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Completions of Normal Affine Surfaces with a Trivial Makar-Limanov Invariant

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Introduction

For a connected normal affine surface $V = \mathrm{Spec}(A)$ over $\mathbb{C}$, the Makar-Limanov invariant of V [10] is the subalgebra $\mathrm{ML}(V) \subset A$ of all regular functions invariant under every algebraic $\mathbb{C}_+$-action on V. Constant functions are certainly contained in $\mathrm{ML}(V)$, and we say that the Makar-Limanov invariant of V is *trivial* (or that V is an *ML-surface*) if $\mathrm{ML}(V) = \mathbb{C}$. In [1], Bandman and Makar-Limanov have re-discovered a link between nonsingular ML-surfaces and *geometrically quasihomogeneous* surfaces studied by Gizatullin in [6]—that is, surfaces whose automorphism group has a Zariski open orbit with a finite complement. More precisely, they have established that, on a nonsingular ML-surface V, there exist at least two nontrivial algebraic $\mathbb{C}_+$-actions that generate a subgroup H of the automorphism group $\mathrm{Aut}(V)$ of V such that the orbit $H.v$ of a general closed point $v \in V$ has finite complement. By Gizatullin [6], such a surface is rational and is either isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or can be obtained from a nonsingular projective surface $\bar{V}$ by deleting an ample divisor of a special form, called a *zigzag*. This is just a linear chain of nonsingular rational curves. Conversely, a nonsingular surface V completable by a zigzag is rational and geometrically quasihomogeneous (see [6]). In addition, if V is not isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ then it admits two independent $\mathbb{C}_+$-actions. More precisely, Bertin [2] showed that if V admits a $\mathbb{C}_+$-action then this action is unique unless V is completable by a zigzag. Altogether, this leads to the following result.

Theorem [1; 2; 6]. *A nonsingular affine surface V that is nonisomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ has a trivial Makar-Limanov invariant if and only if V is completable by a zigzag.*

More generally, in this paper we prove the following theorem.

Theorem. *A normal affine surface V that is nonisomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ has a trivial Makar-Limanov invariant if and only if V is completable by a zigzag.*

We are grateful to the referee for pointing out that closely related results are proved in two recent preprints [3; 14], under the additional assumption that V is rational.

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However, we think it might be useful to provide a self-contained and straightforward proof of this theorem in order to set up a good framework for a more detailed study of ML-surfaces.

1. Rulings and Completions of Normal Surfaces

We use the following terminology.

- A *surface* is a connected, reduced, normal $\mathbb{C}$-scheme of finite type and of dimension 2.
- The intersection number of two divisors D_1 and D_2 on a surface V regular at the points of $D_1 \cap D_2$ is denoted by $(D_1 \cdot D_2)$. The self-intersection number of a divisor $D \subset V_{\mathrm{reg}}$ is denoted by $(D^2) = (D \cdot D)$.
- For a morphism $f : W \to V$ between normal varieties and for a divisor D on V, we denote by $q^{-1}(D)$ the set-theoretic preimage of D, while $q^*(D)$ denotes its preimage considered as a cycle.
- An $\mathbb{A}^1$-fibration (a $\mathbb{P}^1$-fibration) on a surface V is a surjective morphism $\rho : V \to Z$ on a nonsingular curve Z with general fibers isomorphic to the affine line $\mathbb{A}^1$ (to the projective line $\mathbb{P}^1$, respectively). The fibers of ρ that are either not isomorphic to $\mathbb{A}^1$ (resp., $\mathbb{P}^1$) or not reduced are called *degenerate*.
- An *SNC-divisor* D on a surface is a divisor with normal crossing singularities whose irreducible components are nonsingular.
- For a normal affine surface V, we call a *completion* of V an open embedding $i : V \hookrightarrow \bar{V}$ of V into a normal projective surface $\bar{V}$, nonsingular along $B = \bar{V} \setminus i(V)$ and such that B is an SNC-divisor. We say that the completion is *minimal* if B contains no (-1)-curve that meets at most two other components transversally in a single point.
- For an isolated singularity (V, P) of a normal surface, a *minimal embedded resolution* of p is a birational morphism $\pi : W \to V$ such that W is nonsingular, $W \setminus \pi^{-1}(P) \simeq V \setminus \{P\}$, and $\pi^{-1}(P)$ is an SNC-divisor that contains no (-1)-curve meeting at most two other components transversally in a single point.

