The map $\mathcal{M} \to \mathcal{M}/G$ is a G-torsor and hence is representable, separated, and fp (Theorem 4.1).

Finally in Section 5, as a preparation to Part B, we recall the concept of rigidification of a stack and look at its behavior with respect to group actions. The result is that rigidification along a group H and quotient by a group G commute under the natural hypotheses (Theorem 5.1). As far as fixed points are concerned, no such commutation holds in general, but we are able to prove a special case that is sufficient for what we need later on (Proposition 5.3).

In Part B we describe applications to moduli of maps of curves. Let G be a given finite monodromy group with order $|G|$. Our constructions rely on the stacks of curves with level structures. A Teichmüller structure on a curve (in the sense of [DMu]) is essentially given by a torsor over the curve, which is itself a curve. This points out a link with the moduli stack of curves with G action, denoted $\mathcal{H}_{g,G}$. We are therefore led to recall, in Section 6, what is known about $\mathcal{H}_{g,G}$ in the tame case. In Section 7 we derive a definition of a proper $\mathbb{Z}[1/|G|]$-stack $\bar{\mathcal{M}}_g(G)$ that we wish to compare with the normalization of $\bar{\mathcal{M}}_g$ in $\mathcal{M}_g(G)$ as introduced by Deligne–Mumford. Denote the latter normalization by $\tilde{\mathcal{M}}_g(G)$. We show that

- $\bar{\mathcal{M}}_g(G)$ is a "modular" desingularization of $\tilde{\mathcal{M}}_g(G)$ over $\mathbb{Z}[1/|G|]$ (Theorem 7.2.3).

This is, in spirit, close to [ACV, Sec. 5.2], but our construction is more straightforward. In Section 8, we proceed to find a presentation of $\mathcal{H}_{g,G}$ as a quotient of a scheme by a finite group. We introduce $\mathbb{Z}[1/n]$-stacks, denoted $\mathcal{X}^n_{g,G}$ and $\bar{\mathcal{X}}^n_{g,G}$, that are built up from stacks $\bar{\mathcal{M}}_g(G)$. The parameter n is an arbitrary integer, prime to $|G|$. The meaning of these stacks is more visible upon their definition (Definition 8.2.1), and we may just mention here that $\mathcal{X}^n_{g,G}$ is a sum of quotients of a scheme by a finite group, as desired. We prove that there is an isomorphism $\mathcal{X}^n_{g,G} \simeq \mathcal{H}_{g,G}$ over $\mathbb{Z}[1/n]$ (Theorem 8.2.2). In particular, $\mathcal{H}_{g,G}$ is described by a quotient presentation, valid also at characteristics that divide $|G|$.

NOTATION. We have already used some standard notation; observe that $|E|$ denotes the cardinality of a finite set E. Algebraic stacks are taken in the sense of [L-MB, Def. 4.1]. If n is a fixed integer and $\mathcal{M}$ is an algebraic stack then we denote $\mathcal{M}[1/n] = \mathcal{M} \otimes \mathbb{Z}[1/n]$. For instance, we should have written $\mathcal{H}_{g,G}[1/n]$ a few lines back; we shall be careful always to do so. The residue field of a point x in a scheme X is denoted by k_x. In Part A, a scheme S is fixed and we work in the setting of S-stacks, so we often omit S from the notation (e.g., in fibred products). The base change of X/S by T/S is written $X_T := X \times_S T$. We often abbreviate "locally of finite presentation" by lfp. We write categories in calligraphic letters (such as $\mathcal{C}$) and 2-categories in fraktur letters (such as $\mathfrak{C}$). A category fibred in groupoids over S is simply called a groupoid over S. In such a groupoid $\mathcal{M}$, the functor of isomorphisms between two objects $x, y \in \mathcal{M}(T)$ is denoted by $\mathrm{Isom}_T(x, y)$ or by $\mathrm{Isom}_{\mathcal{M}_T}(x, y)$ if mention of $\mathcal{M}$ is needed.

Part A. Group Actions on Stacks

1. Operations in a 2-Category

1.1. We first recall some basics concerning diagrams in a 2-category (in a very sketchy way). Chapter I of [Ha] is a good reference for all we need about this. Loosely speaking, a *diagram* in a 2-category $\mathfrak{C}$ is a set of objects with a set of 1-morphisms between certain pairs of objects and a set of 2-morphisms between certain pairs of 1-morphisms with same source and target. We write $\mathcal{D} = \{\mathcal{M}, f, \alpha\}$ to indicate that $\mathcal{M}$ (resp. f, α) ranges through the set of objects (resp. 1-morphisms, 2-morphisms) of the diagram $\mathcal{D}$. The set of i-morphisms ($i = 1, 2$) of the diagram is assumed to be saturated under composition—that is, including all possible compositions between i-morphisms. We call a pair of i-morphisms of $\mathcal{D}$ with same source and target an *i-circuit*; it *commutes* if its two morphisms coincide. A diagram in $\mathfrak{C}$ is said to be *i-commutative* if any of its i-circuits commutes.

In most diagrams we will consider, to any 1-circuit will be attached a 2-morphism in a natural way, so we will often not bother to draw it. For instance, in the diagram $\mathcal{D}$ pictured below, with our convention it is understood that there is a 2-morphism $\alpha''\colon g'gh \Rightarrow h''f'f$ attached to the exterior rectangular 1-circuit. If $*$ denotes the composition of 2-morphisms between adjoined diagrams [Ha, I, (1.5)], the 2-circuit pictured on the right is simply written $(\alpha'', \alpha' * \alpha)$. Thus $\mathcal{D}$ is 2-commutative if and only if $\alpha'' = \alpha' * \alpha$.

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{M}' \arrow[r, "f'"] \arrow[d, "h'"] & \mathcal{M}'' \arrow[d, "h''"] \\
\mathcal{N} \arrow[r, "g"'] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{N}' \arrow[r, "g'"'] \arrow[ur, Rightarrow, "\alpha'"] & \mathcal{N}''
\end{tikzcd}
\qquad \text{together with the 2-circuit} \qquad
\begin{tikzcd}
& g'h'f \arrow[dr, Rightarrow, "\alpha' f"] & \\
g'gh \arrow[ur, Rightarrow, "g'\alpha"] \arrow[rr, Rightarrow, "\alpha''"'] & & h''f'f
\end{tikzcd}
$$

1.2. We now look at the 2-category $\mathfrak{C} = \mathfrak{Grpd}/S$ of groupoids over S. In this 2-category all 2-morphisms are isomorphisms, so that 2-commutativity of diagrams means "(1-)commutativity up to (given) isomorphisms". Let $\mathcal{M}$ be such a groupoid and G a functor in groups over S. We denote by m the multiplication of G and by e (or sometimes simply 1) its unit section. The weakest possible definition of an action of G on $\mathcal{M}$ is a morphism of groupoids $\mu\colon G \times \mathcal{M} \to \mathcal{M}$ satisfying 2-commutative diagrams concerning compatibility with respect to the multiplication and the unit section of G:

$$
\begin{tikzcd}
G \times G \times \mathcal{M} \arrow[r, "m \times \mathrm{id}_{\mathcal{M}}"] \arrow[d, "\mathrm{id}_G \times \mu"'] & G \times \mathcal{M} \arrow[d, "\mu"] \\
G \times \mathcal{M} \arrow[r, "\mu"'] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{M}
\end{tikzcd}
\quad , \qquad
\begin{tikzcd}
G \times \mathcal{M} \arrow[r, "\mu"] \arrow[dr, phantom, "\mathfrak{a}"] & \mathcal{M} \\
\mathcal{M} \arrow[u, "e \times \mathrm{id}_{\mathcal{M}}"] \arrow[ur, "\mathrm{id}_{\mathcal{M}}"'] &
\end{tikzcd}
\quad . \tag{1}
$$

Given a scheme T, sections $x, y \in \mathcal{M}(T)$ and $g \in G(T)$, and an arrow $\varphi \colon x \to y$, as usual we abbreviate $\mu(g,x)$ by $g.x$ and $\mu(g,\varphi)$ by $g.\varphi$. Then α and $\mathfrak{a}$ are merely given by isomorphisms that are natural in (g,h,x):

$$\alpha^x_{g,h} \colon g.(h.x) \xrightarrow{\sim} (gh).x \quad \text{and} \quad \mathfrak{a}^x \colon 1.x \xrightarrow{\sim} x.$$

Actually, this is not enough to give a reasonable definition because the presence of α and $\mathfrak{a}$ introduces two novelties. Namely, we must make sure that they are compatible with each other and also that "higher associativity" (the different ways to pass from an expression such as $g_1.(g_2.(\dots(g_n.x)))$ to $(g_1\dots g_n).x$ using α) holds. Similarly, a morphism of G-groupoids $f \colon \mathcal{M} \to \mathcal{N}$ should be given by a 2-commutative diagram

$$\begin{tikzcd} G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "\mathrm{id}_G \times f"'] & \mathcal{M} \arrow[d, "f"] \\ G \times \mathcal{N} \arrow[r, "\nu"] \arrow[ur, Rightarrow, "\sigma"] & \mathcal{N} \end{tikzcd} \tag{2}$$

with some "higher associativity" condition on σ (more concretely, we may write $\sigma^x_g \colon g.f(x) \xrightarrow{\sim} f(g.x)$). Sorting this all out, we arrive at the following definition.

DEFINITION 1.3. Let $\mathcal{M}$ be a groupoid over S, and let G be a functor in groups over S.

(i) An *action* of G on $\mathcal{M}$ is a triple $(\mu, \alpha, \mathfrak{a})$, where $\mu \colon G \times \mathcal{M} \to \mathcal{M}$ is a morphism of groupoids satisfying the 2-commutative diagrams in (1) and such that, for all x and all g, h, k we have

$$\alpha^x_{g,hk} \circ g.\alpha^x_{h,k} = \alpha^x_{gh,k} \circ \alpha^{k.x}_{g,h} \quad \text{and} \quad [g.\mathfrak{a}^x = \alpha^x_{g,1} \text{ and } \mathfrak{a}^{h.x} = \alpha^x_{1,h}].$$

We say that $(\mathcal{M}, \mu, \alpha, \mathfrak{a})$, or simply $\mathcal{M}$, is a *G-groupoid*. If α and $\mathfrak{a}$ are the identity 2-isomorphisms, we say that the action (or the G-groupoid) is *strict*.

(ii) A *morphism of G-groupoids* between $(\mathcal{M}, \mu, \alpha, \mathfrak{a})$ and $(\mathcal{N}, \nu, \beta, \mathfrak{b})$ is a pair (f, σ), where $f \colon \mathcal{M} \to \mathcal{N}$ is a morphism of groupoids over S satisfying the 2-commutative diagram in (2) and such that, for all x and all g, h we have $f(\alpha^x_{g,h}) \circ \sigma^{h.x}_g \circ g.\sigma^x_h = \sigma^x_{gh} \circ \beta^{f(x)}_{g,h}$ and $f(\mathfrak{a}^x) \circ \sigma^x_1 = \mathfrak{b}^{f(x)}$.

(iii) An *isomorphism of G-groupoids* is a morphism of G-groupoids that is also an equivalence of categories fibred over S.

REMARK 1.4. To justify Definition 1.3(iii), one must check that if (f, σ) is an isomorphism of G-groupoids then it has a quasi-inverse that is also a morphism of G-groupoids. This is rather straightforward: one just transports the 2-morphism of G-equivariance σ to a given quasi-inverse $e \colon \mathcal{N} \to \mathcal{M}$. Namely, fix some 2-morphisms $\varphi \colon ef \Rightarrow \mathrm{id}_{\mathcal{M}}$ and $\psi \colon fe \Rightarrow \mathrm{id}_{\mathcal{N}}$. Assume we are given a section y of $\mathcal{N}$ and a section g of G. Put $x = e(y)$. Then consider $\sigma^x_g \colon g.f(x) \xrightarrow{\sim} f(g.x)$, apply e to it, and use φ, ψ to derive an isomorphism $\tau^y_g \colon g.e(y) \xrightarrow{\sim} e(g.y)$. One may check that (e, τ) is a morphism of G-groupoids.

The 2-category of G-groupoids over S is denoted G-$\mathfrak{Grpd}/S$. Expert readers recognize that the data $(\mathcal{M}, \mu, \alpha, \mathfrak{a})$ determine exactly what is called a *lax presheaf in* *groupoids $\mathcal{F}$ over $\mathcal{C} = B_0G$*, where B_0G is the groupoid associated to G (i.e., the groupoid whose fibre over a scheme T/S has only one object and "morphisms" the elements of $G(T)$). This lax presheaf (see [Ho, Apx. B]) is described as follows.

- To an object of B_0G (i.e., a scheme T over S) is associated the groupoid $\mathcal{F}(T) = \mathcal{M}(T)$.
- To a morphism of B_0G (i.e., a section $g \in G(T)$) is associated the functor $\mu_g := \mu(g^{-1}, \cdot)\colon \mathcal{M}(T) \to \mathcal{M}(T)$.
- For each $g, h \in G(T)$, there is a natural transformation $\mu_g \circ \mu_h \xrightarrow{\sim} \mu_{hg}$ given by $\alpha_{g^{-1},h^{-1}}$.

So now, Definitions 1.3(i) and (ii) are simply translations of the definitions we find in [Ho, Apx. B]. This link with lax presheaves also explains why, given a G-groupoid $\mathcal{M}$, we will always be able to find an equivalent G-groupoid $\mathcal{M}^{\mathrm{str}}$ such that the 2-isomorphisms α and $\mathfrak{a}$ are the identities.

Proposition 1.5. *The inclusion functor of the 2-category of strict G-groupoids, as a fully faithful sub-2-category of G-$\mathfrak{Grpd}/S$, is a 2-equivalence. More precisely, there is a "strictification" functor G-$\mathfrak{Grpd}/S \to G$-$\mathfrak{Grpd}/S$ sending any G-groupoid to an isomorphic G-groupoid with strict action.*

Definitions of 2-functors and 2-equivalences can be found in [Ha, Chap. I, (1.5)–(1.8)]. However, the reader may safely rely on intuition and hence skip those definitions (and is advised to do so).

Proof. Let $\mathcal{M}$ be a G-groupoid, and define a G-groupoid $\mathcal{M}^{\mathrm{str}}$ in the following way:

- the sections of $\mathcal{M}^{\mathrm{str}}$ over a scheme T are pairs (g, x) with $g \in G(T)$ and $x \in \mathcal{M}(T)$;
- the arrows in $\mathcal{M}^{\mathrm{str}}$ between (g, x) and (h, y) are arrows $\varphi\colon x \to (g^{-1}h).y$ in $\mathcal{M}(T)$; and
- composition of two arrows $\varphi\colon (g, x) \to (h, y)$ and $\psi\colon (h, y) \to (k, z)$ is given by

$$x \xrightarrow{\varphi} (g^{-1}h).y \xrightarrow{(g^{-1}h).\psi} (g^{-1}h).((h^{-1}k).z) \xrightarrow{\alpha^{z}_{g^{-1}h,h^{-1}k}} (g^{-1}k).z.$$

There is a strict action of G on $\mathcal{M}^{\mathrm{str}}$: an element $\gamma \in G(T)$ sends an object (g, x) to $(\gamma g, x)$ and sends an arrow $\varphi\colon x \to (g^{-1}h).y$ to the same arrow as a morphism between $(\gamma g, x)$ and $(\gamma h, y)$.

It remains to check that $\mathcal{M}$ and $\mathcal{M}^{\mathrm{str}}$ are isomorphic. We define a morphism of groupoids $u\colon \mathcal{M}^{\mathrm{str}} \to \mathcal{M}$ by mapping (a) an object (g, x) to $g.x$ and (b) an arrow $(g, x) \to (h, y)$, represented by $\varphi\colon x \to (g^{-1}h).y$, to $\alpha^{y}_{g,g^{-1}h} \circ (g.\varphi)$. Clearly, u is a G-morphism; furthermore, it is essentially surjective because any object x in $\mathcal{M}$ is isomorphic via $\mathfrak{a}^x$ to $1.x$. Finally, it is straightforward to see that u is fully faithful, so it must be an isomorphism. □

Because of this proposition, our point of view in the next section will be limited to considering strict actions. For Sections 2–5 (inclusive), we assume that the scheme S, viewed as the category of S-schemes, is endowed with the étale or fppf topology (fppf = *fidèlement plat de présentation finie,* faithfully flat of finite presentation). Since stackification commutes with finite fibred products, it is clear that an action of G on a groupoid $\mathcal{M}$ extends uniquely to an action on the associated stack $\tilde{\mathcal{M}}$.

2. Definitions

DEFINITION 2.1. Let $\mathcal{M}$ be a stack over S, and let G be a sheaf in groups over S. Let m denote the multiplication of G and let e be its unit section. Let T be an S-scheme.

(i) An *action* of G on $\mathcal{M}$ is a *strict* action (as in Definition 1.3). It is given by a morphism of stacks $\mu\colon G \times \mathcal{M} \to \mathcal{M}$ with the 1-commutative diagrams in (1), where α and $\mathfrak{a}$ equal identities. The pair $(\mathcal{M}, \mu)$ is called a *G-stack*; by abuse of notation, it is sometimes denoted by $\mathcal{M}$.

(ii) A *1-morphism of G-stacks* between $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ is a morphism of G-groupoids (as in Definition 1.3). It is given by a pair (f, σ) with the 2-commutative diagram in (2) such that, for all sections $x \in \mathcal{M}(T)$ and $g, h \in G(T)$, we have $\sigma_g^{h.x} \circ g.\sigma_h^x = \sigma_{gh}^x$. We say that the morphism is *strict* if σ is the identity.

(iii) A *2-morphism of G-stacks* between 1-morphisms (f_1, σ_1) and (f_2, σ_2) is a 2-morphism of stacks $\tau\colon f_1 \Rightarrow f_2$ compatible with the σ_i—that is, such that, for all sections $x \in \mathcal{M}(T)$ and $g \in G(T)$ over a scheme T, we have $\sigma_{2,g}^x \circ g.\tau^x = \tau^{g.x} \circ \sigma_{1,g}^x$. In this way we define a 2-category of G-stacks over S, which will be denoted by G-$\mathfrak{St}/S$ or simply by G-$\mathfrak{St}$ if the base S is understood. In particular, given two G-stacks $\mathcal{M}, \mathcal{N}$, there is the stack $\mathcal{H}om_{G\text{-}\mathfrak{St}}(\mathcal{M}, \mathcal{N})$ of 1-morphisms and 2-morphisms between them.

(iv) An *isomorphism of G-stacks* is a 1-G-morphism that is also an equivalence of groupoids over S. A monomorphism is a fully faithful 1-G-morphism, and an epimorphism is a 1-G-morphism that is locally essentially surjective [L-MB].

It turns out that there is an equivalent way to define morphisms for G-stacks in the sense of (ii) and (iii). This alternative definition requires that we introduce in the first place commutative diagrams in the 2-category G-$\mathfrak{St}/S$. The following remark explains this.

REMARK 2.2. Let $\mathcal{D} = \{\mathcal{M}, f, \alpha\}$ be a diagram of stacks, where $\mathcal{M}$, f, and α range through objects, 1-morphisms, and 2-morphisms of $\mathcal{D}$ (respectively). Consider

- the diagram $G \times \mathcal{D} := \{G \times \mathcal{M},\ \mathrm{id}_G \times f,\ \mathrm{id}_{\mathrm{id}_G} \times \alpha\}$ and
- the diagram $G \times G \times \mathcal{D} := \{G \times G \times \mathcal{M},\ \mathrm{id}_{G\times G} \times f,\ \mathrm{id}_{\mathrm{id}_{G\times G}} \times \alpha\}$.

Assume that the objects are in fact G-stacks $(\mathcal{M}, \mu)$ and that, for any objects $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ and for any 1-morphism $f\colon \mathcal{M} \to \mathcal{N}$, we are given a 2-isomorphism $\sigma\colon \nu \circ (\mathrm{id}_G \times f) \Rightarrow f \circ \mu$. Then we can form a new diagram $G \times G \times \mathcal{D} \to G \times \mathcal{D} \to \mathcal{D}$. Precisely, at the stage $G \times \mathcal{D} \to \mathcal{D}$ the 1-morphisms are the μ, the 2-morphisms are the σ. And at the stage $G \times G \times \mathcal{D} \to G \times \mathcal{D}$ the

1-morphisms are the $(\mathrm{id}_G \times \mu)$, the 2-morphisms are the $(\mathrm{id}_G \times \sigma)$. Hence we can define a *2-commutative diagram of G-stacks* to be given by data $(\mathcal{D}, \{\mu\}, \{\sigma\})$ such that $G \times G \times \mathcal{D} \to G \times \mathcal{D} \to \mathcal{D}$ is a 2-commutative diagram of stacks. In particular, we can now define the notions of *1-morphisms of G-stacks* and *2-morphisms between 1-morphisms of G-stacks* via the 2-commutativity of the two elementary diagrams

$$(\mathcal{M}, \mu) \xrightarrow{(f,\sigma)} (\mathcal{N}, \nu) \quad \text{and} \quad \begin{tikzcd} (\mathcal{M}, \mu) \arrow[r, bend left, "{(f_1,\sigma_1)}"] \arrow[r, bend right, "{(f_2,\sigma_2)}"'] \arrow[r, phantom, "\Downarrow\tau"] & (\mathcal{N}, \nu). \end{tikzcd}$$

Checking 2-commutativity means checking 2-commutativity of the following "prisms" in $\mathfrak{St}/S$:

$$\begin{tikzcd} G \times G \times \mathcal{M} \arrow[r, "1\times\mu"] \arrow[d, "1\times1\times f"'] & G \times \mathcal{M} \arrow[r, "\mu"] \arrow[d, "1\times f"'] & \mathcal{M} \arrow[d, "f"] \\ G \times G \times \mathcal{N} \arrow[r, "1\times\nu"'] \arrow[ur, Rightarrow, "1\times\sigma"] & G \times \mathcal{N} \arrow[r, "\nu"'] \arrow[ur, Rightarrow, "\sigma"] & \mathcal{N} \end{tikzcd}$$

and

$$\begin{tikzcd} G \times G \times \mathcal{M} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[d] & & G \times G \times \mathcal{N} \arrow[d] \\ G \times \mathcal{M} \arrow[rr, bend left, dotted] \arrow[rr, bend right] \arrow[d] & & G \times \mathcal{N} \arrow[d] \\ \mathcal{M} \arrow[rr, bend left, dotted] \arrow[rr, bend right] & & \mathcal{N} \end{tikzcd} .$$

The 2-category G-$\mathfrak{St}/S$ has arbitrary projective and inductive limits. In particular, G-$\mathfrak{St}/S$ has fibred products. Any stack $\mathcal{M}$ over S gives a trivial G-stack $(\mathcal{M}, \mathrm{pr}_2)$, and this gives a 2-functor $\iota : \mathfrak{St} \to G$-$\mathfrak{St}$. The invariants and coinvariants are the 2-adjoints of this functor.

DEFINITION 2.3. Let G be a sheaf in groups over S and let $\mathcal{M}$ be a G-stack over S.

(i) A *stack of fixed points* $\mathcal{M}^G$ is a stack that 2-represents the 2-functor $\mathfrak{St}^\circ \to \mathfrak{Cat}$ defined by

$$F(\mathcal{N}) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\iota(\mathcal{N}), \mathcal{M})$$

(the latter is the stack of Definition 2.1(iii), and $\mathfrak{Cat}$ is the 2-category of categories).

(ii) A *quotient stack* $\mathcal{M}/G$ is a stack that 2-represents the 2-functor $\mathfrak{St} \to \mathfrak{Cat}$ defined by

$$F(\mathcal{N}) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\mathcal{M}, \iota(\mathcal{N})).$$

REMARK 2.4. Assume that $\mathcal{M}$ is a G-stack and that $H \lhd G$ is a normal subgroup. When all that follows make sense, we expect G/H to act on $\mathcal{M}/H$ and $\mathcal{M}^H$ and to have transitivity isomorphisms $(\mathcal{M}/H)/(G/H) \simeq \mathcal{M}/G$ and $(\mathcal{M}^H)^{G/H} \simeq \mathcal{M}^G$. This is indeed the case, but one must be careful and it is appropriate here to stress the weak-versus-strict problem. Denote by $\pi : \mathcal{M} \to \mathcal{M}/H$ the quotient map and, for each section $g \in G(T)$, denote by $\mu_g = \mu(g, \cdot)$ the automorphism of $\mathcal{M}$ (in fact $\mathcal{M}_T$). By the 2-universal property we have a 2-commutative diagram,

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\mu_g"] \arrow[d, "\pi"'] & \mathcal{M} \arrow[d, "\pi"] \\ \mathcal{M}/H \arrow[r, dashed, "\nu_g"'] \arrow[ur, Rightarrow, "\varepsilon_g"] & \mathcal{M}/H \end{tikzcd} .$$

Here ν_g is unique up to a unique 2-isomorphism, so that for any two sections g_1, g_2 there is a 2-isomorphism $\alpha_{g_1,g_2} : \nu_{g_1} \circ \nu_{g_2} \Rightarrow \nu_{g_1 g_2}$ satisfying the cocycle condition. This makes $(\mathcal{M}/H, \nu, \alpha, \mathfrak{a} = \mathrm{id})$ into a weak G-stack. Of course, for $h \in H$ we can choose $\nu_h = \mathrm{id}$ (with nontrivial ε_h) and then by unicity $\alpha_{g_1,g_2} = \mathrm{id}$ as soon as $g_1 \in H$ or $g_2 \in H$. So if we denote by $g \mapsto \bar{g}$ the projection $G \to G/H$, it makes sense to define $\bar{\nu}_{\bar{g}} = \nu_g$ and $\bar{\alpha}_{\bar{g}_1,\bar{g}_2} = \alpha_{g_1,g_2}$. Thus we get a weak G/H-stack $(\mathcal{M}/H, \bar{\nu}, \bar{\alpha}, \mathrm{id})$, which we "strictify" (Proposition 1.5) to obtain a strict G/H-stack and we are done. Note that if we first strictified the G-stack $(\mathcal{M}/H, \nu, \alpha, \mathrm{id})$ then we would lose the possibility of choosing $\nu_h = \mathrm{id}$ and could no longer induce a G/H-action. It is now a simple matter to derive the transitivity isomorphism. All we said applies also to $\mathcal{M}^H$.