DEFINITION 1.1. A *zigzag* B on a normal projective surface $\bar{V}$ is a connected SNC-divisor with nonsingular rational curves as irreducible components and whose dual graph is a linear chain. If $\mathrm{Supp}(B) = \bigcup_{i=1}^{n} B_i$, then the irreducible components B_i $(1 \le i \le n)$ of B can be ordered in such a way that

$$(B_i \cdot B_j) = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

A zigzag with such an ordering on the set of its components is called *oriented* and the sequence $((B_1^2), \dots, (B_n^2))$ is called the *type* of B. For an oriented zigzag B, the components B_1 and B_n are called the *boundaries* of B. Given an irreducible component B_{i_0} of B, we denote by $B_{i_0}^{\pm}$ the component $B_{i_0 \pm 1}$ (provided it does exist). A zigzag B is called minimal if it contains no (-1)-curve.

Let $C \subset \bar{V}$ be an SNC-divisor. A zigzag B of C is a zigzag with support contained in C and such that no irreducible component of B corresponds to a ramification vertex of the dual graph of C. A zigzag B that is maximal for the inclusion of supports is called maximal. If C itself is not a zigzag, then we call a maximal zigzag B of C *simple* if only one boundary of B meets a ramification vertex of the dual graph of C. We call it *double* if this happens for both boundaries of B.

We say that a normal affine surface V is *completable* by a zigzag if there exists a completion $\bar{V}$ of V such that $B := \bar{V} \setminus V$ is a zigzag.

Properties of $\mathbb{P}^1$-Fibrations on Normal Projective Surfaces

We recall some properties of $\mathbb{P}^1$-fibrations on a normal projective surface. The following lemma is well known for a nonsingular surface $\bar{V}$ (see [13, Lemma 1.4.1, p. 195]).

LEMMA 1.2. *Let $\bar{q}\colon \bar{V} \to \bar{Z}$ be a $\mathbb{P}^1$-fibration. If $F := \sum_{1=1}^{p} n_i C_i$ is a fiber of $\bar{q}$ with irreducible components C_i, then*:

(1) *the morphism $\bar{q}$ admits a section $S \subset \bar{V}$; and*
(2) *if F is irreducible and $P = F \cap S$ is a regular point of $\bar{V}$, then F is nondegenerate.*

Now assume that F is degenerate. Then the following statements also hold.

(3) *The support of F is connected.*
(4) *If a singular point P of $\bar{V}$ is contained in a unique curve C_i, then it is a cyclic quotient singularity. In this case, the proper transform of C_i in a minimal embedded resolution $\pi : \bar{W} \to \bar{V}$ of P meets a terminal component of $\pi^{-1}(P)$.*
(5) *If C_i does not contain any singular point of $\bar{V}$, then it is nonsingular ($C_i \simeq \mathbb{P}^1$) and $(C_i^2) < 0$.*
(6) *If C_i and C_j $(i \neq j)$ are nonsingular and do not contain any singular point of $\bar{V}$, then $(C_i \cdot C_j) = 0$ or 1.*
(7) *For any three distinct indices i, j, l, either $C_i \cap C_j \cap C_l = \emptyset$ or $C_i \cap C_j \cap C_l$ is a singular point P of $\bar{V}$.*
(8) *If F is contained in $\bar{V} \setminus \operatorname{Sing}(\bar{V})$ then at least one of the C_i, say C_1, is a (-1)-curve. If $\tau : \bar{V} \to \bar{V}_1$ denotes the contraction of C_1 then $\bar{q}$ factors as*
$$\bar{q}\colon \bar{V} \xrightarrow{\tau} \bar{V}_1 \xrightarrow{\bar{q}_1} \bar{Z},$$
where $\bar{q}_1\colon \bar{V}_1 \to \bar{Z}$ is a $\mathbb{P}^1$-fibration. Hence all but one irreducible component of F can be contracted successively to obtain a nondegenerate fiber. Therefore, F is an SNC-divisor whose dual graph $\Gamma(F)$ is a tree.
(9) *If F is contained in $\bar{V} \setminus \operatorname{Sing}(\bar{V})$ and if one of the n_i, say n_1, is equal to 1, then there exists a (-1)-curve among the C_i, $2 \le i \le p$.*

Proof. We let $\phi\colon \bar{W} \to \bar{V}$ be a minimal embedded resolution of singularities. We denote by $\tilde{q}$ the $\mathbb{P}^1$-fibration on $\bar{W}$ lifting $\bar{q}$ and by $\tilde{S}$ a section of $\tilde{q}$. Then $S := \phi(\tilde{S})$ is a section of $\bar{q}$, and so (1) follows. In the nonsingular case, (2) is a consequence of the existence of a section of $\bar{q}$ and (3)–(9) follow from the genus formula.