PROPOSITION 2.5. *Let G be a sheaf in groups over S and let $\mathcal{M}$ be a G-stack over S. Then there exists a stack of fixed points $\mathcal{M}^G$, and its formation commutes with base change on S.*

Proof. From the definition we must have

$$\mathcal{H}om_{\mathfrak{St}}(\mathcal{N}, \mathcal{M}^G) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\iota(\mathcal{N}), \mathcal{M}).$$

From the particular case $\mathcal{N} = S$ we deduce

$$\mathcal{M}^G = \mathcal{H}om_{\mathfrak{St}}(S, \mathcal{M}^G) = \mathcal{H}om_{G\text{-}\mathfrak{St}}(\iota(S), \mathcal{M}).$$

This is the stack of G-invariant sections of $\mathcal{M}$ whose objects over a base T are pairs $(x, \{\alpha_g\}_{g \in G(T)})$, where $x \in \mathcal{M}(T)$ and $\alpha_g : x \to g.x$ are isomorphisms such that $g.\alpha_h \circ \alpha_g = \alpha_{gh}$ for all sections $g, h \in G(T)$. □

PROPOSITION 2.6. *Let G be a sheaf in groups over S and let $\mathcal{M}$ be a G-stack over S. Then there exists a quotient stack $\mathcal{M}/G$, and its formation commutes with base change on S.*

Proof. We define a prestack $\mathcal{P}$ as follows: sections of $\mathcal{P}(T)$ are sections of $\mathcal{M}(T)$, and morphisms in $\mathcal{P}(T)$ between x and y are pairs (g, φ), with $g \in G(T)$ and $\varphi : g.x \to y$ a morphism in $\mathcal{M}(T)$. Let $\mathcal{M}/G$ be the stack associated to $\mathcal{P}$. It is straightforward to check the universal 2-property. □

3. Algebraicity of Fixed Points

From this section on, we consider only *algebraic* stacks. The category G-$\mathfrak{AlgSt}/S$ of algebraic G-stacks over S is defined to be the full subcategory of G-$\mathfrak{St}/S$ of G-stacks whose underlying stack is algebraic. In particular all definitions of 2.1 apply, so we need not rewrite them. The definitions of 2.3 carry on in an obvious way, namely the *algebraic stack of fixed points* represents a 2-functor $\mathfrak{AlgSt}^\circ \to \mathfrak{Cat}$, and the *quotient algebraic stack* represents a 2-functor $\mathfrak{AlgSt} \to \mathfrak{Cat}$.

In order to derive algebraicity of fixed points we will consider group schemes that are essentially free over the base, a standard assumption in this context (read [SGA3, Exp. VIII, Sec. 6]). Recall that $X \to S$ is essentially free if, possibly after an fppf extension $S' \to S$, there is a covering of S by open affines S_i and, for all i, a covering of $X \times S_i$ by open affines $X'_{i,j}$ such that the function ring of $X'_{i,j}$ is a free module over the function ring of S_i.

Lemma 3.1. *Let $X' \to S' \to S$ be morphisms of schemes, with $X' \to S'$ unramified and separated and with $S' \to S$ essentially free and fppf. Let $X = \prod_{S'/S} X'$ be the Weil restriction functor defined by $X(T) = X'(T \times_S S')$ for all S-schemes T. Then X is representable by an unramified and separated S-scheme.*

Proof. The question of representability by an algebraic space is local for the fppf topology on S. Thus we may assume S affine and S' quasi-compact. Then there is an étale extension $S'' \to S'$ such that $X' \times_{S'} S''$ is a closed subscheme of a finite disjoint union $S'' \amalg \cdots \amalg S''$. So, after we perform the base change $S'' \to S$, the result is a consequence of [SGA3, Exp. VIII, Thm. 4.6] given that the Weil restriction of $S' \amalg \cdots \amalg S'$ is just $S \amalg \cdots \amalg S$. Therefore, X is representable by an algebraic space that is unramified and separated over S. It follows that X is even a scheme [Kn, II, 6.16]. □

Example 3.2. By the theory of the Hilbert scheme it is known that the Weil restriction X is also representable when $S' \to S$ is proper, a fact that we shall use below. We give a counterexample to representability when $S' \to S$ is not proper and $X' \to S'$ is ramified. Namely, we take $S = \mathrm{Spec}(k)$ the spectrum of a field, $S' = \mathrm{Spec}(k[x])$ the affine line over k, and $X' = \mathrm{Spec}(k[x][\varepsilon])$ with $\varepsilon^2 = 0$. Then for any k-algebra A we have

$$X(A) = \mathrm{Hom}_{k[x]}(k[x][\varepsilon], A[x]) = \left\{ P = \sum_{i=0}^{n} a_i x^i \in A[x],\ P^2 = 0 \right\}.$$

In degree n the relations $P^2 = 0$ are $a_0^2 = 2a_0a_1 = \cdots = a_n^2 = 0$, and the data of the coefficients a_i can be viewed as an A-point of

$$X_n = \mathrm{Spec}(k[a_0, \dots, a_n]/(a_0^2, 2a_0a_1, \dots, a_n^2)).$$

This is a local Artinian scheme and hence topologically a point. It is not hard to see that, if $X = \varinjlim X_n$ is representable, then its underlying topological space is still just a point. It follows that X is the spectrum of an Artinian local k-algebra, which can only be

$$M = \varprojlim k[a_0,\dots,a_n]/(a_0^2, 2a_0a_1,\dots,a_n^2).$$

This is impossible, because points of $\mathrm{Spec}(M)$ with values in A do not necessarily have finitely many nonzero coefficients (for instance, take the universal point!).

THEOREM 3.3. *Let G be a group scheme that is essentially free and locally of finite presentation* (*lfp*) *over S. Let $\mathcal{M}$ be an algebraic G-stack with diagonal lfp over S. Assume, furthermore, that either*

(i) *G is proper or*
(ii) *$\mathcal{M}$ is a Deligne–Mumford stack.*

Then the fixed point stack $\mathcal{M}^G$ (*Proposition 2.5*) *is algebraic—so it is a fixed point stack in $\mathfrak{AlgSt}$. Its formation commutes with base change on S. The morphism $\varepsilon\colon \mathcal{M}^G \to \mathcal{M}$ is representable and separated. In case* (ii), *ε is also formally unramified*; *moreover, if $\mathcal{M}$ is separated then ε is finite.*

Proof. It is enough to show that the morphism $\mathcal{M}^G \to \mathcal{M}$ is representable with the desired properties. So let $f\colon T \to \mathcal{M}$ be a 1-morphism corresponding to an object $x \in \mathcal{M}(T)$. The fibre product $\mathcal{M}^G \times_{\mathcal{M}} T$ is the sheaf whose sections over T'/T are collections of isomorphisms $\{\alpha_g : x \simeq g.x\}_{g\in G(T')}$ such that, for all sections $g,h \in G(T')$, we have $g.\alpha_h \circ \alpha_g = \alpha_{gh}$ (cocycle condition).

Denote by x_1 and x_2 the objects of $\mathcal{M}(G\times T)$ corresponding to the 1-morphisms $\mathrm{pr}_2 \circ (\mathrm{id}_G \times f)$ and $\mu \circ (\mathrm{id}_G \times f)$, respectively. Then $\mathcal{M}^G \times_{\mathcal{M}} T$ can again be expressed as the subfunctor of the Weil restriction

$$W = \prod_{G_T/T} \mathrm{Isom}_{G_T}(x_1,x_2)$$

defined by the cocycle condition. In case (i), W is representable by an algebraic space separated and lfp, by the theory of the Hilbert scheme (or, rather, its extension by Artin [Ar, Cor. 6.2]). In case (ii), the functor $\mathrm{Isom}_{G_T}(x_1,x_2)$ is a scheme unramified and separated over G_T, so W is representable by a scheme unramified and separated (by Lemma 3.1). Since G is essentially free, the cocycle condition defines a closed subscheme of W by [SGA3, Exp. VIII, Ex. 6.5.e]. This concludes the proof. □

REMARKS 3.4. (i) An example of the theorem's application to nonproper groups is to Gromov–Witten theory. Given a nonsingular variety V, having a group action on V helps to compute its Gromov–Witten invariants by localization formulas. A classical situation is where a torus T acts (e.g., $V = G/P$ is a projective homogeneous space and T is a maximal torus) and we are led to consider its induced action on the Deligne–Mumford stack of stable maps $\bar{\mathcal{M}}_{g,n}(V,\beta)$ (see e.g. [GrPa]).

(ii) If $\mathcal{M}$ is representable, then $\mathcal{M}^G$ is representable also; hence the fixed points of $\mathcal{M}$ as a space or as a stack are the same (the Yoneda functor from spaces into stacks commutes with projective limits when they exist, but not with inductive limits).

As indicated in Example 3.2, some problems arise if we want to prove a general result of algebraicity for nonproper groups acting on Artin stacks. In the opposite situation, when the group G is *finite* it is possible to give more properties of the morphism $\varepsilon\colon \mathcal{M}^G \to \mathcal{M}$. For a separated (Artin) stack $\mathcal{M}$, properness of ε would follow simply if we knew that this property is preserved under Weil restriction by a finite flat morphism. Unfortunately, Weil restriction does not behave so well, at least if the restriction morphism is ramified (see however [BLR, Chap. 7, Prop. 7.6/5(f)]). We shall deduce the corresponding property for ε by giving a slightly different construction of $\mathcal{M}^G$. We start with a lemma.

LEMMA 3.5. *Let Q be a finite flat scheme lfp over S, and let $\mathcal{M}$ be an algebraic stack lfp over S. Then the stack $\mathcal{H}om_S(Q,\mathcal{M})$ of morphisms of stacks from Q to $\mathcal{M}$ is algebraic and lfp over S.*

Proof. Let us set $\mathcal{H} := \mathcal{H}om_S(Q,\mathcal{M})$ and $n = [Q : S]$. Given an S-scheme T, we have $\mathcal{H}(T) = \mathcal{M}(Q\times T)$. From this and the fact that Q is affine, after algebraicity is proved it will follow that $\mathcal{H}$ is lfp over S because, given a filtering inductive system of S-algebras A_i, we have isomorphisms

$$\begin{aligned}\varinjlim \mathcal{H}(A_i) &\simeq \varinjlim \mathcal{M}(\mathcal{O}_Q \otimes A_i) \simeq \mathcal{M}(\varinjlim \mathcal{O}_Q \otimes A_i)\\ &\simeq \mathcal{M}(\mathcal{O}_Q \otimes \varinjlim A_i) \simeq \mathcal{H}(\varinjlim A_i).\end{aligned}$$

Now we will show that the diagonal of $\mathcal{H}$ is representable, separated, and quasi-compact. It is enough to study the sheaf $\mathrm{Isom}_{\mathcal{H}_T}(x,y)$ for two fixed objects $x,y \in \mathcal{H}(T)$. These correspond to objects $\eta \in \mathcal{M}(Q\times T)$ and $\xi \in \mathcal{M}(Q\times T)$, and

$$\mathrm{Isom}_{\mathcal{H}_T}(x,y) = \mathrm{Hom}_T(Q_T, \mathrm{Isom}_{\mathcal{M}_{Q\times T}}(\eta,\xi)).$$

Here the sheaf $I := \mathrm{Isom}_{\mathcal{M}_{Q\times T}}(\eta,\xi)$ is representable and of finite presentation over Q_T (it is lfp because $\mathcal{M}$ is, by [EGA, I, 6.2.6], which extends to stacks). It keeps these properties as a T-sheaf. Let us introduce the functor H_n, which is the component of the full Hilbert functor of $Q_T \times I$ parameterizing 0-dimensional subspaces of length n. It is representable by a separated algebraic space that is lfp [Ar, Cor. 6.2], and in fact the length-n component is quasi-compact because $Q_T \times I$ is. Now, the graph of a morphism $Q_T \to I$ defines a point in H_n (by separation of I) such that the restriction of the first projection $Q_T \times I \to Q_T$ is an isomorphism. The sheaf $\mathrm{Isom}_{\mathcal{H}_T}(x,y)$ is thus isomorphic to the corresponding constructible open subspace of H_n. By constructibility, this open immersion is quasi-compact [EGA, 0_{III}, 9.1.5] and, of course, separated.

Now let $U \to \mathcal{M}$ be an atlas; we can choose U separated. Then I claim that $V := \mathrm{Hom}_S(Q,U)$ will be an atlas for $\mathcal{H}$. First, by Artin's result again V is representable and lfp. Since $\mathcal{H}$ is also lfp, this shows that the map $V \to \mathcal{H}$ has the same property. Thus we only have to prove that it is formally smooth and surjective. To prove surjectivity, take an algebraically closed field k and a morphism

$\mathrm{Spec}(k) \to \mathcal{H}$ (i.e., a morphism $f\colon Q_k \to \mathcal{M}_k$). Then Q_k is an Artinian scheme and hence a sum of local Artinian k-schemes, so we reduce to the local case. By surjectivity of $U \to \mathcal{M}$, the image of the underlying point of Q_k lifts to U_k, and by smoothness the whole morphism lifts. It remains to prove formal smoothness. Let $A \to A_0$ be a surjection of Artinian rings with nilpotent kernel. Assume we have a 2-commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[dr, Rightarrow] & \mathcal{H} \\ \mathrm{Spec}(A_0) \arrow[u] \arrow[ur] & \end{tikzcd} \quad , \quad \text{meaning that we have} \quad \begin{tikzcd} U_{A_0} \arrow[r] \arrow[dr, Rightarrow] & \mathcal{M}_{A_0} \\ Q_{A_0} \arrow[u] \arrow[ur] & \end{tikzcd} .$$

Since Q_{A_0} is Artinian, it follows by the smoothness of $U_A \to \mathcal{M}_A$ that the map $Q_{A_0} \to U_{A_0} \to U_A$ immediately lifts to $Q_A \to U_A$, and we are done. $\square$

EXAMPLE 3.6. If $Q = S \otimes \mathbb{Z}[\varepsilon]/\varepsilon^2$ we recover the tangent stack $T(\mathcal{M}/S)$, and the lemma gives a proof of its algebraicity that is simpler than in [L-MB, Chap. 17].

PROPOSITION 3.7. *Let G be a finite, flat group scheme lfp over S. Let $\mathcal{M}$ be an algebraic G-stack that is lfp over S. Then the morphism $\varepsilon\colon \mathcal{M}^G \to \mathcal{M}$ is quasi-compact. Furthermore, consider a property of morphisms of schemes that is enjoyed by closed immersions and is stable by composition. Then, if the diagonal of $\mathcal{M}$ has this property, the morphism ε has this property. In particular, ε is proper if $\mathcal{M}$ is separated.*

Proof. Throughout, we will omit the description of the morphisms of the different stacks introduced, since they are obvious and quite lengthy to write completely. By the lemma applied to $Q = G$, the stack $\mathcal{H} = \mathcal{H}om(G, \mathcal{M})$ is algebraic. We now define two morphisms $a, b\colon \mathcal{M} \to \mathcal{H}$. Let $x \in \mathcal{M}(T)$, corresponding to a morphism $f\colon T \to \mathcal{M}$, and look at the compositions

$$\begin{tikzcd} G \times T \arrow[r, "\mathrm{id}\times f"] & G \times \mathcal{M} \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\mathrm{pr}_2"'] & \mathcal{M}. \end{tikzcd}$$

Then we define $a(x) = (\mu \circ (\mathrm{id}_G \times f))^*(x)$ and $b(x) = (\mathrm{pr}_2 \circ (\mathrm{id}_G \times f))^*(x) = x_{G_T}$. In more naive terms, $a(x) = (g \mapsto g.x)$ and $b(x) = (g \mapsto x)$. Now look at the fibre product defined by the diagram

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "b"] \\ \mathcal{M} \arrow[r, "a"] & \mathcal{H} \end{tikzcd} .$$

An object of $\mathcal{N}$ is a pair $(x, \psi^x\colon a(x) \simeq b(x))$, where ψ^x consists of isomorphisms $\psi^x_g\colon g.x \simeq x$. We define a closed substack $\mathcal{Z} \subset \mathcal{H}$ by considering the morphisms $\psi\colon G \to \mathcal{M}$ such that, for all sections $g, h \in G(T)$, we have $g(\psi_h) \circ \psi_g = \psi_{gh}$. The stack $\mathcal{M}^G$ is isomorphic to the preimage of $\mathcal{Z}$ in $\mathcal{N}$. The morphism $\varepsilon\colon \mathcal{M}^G \to \mathcal{M}$ is the first projection. Finally, it is not hard to check that $\mathcal{M}^G$ is lfp, using that this is the case for a as well as for $\mathcal{H}$ and its diagonal.

It remains to prove the properties of the morphism $\mathcal{M}^G \to \mathcal{M}$. First we look at the morphism $b\colon \mathcal{M} \to \mathcal{H}$. Let $U \to \mathcal{H}$ be a morphism corresponding to an object $\xi \in \mathcal{M}(G \times U)$. The fibre product $\mathcal{M} \times_{\mathcal{H}} U$ is the stack of triples (T, η, α) composed of a map of schemes $T \to U$, an object $\eta \in \mathcal{M}(T)$, and an isomorphism α between η_{G_T} and ξ_{G_T}. By fppf descent, this is none other than the functor of descent data for ξ with respect to the fppf covering $G_U \to U$. It is represented by a closed subalgebraic space of $\mathrm{Isom}_{G_U \times_U G_U}(\mathrm{pr}_1^* \xi, \mathrm{pr}_2^* \xi)$, and it inherits such properties as quasi-compactness and separatedness of the diagonal of $\mathcal{M}$. It follows that b has these properties, and similarly for $\mathcal{N}$ and $\mathcal{M}^G$. □

Example 3.8. The morphism $\mathcal{M}^G \to \mathcal{M}$ need not be a monomorphism of algebraic stacks, although it is a monomorphism of G-algebraic stacks (because of the 2-universal property). Let $\mathcal{M}_{g,2}$ be the stack of smooth 2-pointed curves of genus g; it has an action of the symmetric group $\mathfrak{S}_2$. Let (C, a, b) be a curve over a base S, and suppose that C has two distinct automorphisms σ_1 and σ_2 that exchange the marked points. Then these give two morphisms $S \to (\mathcal{M}_{g,2})^{\mathfrak{S}_2}$, and the compositions $S \to \mathcal{M}_{g,2}$ are equal as morphisms of algebraic stacks. However, they are not equal as morphisms of $\mathfrak{S}_2$-algebraic stacks because the maps σ_1, σ_2 enter in the definition of such a morphism.

Example 3.9. "Fixed points" and "coarse moduli space" do not commute. Assume that $\mathcal{M}$ and $\mathcal{M}^G$ admit coarse moduli spaces M and N. Then M acquires an action of G and there is a map $N \to M^G$. The case where the original action of G on $\mathcal{M}$ is trivial shows that N might be "bigger" than M^G. For a somewhat opposite example, let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. Its unique involution generates its center Z, and $G = Q/Z \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not isomorphic to a subgroup of Q. There is a faithful action of G on Q by conjugation, whence an action of G on BQ. Then $(BQ)^G$ is empty, whereas the moduli space of BQ is S and we have $S^G = S$ for the induced action.

Example 3.10 (continuation of 3.2). The following example (suggested to me by B. Toen) shows that $\mathcal{M}^G$ may not be algebraic when G is not proper. If H is a commutative positive-dimensional group scheme and if G is a group scheme acting trivially on BH, then an object of $(BH)^G$ is an H-torsor x together with a morphism $G \to \mathrm{Aut}(x) = H$, so $(BH)^G = BH \times \mathrm{Hom}(G, H)$. This stack is not algebraic in general, though it may be for special groups G, H (e.g., if both G and H are of multiplicative type; see [SGA3]).

Finally we mention a simple application of Theorem 3.3. Let G and $\mathcal{M}$ be as in the statement of the theorem, with G proper. Assume that $\mathcal{M}$ is endowed with the trivial action. Then one sees that $\mathcal{M}^G$ is the stack of "G-objects in $\mathcal{M}$"—that is, of objects $x \in \mathcal{M}(T)$ together with a group homomorphism $\rho\colon G \to \mathrm{Aut}_T(x)$. We denote by $\mathcal{M}\{G\}$ the full faithful subcategory corresponding to the pairs (x, ρ) such that ρ is a faithful action.

Corollary 3.11. *Let $\mathcal{M}$ be an algebraic stack with diagonal lfp over S, and let G be a proper group scheme that is essentially free and of finite presentation over*

S. Then the morphism $\mathcal{M}\{G\} \to \mathcal{M}^G$ defined previously is an open quasi-compact immersion, so $\mathcal{M}\{G\}$ is algebraic. In particular, if $\mathcal{M}$ is a smooth (*resp. proper*) *Deligne–Mumford stack and if G is a finite étale group such that its order is invertible in $\mathcal{O}_S$, then $\mathcal{M}\{G\}$ is a smooth* (*resp. proper*) *Deligne–Mumford stack.*

Proof. In order to prove that $\mathcal{M}\{G\} \to \mathcal{M}^G$ is an open quasi-compact immersion, we take $T \to \mathcal{M}^G$ a morphism from a scheme that corresponds to an object $(x, \rho\colon G \to \mathrm{Aut}_T(x))$. Under our assumptions, the kernel $K = \ker(\rho)$ is a proper T-group scheme of finite presentation. Then $T^0 := T \times_{\mathcal{M}^G} \mathcal{M}\{G\}$ is the functor $\mathrm{Isom}_{T\text{-gr}}(1_G, K)$, which has an open, quasi-compact immersion into the scheme $\mathrm{Hom}_{T\text{-gr}}(1_G, K) = T$. (In other words, T^0 is the locus of points in T such that the unit section $\{1\} \to K$ is an isomorphism; checking openness boils down to proving that, if a proper scheme over a discrete valuation ring R has a section to which it is equal on the special fibre, then it must be $\mathrm{Spec}(R)$.)

In the Deligne–Mumford case, if $\mathcal{M}$ is smooth then $\mathcal{M}^G$ and hence $\mathcal{M}\{G\}$ are also smooth (this is a property of reductive groups). If $\mathcal{M}$ is proper then the morphism $\mathcal{M}\{G\} \to \mathcal{M}^G$ is an open and closed embedding, because K is open and closed in G (since the intertia stack is unramified). Thus $\mathcal{M}\{G\}$ is proper. □

EXAMPLE 3.12. Let $g \geq 0$ be an integer and let G be a finite group. Then $\mathcal{H}_{g,G} := \mathcal{M}_g\{G\}$ is the stack of smooth genus-g curves with faithful G-action. It is a Deligne–Mumford stack over $\mathrm{Spec}(\mathbb{Z})$, smooth over $\mathrm{Spec}(\mathbb{Z}[1/|G|])$. The stack $\bar{\mathcal{M}}_g\{G\}$ is the stack of stable curves with action; it is proper over $\mathrm{Spec}(\mathbb{Z})$ and smooth over $\mathrm{Spec}(\mathbb{Z}[1/|G|])$.

4. Algebraicity of Quotients

Let G be a flat, separated group scheme of finite presentation over S. By a G-torsor over an S-scheme T we will mean an algebraic space with G-action $p\colon E \to T$ that locally on T is isomorphic to the trivial G-space $G \times T$. In general such a torsor will not be a scheme, unless (for example) G is quasi-affine.

Let $\mathcal{M}$ be a G-algebraic stack over S. In case $\mathcal{M} = X$ is an algebraic space, the quotient of Proposition 2.6 is known under the more familiar decription of the stack of G-torsors with an equivariant morphism to X. It is traditionally denoted $[X/G]$ to avoid confusion with a hypothetical *quotient algebraic space,* but when $\mathcal{M}$ is a general stack no such confusion is possible and so it is natural to omit the brackets.

For general $\mathcal{M}$ we can still define a stack whose objects are G-torsors $p\colon E \to T$ with an equivariant morphism $(f,\sigma)\colon E \to \mathcal{M}$. More precisely, we define a stack $(\mathcal{M}/G)^*$ whose sections over T are triples $t = (p, f, \sigma)$ as before, and the isomorphisms between t and t' in $(\mathcal{M}/G)^*$ are pairs (u, α) with a G-morphism $u\colon E \to E'$ and a 2-commutative diagram of G-stacks (see Remark 2.2):

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "{(f,\sigma)}"'] & \arrow[d, phantom, "\alpha"] & E' \arrow[dl, "{(f',\sigma')}"] \\ & \mathcal{M} & \end{tikzcd} .$$

THEOREM 4.1. *Let G be a flat, separated group scheme of finite presentation over S. Let $\mathcal{M}$ be a G-algebraic stack over S. Then the quotient stack $\mathcal{M}/G$ (Proposition 2.6) is isomorphic to the stack of G-torsors $(\mathcal{M}/G)^*$, and it is algebraic (so it is a quotient stack in $\mathfrak{AlgSt}$). The canonical morphism $\pi : \mathcal{M} \to \mathcal{M}/G$ is the universal torsor over $\mathcal{M}/G$. The formation of $\mathcal{M}/G$ commutes with base change on S.*

Proof. There are two things to show. First, we explain why $\mathcal{M}/G \simeq (\mathcal{M}/G)^*$. Let $\mathcal{M}/G$ be the quotient as described in Proposition 2.6, which is the stack associated to a prestack $\mathcal{P}$. We define a morphism $u\colon \mathcal{P} \to (\mathcal{M}/G)^*$ by sending an object $x \in \mathcal{M}(T)$, viewed as a map $x\colon T \to \mathcal{M}$, to the trivial torsor together with the map $G \times T \to \mathcal{M}$ given by $\mu \circ (\mathrm{id} \times x)$, which is clearly equivariant. The image of a morphism $(g, \varphi)\colon x \to y$ in $\mathcal{P}$ is the multiplication by g (as a map of torsors). This morphism u extends to a morphism of stacks $u'\colon \mathcal{M}/G \to (\mathcal{M}/G)^*$. It is clearly fully faithful and also locally essentially surjective, by the definition of a torsor. Hence it is an isomorphism of stacks. From now on we identify $\mathcal{M}/G$ and $(\mathcal{M}/G)^*$.