In the normal case, (3) and (5)–(9) follow at once from the nonsingular case and (4) can be proved in the same way as Lemma 1.4.4 in [13, p. 196]. To show (2), we let $F = \bar{q}^{-1}(z_0)$, $z_0 \in Z$, be an irreducible fiber of $\bar{q}$. Its total transform $\phi^{-1}(F)$ is the fiber $\tilde{F} = \tilde{q}^{-1}(z_0)$ of $\tilde{q}$. If $P \in F$ is a singular point of $\bar{V}$, then $\phi^{-1}(P) \subset \bar{W}$ contains no (-1)-curve that meets at most two other components transversally in a single point. Then assertions (7) and (8) on $\bar{W}$ imply that $\phi^{-1}(P)$ contains no (-1)-curve at all. It follows from (8) that the proper transform F' of F is the unique (-1)-curve in $\tilde{F}$. Hence F must be a nonreduced fiber of $\bar{q}$, for otherwise F' has multiplicity 1 in $\tilde{F}$, which contradicts (9). Provided that $P_0 = S \cap F$ is a regular point of $\bar{V}$, F does not contain any singular point of $\bar{V}$ and so is nondegenerate, which proves (2). □

REMARK 1.3. Note that by (7) and (8), a (-1)-curve E contained in a degenerate fiber $F \subset \bar{V}_{\mathrm{reg}}$ of $\bar{q}$ cannot be a ramification vertex of the dual graph of $F \cup S$.

DEFINITION 1.4. Let $F \subset \bar{V}_{\mathrm{reg}}$ be a degenerate fiber of a $\mathbb{P}^1$-fibration $\bar{q}\colon \bar{V} \to \bar{Z}$ over a nonsingular projective curve $\bar{Z}$, and let S be a section of $\bar{q}$. A maximal zigzag D of F (see Definition 1.1) is called *terminal* if either $D = F$ or D is a maximal simple zigzag of F that does meet S.

In the following lemma we specify the position of (-1)-curves in a degenerate fiber of a $\mathbb{P}^1$-fibration.

LEMMA 1.5. *Let $\bar{q}\colon \bar{V} \to \bar{Z}$ be a $\mathbb{P}^1$-fibration on a normal projective surface $\bar{V}$ over a nonsingular projective curve $\bar{Z}$. Let S be a section of $\bar{q}$, and let $F \subset \bar{V}_{\mathrm{reg}}$ be a degenerate fiber of $\bar{q}$. If $F \cup S$ is not a zigzag then the following assertions hold*:

(1) *at least one (-1)-curve E in F is contained in a maximal terminal zigzag of F*;
(2) *if all such (-1)-curves are contained in the same maximal terminal zigzag D of F, then every ramification vertex of the dual graph $\Gamma(F \cup S)$ of $F \cup S$ belongs to the shortest path in $\Gamma(F \cup S)$ that joins D and S.*

Proof. Given a (-1)-curve E in F, we let $\tau_E\colon \bar{V} \to \bar{V}_1$ be the contraction of E. Consider the factorization

$$\bar{q}\colon \bar{V} \xrightarrow{\tau_E} \bar{V}_1 \xrightarrow{\bar{q}_1} \bar{Z},$$

where $\bar{q}_1\colon \bar{V}_1 \to \bar{Z}$ is a $\mathbb{P}^1$-fibration with a degenerate fiber $F_1 := \tau_E(F) \subset (\bar{V}_1)_{\mathrm{reg}}$ and a section $S_1 = \tau_E(S)$. By assumption the graph $\Gamma(F \cup S)$ has a ramification vertex, so $F \cup S$ has at least four irreducible components. By Remark 1.3, E is a component of a maximal zigzag D of F.

We consider first the case that $F \cup S = E_1 \cup E_2 \cup E_S \cup S$ has four irreducible components where E_S meets S. It is easily seen that E_S corresponds to a ramification vertex of $\Gamma(F \cup S)$. Then E_1 and E_2 are both maximal terminal zigzags of F and at least one of them is a (-1)-curve, which proves the first assertion in this case. The second assertion follows then at once because E_S is a unique ramification vertex of $\Gamma(F \cup S)$.