Second, we prove algebraicity. We keep our notation $t = (p, f, \sigma)$ for sections of $\mathcal{M}/G$ and $\varphi = (u, \alpha)$ for morphisms between t and t'. Note that there is a morphism $\omega\colon \mathcal{M}/G \to BG$ obtained by forgetting the maps to $\mathcal{M}$. To study the diagonal of $\mathcal{M}/G$, we take $t, t' \in (\mathcal{M}/G)(T)$; then ω induces a morphism $\omega_\circ\colon \mathrm{Isom}_T(t, t') \to \mathrm{Isom}_{BG}(E, E')$ given by $(u, \alpha) \mapsto u$. The target space is algebraic, and the fibre of $\omega_\circ$ over u is the closed (hence algebraic) subspace of $\mathrm{Isom}_{\mathcal{M}_T}(E, u^*E')$ of 2-G-isomorphisms. This shows that $\mathrm{Isom}_T(t, t')$ is representable, separated, and quasi-compact. The morphism $S \to BG$ is fppf and so, from the 2-Cartesian diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{M}/G \arrow[r] & BG \end{tikzcd},$$

we deduce that $\mathcal{M} \to \mathcal{M}/G$ is fppf. By composition with an atlas of $\mathcal{M}$ we obtain an fppf presentation of $\mathcal{M}/G$, whence the result by Artin's theorem [L-MB, Thm. 10.1]. □

REMARK 4.2. Let $\mathcal{M}$ be a G-algebraic stack as before, and let $x\colon T \to \mathcal{M}$ be a point. We define the *inertia subgroup* of x, as a sheaf on T, by $\mathrm{In}_T(x)(T') = \{(g, \alpha) \mid g \in G(T'), \alpha : g(x_{T'}) \simeq x_{T'}\}$ for all T'/T. Observe that $\mathrm{In}_T(x) \simeq (G \times T) \times_{\mathcal{M}} T$ and hence it is algebraic. If we are under the conditions of the theorem, so that a quotient $\pi : \mathcal{M} \to \mathcal{M}/G$ exists, then $\mathrm{In}_T(x) \simeq \mathrm{Aut}_T(\pi x)$ canonically. The quotient sheaf $\mathrm{St}_T(x) = \mathrm{In}_T(x)/\mathrm{Aut}_T(x)$, which is not algebraic a priori, measures the default of the action to be free. Locally this sheaf is just the set of sections of G such that $gx \simeq x$ (so we might call it the *stabilizer* of x).

EXAMPLE 4.3. Let $\mathcal{M}$ be a G-algebraic stack over S, so we have morphisms $G \times \mathcal{M} \overset{\mu,\, \mathrm{pr}_2}{\rightrightarrows} \mathcal{M}$. Given a sheaf F on the smooth-étale site of $\mathcal{M}$, a G-linearization of F is an isomorphism $\alpha\colon \mu^*F \simeq \mathrm{pr}_2^*F$, which is compatible with associativity: $(m \times \mathrm{id}_{\mathcal{M}})^*\alpha = (\mathrm{id}_G \times \mu)^*\alpha$. We define a (smooth-étale) *G-sheaf* on $\mathcal{M}$ to be a pair (F,α) as just described. We can look at the stack of invertible G-sheaves (with obvious isomorphisms of G-sheaves between them), denoted $\mathcal{P}ic^G(\mathcal{M})$, and it is easy to see that we have canonical isomorphisms of stacks $\mathcal{P}ic(\mathcal{M}/G) \simeq \mathcal{P}ic^G(\mathcal{M}) \simeq \mathcal{P}ic(\mathcal{M})^G$. In particular, if $\mathcal{P}ic(\mathcal{M})$ is algebraic and if G is proper, essentially free, and of finite presentation, then by Theorem 3.3 we obtain algebraicity of the first two stacks. $\square$

5. Rigidifications

To conclude, we examine how group actions behave with respect to *rigidifications*—the case when there is a fixed flat group lying inside all automorphism groups of objects. Then, in some sense this group can be removed so as to obtain an algebraic stack $\mathcal{M} /\!\!\!/ H$ called the *rigidification of* $\mathcal{M}$ *along* H, following [ACV]. (Notational remark: [ACV] write $\mathcal{M}^H$, but we have already used this notation for fixed points, as is natural. Since rigidification is a kind of 2-quotient, the double bar // would be properly suggestive yet confusing, because it is already used in algebraic geometry to denote a GIT quotient and in topology to denote a homotopy quotient. The `\fatslash` symbol "$/\!\!\!/$" seemed appropriate.)

More precisely, let $\mathcal{M}$ be an algebraic stack and let H be a group scheme that is flat, separated, and of finite presentation over the base. We assume that, for any object $x \in \mathcal{M}(T)$, there is an injective morphism $i_x \colon H_T \hookrightarrow \mathrm{Aut}_T(x)$ whose formation is compatible with base change. In this situation, for any $x, y \in \mathcal{M}(T)$ the group H_T acts on the left and right on the sheaf $\mathrm{Hom}_T(x, y)$ by the rule $h_1.u.h_2 := i_y(h_1) \circ u \circ i_x(h_2)$. We assume (with self-explanatory notation) that, for any T'/T,

$$\text{for any } h \in H(T') \text{ and } u \in \mathrm{Hom}(x_{T'}, y_{T'}),\ u^{-1}hu \in H(T'), \tag{3}$$

and we say that H *is normal in the sheaves* $\mathrm{Hom}_{\mathcal{M}}(x, y)$.

THEOREM 5.1. *With the foregoing assumptions, there exist a stack* $\mathcal{M} /\!\!\!/ H$ *and a smooth surjective morphism of finite presentation* $f \colon \mathcal{M} \to \mathcal{M} /\!\!\!/ H$ *whose formation commutes with base change and such that the following statements hold.*

(i) *Via* f, *elements of* $H \subset \mathrm{Aut}_T(x)$ *map to the identity, and* f *is universal for this property.*
(ii) f *is a gerbe*; *if* $\mathcal{M}$ *is Deligne–Mumford, then* $\mathcal{M} /\!\!\!/ H$ *also is and then* f *is étale.*
(iii) *If* $\mathcal{M}$ *is separated or proper, then* $\mathcal{M} /\!\!\!/ H$ *also is.*
(iv) $\mathcal{M} /\!\!\!/ H$ *has a coarse moduli space if and only if* $\mathcal{M}$ *has one* (*then they are the same*).

Now let G be a flat, separated group scheme of finite presentation. Assume, moreover, that $\mathcal{M}$ *is endowed with an action of G such that the subgroups* $H_T \hookrightarrow \mathrm{Aut}_T(x)$ *are stable. Then*:

(v) *the injections* $j_x \colon H_T \hookrightarrow \mathrm{Aut}_{\mathcal{M}_T}(x) \hookrightarrow \mathrm{Aut}_{(\mathcal{M}/G)_T}(x)$ *make* H *normal in the sheaves* $\mathrm{Hom}_{\mathcal{M}/G}(x, y)$;
(vi) *the action of G on* $\mathcal{M}$ *induces an action on* $\mathcal{M} /\!\!\!/ H$;

and we have a canonical isomorphism $\iota \colon (\mathcal{M}/G) /\!\!\!/ H \xrightarrow{\sim} (\mathcal{M} /\!\!\!/ H)/G$.

Proof. Parts (i)–(iv) are proved in [ACV, 5.1.5] (with complements in [R, I, Sec. 3] for the normality assumption on H). For (v) one need only express the group law of the sheaves $\mathrm{Aut}_{(\mathcal{M}/G)_T}(x)$ using the description via the inertia sheaf $\mathrm{In}_T(x)$ of Remark 4.2. Part (vi) is straightforward. It remains to check the final statement: from the universal property (i), we obtain morphisms in both ways that are obviously inverse to each other. □

REMARKS 5.2. (i) If k is an algebraically closed field, then the k-points of $\mathcal{M}/\!\!/H$ are the same as those of $\mathcal{M}$ and, for one such point x, we have $\mathrm{Aut}_{\mathcal{M}/\!\!/H}(f(x)) = \mathrm{Aut}_{\mathcal{M}}(x)/H$.

(ii) As an example it is clear that all objects of the stack $\mathcal{M}\{G\}$ of Corollary 3.11 have the center $Z = Z(G)$ in their automorphism group, so we can rigidify and get $\mathcal{M}\{G\}/\!\!/Z$.

Rigidification and fixed points do not commute in general: because the map $\varepsilon\colon \mathcal{M}^G \to \mathcal{M}$ is representable, it simply doesn't make sense in general to try to form something such as $\mathcal{M}^G/\!\!/H$. However, in the example of a stack of the form $\mathcal{M}\{G\}$ as in (ii), we may obtain such a result as follows.

PROPOSITION 5.3. *Let $\mathcal{M}$ be an algebraic stack with diagonal lfp over S. Let G be a finite (ordinary) group and let H be a G-module that is a finite abelian group. Then G acts on $\mathcal{M}\{H\}$ by twisting the actions, namely by the rule $g.(x,\rho) = (x, \rho \circ g^{-1})$. Let $\mathcal{N}$ be any algebraic substack of $\mathcal{M}\{H\}$ that is stable under G. Then there is a morphism $\iota\colon \mathcal{N}^G/\!\!/H^G \to (\mathcal{N}/\!\!/H)^G$, and:*

(i) *if $H^1(G,H) = 0$ then it is a monomorphism;*
(ii) *if $H^2(G,H) = 0$ then it is an epimorphism.*

(*The cohomology here is ordinary group cohomology.*)

Proof. Given T/S, a section of $\mathcal{N}^G(T)$ is a triple $\mathsf{x} = (x, \rho, \{\alpha_g\}_{g\in G})$. Here ρ is a faithful action of H on x and the $\alpha_g\colon x \simeq x$ satisfy $\alpha_g \circ h = g^{-1}(h) \circ \alpha_g$; the cocycle condition reduces to $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$. A morphism $\mathsf{x} \to \mathsf{x}'$ is an H-equivariant map $u\colon x \to x'$ that commutes with the α_g and α'_g; in particular, elements of H^G give automorphisms of the objects of $\mathcal{N}^G$, and this gives the existence of ι.

To check (i) we may work locally over the base T, so that two sections x, x' of $\mathcal{N}^G/\!\!/H^G$ are as just described; we must then check that the map from

$$\mathrm{Hom}_{\mathcal{N}^G/\!\!/H^G}(\mathsf{x},\mathsf{x}') = \{u\colon x \to x' \text{ such that } u \circ \alpha_g = \alpha'_g \circ u \ (\forall g)\}/H^G$$

to

$$\mathrm{Hom}_{(\mathcal{N}/\!\!/H)^G}(\iota(\mathsf{x}),\iota(\mathsf{x}')) = \{\bar u = u \bmod H, \text{ for } u\colon x \to x', \\ \text{such that } \bar u \circ \bar\alpha_g = \bar\alpha'_g \circ \bar u \ (\forall g)\}$$

is bijective. It is injective because, if u, v have the same image (i.e., $v = hu$ for some $h \in H$), then $v\alpha_g = hu\alpha_g = h\alpha'_g u = \alpha'_g g^{-1}(h)u$; on the other hand, it is also equal to $\alpha'_g v = \alpha'_g hu$. It follows that $g^{-1}(h) = h$ for all $g \in G$, so $h \in H^G$. It is surjective because, given $\bar u$ the class of u modulo H, there exists an $h_g \in H$ such that $u\alpha_g = h_g\alpha'_g u$. A straightforward computation gives that h_g is a 1-cocyle. By the assumption $H^1(G, H) = 0$ there is a $k \in H$ such that $h_g = g(k)k^{-1}$, and then $u^* = ku$ lifts $\bar{u}$; that is, $u^* \circ \alpha_g = \alpha'_g \circ u^*$.

To check (ii) we observe that, locally on T, an object of $(\mathcal{N}/\!\!/H)^G$ is a triple $(x, \rho, \bar{\alpha}_g)$, where $\bar{\alpha}_g$ is the class of an isomorphism $\alpha_g\colon (x, \rho \circ g^{-1}) \to (x, \rho)$ and satisfies $\bar{\alpha}_{g_1g_2} = \bar{\alpha}_{g_1} \circ \bar{\alpha}_{g_2}$. This means there exist elements h_{g_1,g_2} such that $\alpha_{g_1g_2} = h_{g_1,g_2}\alpha_{g_1}\alpha_{g_2}$. Computing $\alpha_{g_1g_2g_3}$ in two different ways, one checks that $h = \{h_{g_1,g_2}\}$ is a 2-cocycle. Since $H^2(G, H) = 0$ by assumption, we have that h is a coboundary; that is, it has the form $h_{g_1,g_2} = k_{g_1}g_1(k_{g_2})k^{-1}_{g_1g_2}$ for some collection $\{k_g\}_{g\in G} \in H$. It follows that we can consider $\alpha^*_g = k_g\alpha_g$, which satisfies $\alpha^*_{g_1g_2} = \alpha^*_{g_1}\alpha^*_{g_2}$. Then $\mathsf{x} = (x, \rho, \alpha^*_g)$ defines a preimage in $\mathcal{N}^G/\!\!/H^G$. $\square$

PART B. APPLICATIONS: MODULI FOR COVERS OF CURVES

Here we give applications of the machinery of Part A to simple constructions of moduli stacks. We build on a first construction that is a smooth compactification for the stack of curves with level structures. Then we use fixed points and quotients of various finite group actions in order to obtain the stack $\mathcal{H}_{g,G} = \mathcal{M}_g\{G\}$ (see Example 3.12) as a quotient stack.

6. Preliminaries on Hurwitz Theory of Tame Covers

Here we review briefly the construction of Hurwitz stacks of Galois covers. The theory aims at providing a natural compactification of the stack of tame covers of smooth curves $\mathcal{H}_{g,G} \otimes \mathbb{Z}[1/|G|]$, as defined in Example 3.12. We wish to present just the material necessary to state Theorem 6.5, which will be used in Section 7.

Here, in more detail, are the relevant definitions concerning curves and actions. We fix an integer $g \geq 2$, a finite group G, and a scheme S. Throughout this section, we will always assume that $|G| \in \mathcal{O}_S^\times$. By a *curve over S* we mean a proper flat morphism of finite presentation $f\colon C \to S$ with 1-dimensional reduced and connected fibres. By a *G-curve over S* we mean a curve together with a fibrewise faithful action $\rho\colon G \to \mathrm{Aut}_S(C)$. The map f and the action ρ are often understood and we simply write C. Finally, we say that a curve or a G-curve is *smooth* (resp. *nodal*) when the geometric fibres of $C \to S$ are smooth (resp., have only ordinary double point singularities).

DEFINITION 6.1. Let C be a nodal G-curve over S. For $x \in C$ we denote by G_x its stabilizer. Assume that S is the spectrum of an algebraically closed field. We say that the action is *admissible* if, for every node $x \in C$ and every $g \in G_x$, we have $\det(g) = 1$ if g respects the branches, and -1 otherwise (the determinant is computed with respect to the natural representation of G_x in the 2-dimensional cotangent space T_x^*). For arbitrary S, we say that the action is admissible if it is admissible on each geometric fibre of $C \to S$.

When the base is an algebraically closed field, it has been shown [BeR; E] that C has a universal equivariant deformation with smooth generic fibre if and only if the action of G is admissible. Thus, the obstructions to the possibility of deforming the pair (C,ρ) smoothly are all localized at the nodes. When the action is admissible, we can distinguish between two kinds of fixed points as follows.

- A fixed point x is *of cyclic type* if (a) it is smooth or (b) it is a node and no element of G_x permutes the branches at x. Then G_x is isomorphic to the cyclic group $\mathbb{Z}/e\mathbb{Z}$.
- A fixed point is *of dihedral type* if it is a node and one element of G_x permutes the branches at x. Then G_x is isomorphic to the group $\mathbb{D}_e = \langle a, b \mid a^2 = (ab)^2 = b^e = 1\rangle$ of order $2e$. (This is the dihedral group for $e \geq 2$, while $\mathbb{D}_1 \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{D}_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.)

Let S be arbitrary again, and let $\pi : C \to C/G$ be the quotient map. It is classical that the assumption $|G| \in \mathcal{O}_S^\times$ implies that formation of π commutes with base change. As usual, we denote by ω the dualizing sheaves.

Definition 6.2. The *ramification divisor* is defined by $R = \mathrm{Div}(\mathcal{O}_R)$, where $\mathcal{O}_R = \omega_C^{-1} \otimes \mathcal{C}$ with $\mathcal{C} := \mathrm{coker}(\pi^*\omega_{C/G} \to \omega_C)$ (see [GIT, Chap. 5, Sec. 3] for the operation Div).

The ramification divisor is flat on the base, and its formation commutes with base change. Locally it has an equation given by the determinant of the map $\pi^*\omega_{C/G} \to \omega_C$. A local computation shows that its support $|R|$ is the set of fixed points of the action, excluding nodal points of cyclic type. For instance, look at the family $xy = t$ over the base $\mathbb{C}[[t]]$. Let $\mathbb{Z}/n\mathbb{Z}$ act by $\sigma(x,y) = (e^{2i\pi/n}x, e^{-2i\pi/n}y)$. The fixed point scheme is not flat; it has degree 2 on the generic fibre and degree 3 on the special fibre. If we remove the nodal fixed point from the special fibre, the resulting scheme is flat—it is the support of R.

Definitions 6.3. (i) A *nodal r-marked curve* over a base S is a tuple $(C, x_1, \dots, x_r)$, where C is a nodal curve and the $x_i : S \to C$ are disjoint sections that land in the smooth locus of C. We often denote $\underline{x} := (x_1, \dots, x_r)$ and call the relative divisor $\sum x_i$ the *unordered mark*. The morphisms between $(C, \underline{x})$ and $(C', \underline{x}')$ are morphisms of S-curves $u : C \to C'$ such that $ux_i = x_i'$ for all i. We say that $(C, \underline{x})$ is *stable* if the automorphism groups of the geometric fibres are finite.

(ii) An *admissible G-curve r-marked by ramification* is a triple $(C, \rho, \underline{x})$, where $(C, \underline{x})$ is a stable marked curve and ρ is a fibrewise faithful admissible action on C such that the unordered mark and the ramification divisor are equal. A morphism between such objects is a morphism of marked curves that is G-equivariant. Given an integer $g \geq 2$, we denote by $\bar{\mathcal{I}}_{g,G,r}$ (resp. $\mathcal{I}_{g,G,r}$) the stack of admissible (resp. smooth) G-curves, r-marked by ramification, whose fibres all have arithmetic genus g.

Of course, when C is a smooth G-curve over an algebraically closed field, the conditions on $\sum x_i = R$ in Definition 6.3(i) are empty, so it brings nothing new to look at the G-curve marked by ramification or to look at the G-curve alone (except that one must choose an ordering of the ramification points). But when C is singular, this point of view is very convenient:

- it forbids fixed points of dihedral type (but there may be nodes of cyclic type), and so
- the nodes of C must then lie above nodes of C/G;
- the ramification divisor $|R|$ is then étale.

DEFINITION 6.4. Let $(C, \rho, \underline{x})$ be an admissible G-curve marked by ramification over a base S. Then the isomorphism class of the representation of G in the cotangent space of the unordered marked sections $\bigoplus_{1\leq i\leq r} T^*_{x_i}$ is called the *ramification datum*. We denote it by $\xi = \xi(C, \rho, \underline{x})$.

Observe that $\bigoplus T^*_{x_i}$ is a free $\mathcal{O}_S$-module of rank r. One can show that ξ is locally constant on the base; that is, there is an integral representation V (a projective module of finite type over $\mathbb{Z}[1/|G|]$) such that $\xi \simeq V \otimes \mathcal{O}_S$ as representations. Hence we have splittings into open and closed substacks,

$$\mathcal{H}_{g,G}[1/|G|] = \coprod \mathcal{H}_{g,G,\xi}, \quad \mathcal{I}_{g,G,r} = \coprod \mathcal{I}_{g,G,r,\xi}, \quad \bar{\mathcal{I}}_{g,G,r} = \coprod \bar{\mathcal{I}}_{g,G,r,\xi}.$$

In fact, ξ is clearly induced (in the sense of representations) by the list of pairs (G_x, χ_x), where x ranges through a set of representatives $\{x_{i_1}, \dots, x_{i_b}\}$ of the G-orbits in $\underline{x}$ and where χ_x is the character of the representation of G_x on T^*_x. One can equivalently define ξ to be this list, as in [BeR].

For a G-curve C, arbitrary automorphisms of C may permute the ramification points. More precisely, we can consider the group $\mathfrak{S}_r^\xi$ of permutations of the r ramification points that preserve the ramification datum ξ. Then there is a natural map $\mathcal{I}_{g,G,r,\xi}/\mathfrak{S}_r^\xi \xrightarrow{\sim} \mathcal{H}_{g,G,\xi}$ that is an isomorphism (see Theorem 4.1 for the procedure of quotient). Therefore, we define $\bar{\mathcal{H}}_{g,G,\xi} = \bar{\mathcal{I}}_{g,G,r,\xi}/\mathfrak{S}_r^\xi$. Note that ξ determines two invariants: the genus g' of the quotient curve C/G (by the Riemann–Hurwitz formula); and the number b of G-orbits in $\underline{x}$. The following statement summarizes and completes our discussion.

THEOREM 6.5 [BeR]. *The stack $\bar{\mathcal{H}}_{g,G,\xi}$ is a smooth proper stack over the ring $\mathbb{Z}[1/|G|]$ and is equidimensional of dimension $3g' - 3 + b$. The open substack $\mathcal{H}_{g,G,\xi}$ is fibrewise dense.*

We may view $\bar{\mathcal{H}}_{g,G,\xi}$ as a closed substack of $\bar{\mathcal{M}}_g\{G\} \otimes \mathbb{Z}[1/|G|]$, defined in Example 3.12. Therefore, we recover most assertions of the theorem as formal consequences of Corollary 3.11 (and the dimension may be computed by looking at the map $\bar{\mathcal{H}}_{g,G,\xi} \to \bar{\mathcal{M}}_{g',b}$ taking $(C, \rho, \underline{x})$ to $C/G +$ [branch locus]).

7. Level Structures on Curves

In this section we define a stack of stable curves with level structure. It will be used and studied further in the sequel (Section 8). We first recall briefly the setting from [DMu], including the normal stack $\tilde{\mathcal{M}}_g(G)$. Then we define the proper stack $\bar{\mathcal{M}}_g(G)$ and show that it desingularizes $\tilde{\mathcal{M}}_g(G)$ (Theorem 7.2.3 is a little more precise). At last we prove that stable curves admit level structures locally for the fppf topology (Corollary 7.2.4).

Fix an integer $g \geq 2$ and denote by Π the fundamental group of a compact Riemann surface of genus g. Choose a finite group G such that there exists a surjection $\Pi \to G$, and put $n := |G|$ its order.

7.1. Level Structures on Smooth Curves

Deligne and Mumford defined a stack $\mathcal{M}_g(G)$ over $\mathbb{Z}[1/n]$, parameterizing smooth curves C/S of genus g together with a *Teichmüller structure* of level G. We briefly recall that such a structure is given by a global section of the sheaf of exterior surjective group homomorphisms from $\pi_1(C, *)$ to G, denoted $\mathcal{H}om^{\text{ext}}(\pi_1(C), G)$. We refer to [DMu, (5.5)] for the details. Note that the base point $*$ can be chosen arbitrarily because we "mod out" by inner automorphisms of π_1. When $G = (\mathbb{Z}/n\mathbb{Z})^{2g}$ we recover the stack of curves with full level-n structure, usually denoted $\mathcal{M}_g(n)$. Indeed, a surjective morphism $\pi_1(C, *) \to (\mathbb{Z}/n\mathbb{Z})^{2g}$ determines a torsor over C (up to isomorphism), and this in turn gives a basis for $H^1(C, \mathbb{Z}/n\mathbb{Z})$, which is a full level-$n$ structure in the classical sense.

As a matter of notation, we will use the letter L and its variants to denote level structures, as in $\ell\colon \pi_1(C, *) \to G$ or $(L \to C, \lambda\colon G \to \mathrm{Aut}(L))$ for the G-torsor it determines.

Deligne and Mumford also considered the normalization of $\bar{\mathcal{M}}_g$ with respect to the forgetful morphism $\mathcal{M}_g(G) \to \mathcal{M}_g$ (see [DMu]). We denote it by $\tilde{\mathcal{M}}_g(G)$ and recall the following theorem.