To show (1), we may assume that F is not a zigzag, for otherwise our statement is evidently true. We also suppose that $F \cup S$ has $n > 4$ irreducible components, and we assume on the contrary that every (-1)-curve E in F is contained either in a maximal simple zigzag of F that meets S or in a maximal double zigzag of F. We denote this maximal zigzag by $D = D(E)$. By our assumption, the contraction τ_E of E gives a one-to-one correspondence between the maximal simple zigzags of $F \cup S$ and the maximal simple zigzags of $F_1 \cup S_1$. Moreover, none of the maximal terminal zigzags of F is affected by this contraction. Since F_1 has one fewer irreducible component than F, we may conclude by induction that there is a (-1)-curve E_1 in F_1 that belongs to a maximal terminal zigzag of F_1. Then $\tau_E^{-1}(E_1)$ is a (-1)-curve contained in a maximal terminal zigzag of F, a contradiction. Thus assertion (1) is proved.

To prove (2), we may suppose that F is not a zigzag and that $F \cup S$ has $n > 4$ irreducible components. We let E be a (-1)-curve in D. If $D \neq E$ then the contraction τ_E of E yields a bijection between maximal terminal zigzags of F_1 and those of F. Since D is the only maximal terminal zigzag of F affected by the contraction of E, it follows from (1) that $\tau_E(D)$ contains a (-1)-curve. In fact, it contains all (-1)-curves as in (1), and so we are finished by induction.

In case $D = E$ we let H be a ramification vertex of $\Gamma(F \cup S)$ such that E is a branch of $\Gamma(F \cup S)$ at H. Then H has valency 3, for otherwise $\tau_E(H)$ is a ramification vertex of $\Gamma(F_1 \cup S_1)$ and hence none of the maximal terminal zigzags of F_1 contains a (-1)-curve, which contradicts (1). Thus, if $F_1 \cup S_1$ is a zigzag then we are done. If $F_1 \cup S_1$ is not a zigzag then $\tau_E(H)$ is contained in a maximal zigzag D_1 of F_1. If either D_1 meets S_1 or D_1 is double, then τ_E provides a bijective correspondence between the maximal terminal zigzags of F different from E and those of F_1. Since these maximal zigzags of F were not affected by the contraction of E, it follows that none of the maximal terminal zigzags of F_1 contains a (-1)-curve, which again contradicts (1). Therefore, D_1 is a maximal terminal zigzag of F_1 and by (1) it contains a (-1)-curve E_1. Our induction hypothesis then implies that every ramification vertex of $\Gamma(F_1 \cup S_1)$ belongs to the shortest path from E_1 to S_1 in $\Gamma(F_1 \cup S_1)$. As H is the only ramification vertex of $\Gamma(F \cup S)$ that is eliminated by the contraction of E, we conclude that every such ramification vertex belongs to the shortest path from E to S in $\Gamma(F \cup S)$. This proves the second assertion. □

Properties of $\mathbb{A}^1$-Fibrations on Normal Affine Surfaces

Given a normal affine surface V together with an $\mathbb{A}^1$-fibration $q : V \to Z$ over a nonsingular affine curve Z, we let $\bar{V}$ be a minimal completion of V. Because V is affine, the divisor $B := \bar{V} \setminus V$ is connected. The $\mathbb{A}^1$-fibration q on V induces a rational map $\bar{q} : \bar{V} \dashrightarrow \bar{Z}$, where $\bar{Z}$ denotes a nonsingular projective model of Z. The closures of the fibers of q in $\bar{V}$ define a pencil of nonsingular rational curves with at most one base point on B. If necessary, this base point and all infinitely near ones can be eliminated by a succession of blow-ups with centers outside of V. Thus we may suppose that $\bar{q}$ is a well-defined $\mathbb{P}^1$-fibration on $\bar{V}$.

1.6. In this way we arrive at a completion $\bar{V}$ of V with the following properties.

(1) $\bar{V}$ is a normal projective surface, nonsingular along $B := \bar{V} \setminus V$, with a $\mathbb{P}^1$-fibration $\bar{q} : \bar{V} \to \bar{Z}$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "q"'] & \bar{V} \arrow[d, "\bar{q}"] \\ Z \arrow[r, hook] & \bar{Z}. \end{tikzcd}$$

(2) B is a connected SNC-divisor and can be written as $B = H \cup S \cup G$, where S is a section of $\bar{q}$, $H = \bigcup H_j$ for $H_j := \bar{q}^{-1}(z_j)$ with $z_j \in \bar{Z} \setminus Z$, and the connected components of G are trees of nonsingular rational curves.

(3) We can write $G = \bigcup_{i=1}^{s} G_i$, where $\bar{q}(G_i) = z_i \in Z$ and where $z_1, \dots, z_s \in Z$ are the points such that the fiber $q^{-1}(z_i) \subset V$ is degenerate. Thus $\bar{q}^{-1}(z_i) = G_i \cup \overline{q^{-1}(z_i)}$, $1 \le i \le s$, where $\overline{q^{-1}(z_i)}$ denotes the closure of $q^{-1}(z_i)$ in $\bar{V}$.