THEOREM 7.1.1 [DMu]. *The $\mathbb{Z}[1/n]$-stacks $\mathcal{M}_g(G)$ and $\tilde{\mathcal{M}}_g(G)$ are algebraic, separated, and of finite type. The stack $\tilde{\mathcal{M}}_g(G)$ is proper. It contains $\mathcal{M}_g(G)$ as a smooth, fibrewise dense substack.*

In general $\tilde{\mathcal{M}}_g(G)$ is only normal, but some special cases where it is smooth have been studied in [PdJ] (see also proof of Corollary 7.2.4 to follow). As a complement to the theorem, Deligne extended Serre's lemma to stable curves and proved that, if there exists a surjection $G \to (\mathbb{Z}/m\mathbb{Z})^{2g}$ with $m \geq 3$, then both stacks are representable [D, Lemma 3.5.1].

7.2. Level Structures on Stable Curves

We now derive from Hurwitz theory a stack that should be viewed as a resolution of singularities of $\tilde{\mathcal{M}}_g(G)$. We note that in [ACV] a more general and sophisticated theory of twisted maps into a Deligne–Mumford stack is developed. As far as level structures are concerned, our stack $\bar{\mathcal{M}}_g(G)$ and their stack $\mathcal{B}_g^{\text{tei}}(G)$ (twisted stable maps of degree 0 into BG) are isomorphic; see [ACV, Sec. 5.2].

The data g, G, n remain as before. The genus of a Galois étale cover of a curve of genus g, with Galois group equal to G, is equal to $n(g-1)+1$. We denote this number by γ. We refer to Section 6 for the presentation of the Hurwitz stack with no ramification ($\xi = \emptyset$) and to Section 5 for the operation of rigidification (denoted by a $/\!\!/$ sign). Observe that the center $Z(G) \subset G$ is present in all automorphism groups of curves with G action, so we may rigidify the Hurwitz stack along $Z(G)$ (see Corollary 3.11 and Remark 5.2(ii)).

DEFINITION 7.2.1. We define the stack of *stable curves with (Teichmüller) structure of level G* by

$$\bar{\mathcal{M}}_g(G) := \bar{\mathcal{H}}_{\gamma,G,\emptyset} /\!\!/ Z(G).$$

By Theorem 5.1(ii) and (iii), $\bar{\mathcal{M}}_g(G)$ is a smooth and proper stack over $\mathbb{Z}[1/n]$. We recall from Remark 5.2(i) that, for any algebraically closed field k, the objects of $\bar{\mathcal{M}}_g(G)$ over k are the same as those of $\bar{\mathcal{H}}_{\gamma,G,\emptyset}$, but automorphism groups are divided by $Z(G)$. On the open substack of smooth curves we have the following result.

PROPOSITION 7.2.2. *There is an isomorphism of* $\mathbb{Z}[1/n]$*-stacks* $\mathcal{H}_{\gamma,G,\emptyset} /\!\!/ Z(G) \xrightarrow{\sim} \mathcal{M}_g(G)$.

Proof. An object of $\mathcal{H}_{\gamma,G,\emptyset}$ is a smooth curve L/S with fixed point–free action of the group G. We define a morphism $\mathcal{H}_{\gamma,G,\emptyset} \to \mathcal{M}_g(G)$ by sending L to the base $C = L/G$ of the cover, together with the exterior morphism $\ell\colon \pi_1(C,*) \to G$ that this cover determines by the theory of the fundamental group (we need to have sections of C, which we have locally on the base). This clearly gives a morphism as announced by the universal property of rigidification (see Theorem 5.1).

Given a curve C/S of genus g and with a Teichmüller structure ℓ, what we have just said shows that, étale locally on S, this provides a Galois étale cover L of group G. So our morphism is essentially surjective.

It remains to check that it is also a monomorphism. Toward this end it is enough to look at a base equal to a field $S = \mathrm{Spec}(k)$, to take L, L' two k-curves with action, and to show that the map $\mathrm{Hom}_G(L, L')/Z(G) \to \mathrm{Hom}_k((C,\ell),(C',\ell'))$ is bijective. Injectivity comes from Galois theory of smooth curves, which asserts that the group of automorphisms of L that induce the identity on C is just G. Surjectivity is straightforward from the definitions. $\square$

We now come to the main point.

THEOREM 7.2.3. *There is a proper, birational morphism* $\bar{\mathcal{M}}_g(G) \to \tilde{\mathcal{M}}_g(G)$ *that is an isomorphism on* $\mathcal{U} = \mathcal{M}_g(G)$. *The two stacks have the same coarse moduli spaces; in particular, when* $\tilde{\mathcal{M}}_g(G)$ *is representable, it is the moduli space of* $\bar{\mathcal{M}}_g(G)$.

Thus, in some sense, $\bar{\mathcal{M}}_g(G)$ is a desingularization of $\tilde{\mathcal{M}}_g(G)$. However, $\tilde{\mathcal{M}}_g(G)$ may be smooth without being isomorphic to $\bar{\mathcal{M}}_g(G)$: this is what happens with the example of Pikaart and de Jong [PdJ].

Proof. Consider the stack $\mathcal{X}$ defined by the following fibred product:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "w"] \arrow[d, "v"] & \tilde{\mathcal{M}}_g(G) \arrow[d, "u"] \\ \bar{\mathcal{M}}_g(G) \arrow[r] & \bar{\mathcal{M}}_g \end{tikzcd} .$$

The stacks $\bar{\mathcal{M}}_g(G)$ and $\tilde{\mathcal{M}}_g(G)$ share the common open substack $\mathcal{U} = \mathcal{M}_g(G)$, which therefore embeds diagonally in $\mathcal{X}$. Let $\mathcal{Y}$ be the reduced closed substack structure on the closure of the image of $\mathcal{U}$ in $\mathcal{X}$. The restriction $v|_{\mathcal{Y}} : \mathcal{Y} \to \bar{\mathcal{M}}_g(G)$ is (representable) finite because it is the composition of a closed immersion with the pullback of u. Also, it is birational. It follows that v is an isomorphism: indeed, we can work étale locally on $\bar{\mathcal{M}}_g(G)$, which brings us back to the case of a morphism of schemes, and then Zariski's main theorem [EGA, (IV), (8.12.10)] yields the result. Therefore, $\pi = w \circ (v|_{\mathcal{Y}})^{-1}$ is a morphism as announced.

We now show that the two stacks have the same moduli space. Let $\bar{M}$ and $\tilde{M}$ be the respective moduli spaces, which exist by [KeMo]. The induced map $\bar{M} \to \tilde{M}$ is again finite and birational, landing in a normal space. By Zariski's main theorem, this is an isomorphism. □

The stack $\tilde{\mathcal{M}}_g(G)$ is not smooth (in general) and not modular (i.e., one can hardly say what its objects stand for). Therefore, $\bar{\mathcal{M}}_g(G)$ corrects these deficiencies. We should emphasize that the map $\bar{\mathcal{M}}_g(G) \to \bar{\mathcal{M}}_g[1/n]$ is *not* representable, whereas the map $\tilde{\mathcal{M}}_g(G) \to \bar{\mathcal{M}}_g[1/n]$ is. Usually this is a minor inconvenience; for instance, the problem is easily overcome in the following result of essential interest.

COROLLARY 7.2.4. *Let C/S be a stable curve of genus $g \geq 2$. Assume that $n = |G| \geq 3$ (with $n \in \mathcal{O}_S^\times$ as always). Then C has a level-G structure after a finite faithfully flat extension $S' \to S$.*

Proof. We first provide a finite flat cover of $\bar{\mathcal{M}}_\gamma[1/n]$ by a scheme ($\gamma = n(g-1)+1$). For a well-chosen group Γ, the stack $\tilde{\mathcal{M}}_\gamma(\Gamma)[1/n]$ will answer the question. Let $\Gamma = \pi/\pi^{(k)}\pi^m$ be the quotient of π by the characteristic subgroup generated by $(k+1)$th commutators and mth powers; see [PdJ]. We must ensure that the order of Γ is invertible on $\mathrm{Spec}(\mathbb{Z}[1/n])$. Hence, if n is a power of 2 then we choose $(k,m) = (3,4)$, and if not then we pick an odd prime factor p of n and choose $(k,m) = (3,p)$. By [PdJ, Thm. 3.11], the normalization $\tilde{\mathcal{M}}_\gamma(\Gamma)[1/n]$ is a smooth proper scheme and so the covering map to $\bar{\mathcal{M}}_\gamma[1/n]$ is finite and flat.

By pullback along $\bar{\mathcal{H}}_{\gamma,G,\emptyset} \to \bar{\mathcal{M}}_\gamma[1/n]$, this provides a scheme U with a finite flat cover to both stacks $\bar{\mathcal{H}}_{\gamma,G,\emptyset}$ and $\bar{\mathcal{M}}_g(G)$. Now the curve C gives a morphism $S \to \bar{\mathcal{M}}_g[1/n]$. Consider the following diagram of fibred products:

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] \arrow[d] & S \arrow[d] \\ U \arrow[r] & \bar{\mathcal{M}}_g(G) \arrow[r] & \bar{\mathcal{M}}_g\left[\frac{1}{n}\right] \end{tikzcd}.$$

The stack $\mathcal{V}$ is not representable, whence the need for U. Clearly $S' \to S$ is finite and flat. □

7.2.5. Let us finally describe functoriality with respect to the group G. Let $N \lhd G$ be a normal subgroup and let $G' = G/N$. Let $n' = |G'|$ and $\gamma' = n'(g-1)+1$. For classical level structures, there is a map $\mathcal{M}_g(G) \to \mathcal{M}_g(G')[1/n]$ given by the composition $\pi_1(C) \to G \to G'$. The analogue for our compactifications proceeds as follows. Start from

$$\begin{aligned} \bar{\mathcal{H}}_{\gamma,G,\emptyset} &\longrightarrow \bar{\mathcal{H}}_{\gamma',G',\emptyset}, \\ (L,\lambda) &\longmapsto (L/N, \lambda/N), \end{aligned}$$

where $\lambda/N\colon G/N \to \mathrm{Aut}(L/N)$ is the induced action. Since the center $Z(G)$ maps to the center $Z(G')$, we deduce a morphism (after rigidifications) $\bar{\mathcal{M}}_g(G) \to \bar{\mathcal{M}}_g(G')[1/n]$. This map is proper, quasi-finite, and flat. It is ramified on the boundary and étale on the locus of smooth curves.

NOTATION 7.2.6. To conform with tradition (see Section 7.1), whenever $G = (\mathbb{Z}/n\mathbb{Z})^{2g}$ we use $\bar{\mathcal{M}}_g(n)$ to denote $\bar{\mathcal{M}}_g(G)$.

8. A Presentation of $\mathcal{H}_{g,G}$ Using Level Structures

In this section we wish to show how the tools developed in the previous sections relate to the stack $\mathcal{H}_{g,G} = \mathcal{M}_g\{G\}$ (see Example 3.12). Note that here we include the prime divisors of $|G|$ among the characteristics (this will be made more precise in the sequel). The first step is to get to the definition of certain stacks over $\mathbb{Z}[1/n]$, denoted $\mathcal{X}^n_{g,G}$ and $\bar{\mathcal{X}}^n_{g,G}$. Then we show that there is an isomorphism $\mathcal{X}^n_{g,G} \simeq \mathcal{H}_{g,G}[1/n]$ that gives the desired presentation (Theorem 8.2.2).

From now until the end, we fix a finite group G and an integer $n \geq 3$ prime to $|G|$. In particular, we must warn the reader that now n is *not* the order of G. The reason is that G will come up as a group acting on curves of genus g but no longer as a level structure. The integer n will come up as the abelian level when we will use the smooth stack $\bar{\mathcal{M}}_g(n)$ defined in Section 7.

8.1. G-Module Structures on the n-Torsion

Assume that G acts faithfully on a curve C of genus $g \geq 2$ that is defined over an algebraically closed field of characteristic prime to n. Then it acts (faithfully, by Serre's lemma) on the n-torsion of the Jacobian of the curve $C[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$. We denote by H_n the abstract group $(\mathbb{Z}/n\mathbb{Z})^{2g}$. A G-module structure on H_n is just a group homomorphism $i\colon G \to \mathrm{Aut}(H_n) = \mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$, and we say that H_n is a *faithful* module if i is injective.

NOTATION 8.1.1. We introduce the following finite set:

$$\mathcal{I}_n = \{\text{faithful } G\text{-module structures on } H_n \text{ such that, for all divisors } m|n \\ \text{with } m \geq 3, \text{ the induced } G\text{-module } H_m = H_n \otimes \mathbb{Z}/m\mathbb{Z} \text{ is faithful}\}.$$

It is clearly enough to check the condition for prime divisors m and $m = 4$. By its very definition, $\mathcal{I}$ is a functor from $\mathbb{Z}_{\geq 3}$, anti-ordered by divisibility, to the category of finite sets. Namely, for $m|n$ with $m \geq 3$, we have a morphism $\mathcal{I}_n \to \mathcal{I}_m$ given by $i \mapsto i \otimes \mathbb{Z}/m\mathbb{Z}$.

We remark that if n is a power of a prime $\ell \geq 3$, then the kernel of the reduction map $\mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ is an ℓ-group. Hence, in that case the condition in the definition of $\mathcal{I}_n$ is automatically verified, because G can't meet this kernel ($|G|$ and n are coprime).

8.1.2. The stack of stable curves of genus g with level-n structure has been defined in 7.2.1 by rigidification of the stack $\bar{\mathcal{H}}_{\gamma,H,\emptyset}$ with $H = (\mathbb{Z}/n\mathbb{Z})^{2g}$ and $\gamma = n^{2g}(g-1)+1$. We fix $i \in \mathcal{I}_n$ as introduced previously. Then there is an action μ_i of G on $\bar{\mathcal{H}}_{\gamma,H,\emptyset}$ (see Definition 2.1 for the relevant concepts). It goes as follows: for any $g \in G$,

(a) $g.(L,\lambda) := (L, g.\lambda)$ for an H-curve (L,λ), where $g.\lambda = \lambda \circ g^{-1}$, and
(b) $g.u := u$ for a map $u\colon (L,\lambda) \to (L',\lambda')$.

We use simply $\bar{\mathcal{H}}^i_{\gamma,H,\emptyset}$ to denote the G-stack $(\bar{\mathcal{H}}_{\gamma,H,\emptyset}, \mu_i)$. Now let us describe the fixed point stack (see Proposition 2.5 for a general description; see Theorem 3.3 and Proposition 3.7 for its properties). Let S be a base scheme over $\mathrm{Spec}(\mathbb{Z}[1/n])$.

(i) An object of $(\bar{\mathcal{H}}^i_{\gamma,H,\emptyset})^G$ over S is a triple $t = (L,\lambda,\{\alpha_g\}_{g\in G})$. The pair (L,λ) is a curve with action of H, and $\alpha_g\colon (L, g.\lambda) \to (L,\lambda)$ is an isomorphism such that $\alpha_{g_1g_2} = \alpha_{g_1} \circ g_1.\alpha_{g_2} = \alpha_{g_1} \circ \alpha_{g_2}$, by (b) in the previous listing. In particular, α_g is H-equivariant, which means that for all $h \in H$ we have

$$\alpha_g \circ g^{-1}(h) = h \circ \alpha_g.$$

This is the same as saying that α_g belongs to the normalizer of H in $\mathrm{Aut}_S(L)$.

(ii) A morphism between two objects $t = (L,\lambda,\alpha_g)$ and $t' = (L',\lambda',\alpha'_g)$ is an H-equivariant morphism $u\colon L \to L'$ such that $u \circ \alpha_g = \alpha'_g \circ u$ for all $g \in G$.

8.2. *Stacks $\mathcal{X}^n_{g,G}$*

Clearly the action μ_i induces an action on $\bar{\mathcal{M}}_g(n)$ (defined in 7.2.6) owing to the universal property of rigidification. We write $\bar{\mathcal{M}}^i_g(n)$ for the resulting G-stack. The fixed point stack $\bar{\mathcal{M}}^i_g(n)^G$ enjoys a similar description as $(\bar{\mathcal{H}}^i_{\gamma,H,\emptyset})^G$ in Section 8.1.2. The following definition is justified by the usual fact that if we have a group G and a normal subgroup H then, whenever G acts on a "structure" X, G/H acts on the fixed points X^H. For stacks, this was made precise in Remark 2.4.

DEFINITION 8.2.1. Fix $i \in \mathcal{I}_n$, and let $Z^i_n(G)$ denote the centralizer of G in $\mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$. We define the following $\mathbb{Z}[1/n]$-stacks:

$$\mathcal{X}_{g,G}^n := \coprod_{i \in \mathcal{I}_n} \frac{\mathcal{M}_g^i(n)^G}{Z_n^i(G)}. \tag{1}$$

$$\bar{\mathcal{X}}_{g,G}^n := \coprod_{i \in \mathcal{I}_n} \frac{\bar{\mathcal{M}}_g^i(n)^G}{Z_n^i(G)}. \tag{2}$$

We are mainly interested in $\mathcal{X}_{g,G}^n$. The formation of $\bar{\mathcal{X}}_{g,G}^n$ is functorial with respect to n. Indeed, let $m|n$ with $m,n \geq 3$. Then we pick $i \in \mathcal{I}_n$ and denote $j = i \otimes \mathbb{Z}/m\mathbb{Z} \in \mathcal{I}_m$. By 7.2.5 there is a morphism $\bar{\mathcal{M}}_g(n) \to \bar{\mathcal{M}}_g(m)[1/n]$, and this is equivariant for the actions induced by i and j. We may thus derive morphisms $\bar{\mathcal{M}}_g^i(n)^G/Z_n^i(G) \to \bar{\mathcal{M}}_g^j(m)^G/Z_m^j(G)$ and $\bar{\mathcal{X}}_{g,G}^n \to \bar{\mathcal{X}}_{g,G}^m[1/n]$. This latter map is proper and quasi-finite, but it is not representable in general.

Our interest in these stacks comes from the following result. Its virtue is that, since $\mathcal{M}_g^i(n)^G$ is a scheme, it follows that away from the characteristics that divide n we express explicitly the stack of curves with action as a quotient of a scheme by a finite group (or, more precisely, as a sum of such quotients). The important point here is that the characteristics dividing $|G|$ are included.

THEOREM 8.2.2. *Let $n \geq 3$ be prime to $|G|$. We have an isomorphism of $\mathbb{Z}[1/n]$-stacks $\mathcal{X}_{g,G}^n \xrightarrow{\sim} \mathcal{H}_{g,G}[1/n]$.*

Proof. It is more convenient here to think of $\mathcal{M}_g(n)$ in its classical definition. That is to say, its objects are, locally on the base, pairs (C,ℓ) with a curve and an isomorphism $\ell\colon H^1(C,\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^{2g}$. Morphisms are maps $u\colon C \to C'$ such that $u^*\ell' = \ell$. Given $i \in \mathcal{I}_n$, through which G acts on $(\mathbb{Z}/n\mathbb{Z})^{2g}$, the action on $\mathcal{M}_g(n)$ is given by $g(C,\ell) = (C, g\circ\ell)$. We now define maps from each summand of $\mathcal{X}_{g,G}^n$ to $\mathcal{H}_{g,G}$. Starting from $\mathcal{M}_g^i(n)^G$, the map is just the forgetful morphism that forgets the level structure (note that the G-linearization provides a faithful action of G on the curve). The fact that it passes to the quotient by $Z_n^i(G)$ is obvious. It remains to check that the result is an isomorphism.

That the map is an epimorphism is clear because, given a curve C with action of G, we can choose any level structure ℓ on C and look at how it pulls back via $g\colon C \to C$. This corresponds to an action $i \in \mathcal{I}_n$. Thus we get a preimage in the ith component.

To check that the map is a monomorphism, we may again work locally over the base S. We pick two objects in the same i-component: an object is a triple $t = (C,\ell,\alpha_g)$ over a base E, which is a $Z_n^i(G)$-torsor over S (4.1). Localizing again to trivialize the torsors, we have

$$\begin{aligned}\mathrm{Hom}(t,t') = \{(z,u) \mid z \in Z_n^i(G) \text{ and } u\colon (C, z\circ\ell) \to (C',\ell')\\ \text{is such that } u\circ\alpha_g = \alpha'_g \circ u \ (\forall g)\}\end{aligned}$$

(see Proposition 2.6 and Remark 4.2). The map to $\mathrm{Hom}((C,\alpha_g),(C',\alpha'_g))$ forgets z. Hence it is bijective because, obviously, u determines uniquely the element $z \in Z_n^i(G)$ by $z = (u^*\ell')\circ\ell^{-1}$. $\square$

REMARK 8.2.3. In view of the theorem, a (wrong) candidate to compactify $\mathcal{H}_{g,G}$ is $\bar{\mathcal{X}}^n_{g,G}$. The reason why it is not satisfactory is that, because of the operation of fixed points, $\mathcal{X}^n_{g,G} \subset \bar{\mathcal{X}}^n_{g,G}$ is not dense. However, it is interesting for the results of Part A to note that, because the order of G is prime to n, we are in one of the remarkable cases (Proposition 5.3) where the rigidification procedure at the origin of the definition of $\bar{\mathcal{M}}_g(n)$ goes through the successive operations on $\bar{\mathcal{H}}^i_{\gamma,H,\emptyset}$. Namely, the morphism $\bar{\mathcal{H}}^i_{\gamma,H,\emptyset} \to \bar{\mathcal{M}}^i_g(n)$ induces an isomorphism

$$\frac{(\bar{\mathcal{H}}^i_{\gamma,H,\emptyset})^G}{Z^i_n(G)} /\!\!/ H^G \xrightarrow{\sim} \frac{\bar{\mathcal{M}}^i_g(n)^G}{Z^i_n(G)}$$

(this also holds for the substacks parameterizing smooth curves). This follows because if $|G|$ is prime to n then the cohomology groups $H^i(G,H)$ vanish for $i \geq 1$. In particular, $H^1(G,H) = H^2(G,H) = 0$ and so Proposition 5.3 applies to the stack $\mathcal{N} = \bar{\mathcal{H}}_{\gamma,H,\emptyset}$, which is a substack of the stack of admissible curves with faithful action. It follows that $(\bar{\mathcal{H}}^i_{\gamma,H,\emptyset})^G /\!\!/ H^G \simeq (\bar{\mathcal{H}}^i_{\gamma,H,\emptyset} /\!\!/ H)^G$. Then commutation of $/\!\!/ H^G$ with quotients is a general fact proven in Theorem 5.1.

One can also show that the morphism of Theorem 8.2.2 extends to a morphism $\bar{\mathcal{X}}^n_{g,G} \to \bar{\mathcal{M}}_g\{G\}[1/n]$.

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References

[ACV] D. Abramovich, A. Corti, and A. Vistoli, *Twisted bundles and admissible covers,* Comm. Algebra 31 (2003), 3547–3618.

[Ar] M. Artin, *Algebrization of formal moduli I,* Global Analysis, papers in honor of K. Kodaira, pp. 21–71, Univ. Tokyo Press, Tokyo, 1969.

[BeM] J. Bertin and A. Mézard, *Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques,* Invent. Math. 141 (2000), 195–238.

[BeR] J. Bertin and M. Romagny, *Champs de Hurwitz* (in preparation).

[BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models,* Ergeb. Math. Grenzgeb. (3), 21, Springer-Verlag, Berlin, 1990.

[D] P. Deligne, *Le lemme de Gabber,* Seminar on arithmetic bundles: The Mordell conjecture (Paris, 1983/84), Astérisque 127 (1985), 131–150.

[DMu] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus,* Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.

[SGA3] M. Demazure and A. Grothendieck, *Schémas en groupes,* Séminaire de Géométrie Algébrique du Bois Marie 1962/64, Lecture Notes in Math., 152, Springer-Verlag, Berlin, 1962/1964.

[E] T. Ekedahl, *Boundary behaviour of Hurwitz schemes,* The moduli space of curves (Texel Island, 1994), Progr. Math., 129, pp. 173–198, Birkhäuser, Boston, MA, 1995.

[GIT] J. Fogarty and D. Mumford, *Geometric invariant theory,* 2nd ed., Ergeb. Math. Grenzgeb. (2), 34, Springer-Verlag, Berlin, 1982.

[GrPa] T. Graber and R. Pandharipande, *Localization of virtual classes,* Invent. Math. 135 (1999), 487–518.

[EGA] A. Grothendieck, *Eléments de géométrie algébrique,* Inst. Hautes Études Sci. Publ. Math. 4 (I), 8 (II), 11, 17 (III), 20, 24, 28, 32 (IV) (1961–1967).

[Ha] M. Hakim, *Topos annelés et schémas relatifs,* Ergeb. Math. Grenzgeb., 64, Springer-Verlag, Berlin, 1972.

[Ho] S. Hollander, *A homotopy theory for stacks,* preprint, arXiv: math.AT/0110247.

[KeMo] S. Keel and S. Mori, *Quotients by groupoids,* Ann. of Math. (2) 145 (1997), 193–213.

[Kn] D. Knutson, *Algebraic spaces*, Lecture Notes in Math., 203, Springer-Verlag, New York, 1971.

[Ko] M. Kontsevitch, *Enumeration of rational curves via torus actions,* The moduli space of curves (Texel Island, 1994), Progr. Math., 129, pp. 335–368, Birkhäuser, Boston, 1995.

[L-MB] G. Laumon and L. Moret-Bailly, *Champs algébriques,* Ergeb. Math. Grenzgeb. (3), 39, Springer-Verlag, Berlin, 1990.

[PdJ] M. Pikaart and J. de Jong, *Moduli of curves with non-abelian level structure* (Dijkgraaf et al., eds.), The moduli space of curves (Texel Island, 1994), Progr. Math., 129, pp. 483–509, Birkhäuser, Boston, 1995.

[R] M. Romagny, *Sur quelques aspects des champs de revêtements de courbes algébriques,* Ph.D. thesis, Institut Fourier, Univ. Grenoble 1, 2002.

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