One can, moreover, assume that the boundary divisor B contains no (-1)-curve except perhaps the section S. Since B contains no singular point of $\bar{V}$, it follows that every H_j is a nonsingular rational curve. In the sequel, such a completion will be called a *good completion of V with respect to q*.

For degenerate fibers of an $\mathbb{A}^1$-fibration on a normal affine surface V, we have the following description.

LEMMA 1.7 [13, Lemmas 1.4.2 & 1.4.4, p. 196]. *If $q : V \to Z = \mathbb{A}^1$ is an $\mathbb{A}^1$-fibration, then the following assertions hold.*

(1) *Every irreducible component C of $q^{-1}(z)$ is a connected component of $q^{-1}(z)$ and is a rational curve with only one place at infinity*; *hence C is isomorphic to $\mathbb{A}^1$ provided it is nonsingular.*

(2) *Every such component C contains at most one singular point of V.*

(3) *The surface V has at most cyclic quotient singularities.*

(4) *If C contains a singular point P of V and if $\pi : W \to V$ is a minimal embedded resolution of P, then the closure $\bar{C}'$ in W of the proper transform C' of C meets a terminal component of $\pi^{-1}(P)$.*

2. Completions of ML-Surfaces

This section is devoted to the proof of the following theorem.

THEOREM 2.1. *A normal affine surface V has a trivial Makar-Limanov invariant if and only if it is completable by a zigzag.*

In order to reformulate our statement, we need the following lemma.

LEMMA 2.2 ([5], e.g.). *If V is a normal affine surface, then the following assertions are equivalent*:

(1) *there exists an $\mathbb{A}^1$-fibration $q: V \to Z$ over a nonsingular affine curve Z*;
(2) *the surface V contains a principal Zariski open subset U that is a cylinder, $U \simeq C \times \mathbb{A}^1$*;
(3) *there exists a nontrivial algebraic $\mathbb{C}_+$-action on V.*

As a consequence we obtain the following corollary.

COROLLARY 2.3. *For a normal affine surface V, the following assertions are equivalent*:

(1) *the Makar-Limanov invariant of V is trivial*;
(2) *there exist at least two nontrivial algebraic $\mathbb{C}_+$-actions on V whose general orbits do not coincide*;
(3) *there exist at least two $\mathbb{A}^1$-fibrations, $q_1: V \to Z_1$ and $q_2: V \to Z_2$ over nonsingular affine curves Z_1 and Z_2, such that the general fibers of q_1 and q_2 do not coincide.*

Thus, Theorem 2.1 can be equivalently formulated as follows.

THEOREM 2.4. *A normal affine surface is completable by a zigzag if and only if it admits two $\mathbb{A}^1$-fibrations whose general fibers do not coincide.*

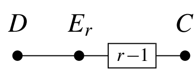
Normal Affine Surfaces Completable by a Zigzag

This section is closely related to the work of Gizatullin [6] and Danilov [7], where the case of nonsingular surfaces completable by a zigzag was treated. Let us mention first some useful technical results about zigzags on normal projective surfaces. The following construction will be frequently used in the sequel.

DEFINITION 2.5. Let $\bar{V}$ be a normal projective surface, and let C and D be two irreducible nonsingular curves on $\bar{V}$ that intersect transversally at a single nonsingular point of $\bar{V}$. By the *iterative modification of $\bar{V}$ with center (C, D), length $r \in \mathbb{N}^*$, and divisors $E_1, \ldots, E_r$* we mean the birational morphism $\sigma: \bar{W} \to \bar{V}$, where $\bar{W}$ is a normal projective surface, obtained by the following blow-up procedure.

- Step 1 is the blow-up $\sigma_1: \bar{W}_1 \to \bar{V}$ of the intersection point of C and D with exceptional curve $E_1 \subset \bar{W}_1$.
- Step k for $2 \leq k \leq r$ is the blow-up $\sigma_k: \bar{W}_k \to \bar{W}_{k-1}$ of the intersection point of E_{k-1} and the proper transform of D in $\bar{W}_{k-1}$, with exceptional curve $E_k \subset \bar{W}_k$.

We let $\sigma := \sigma_r \circ \cdots \circ \sigma_1: \bar{W} := \bar{W}_r \to \bar{V}$. If $C' \subset \bar{W}$ ($D' \subset \bar{W}$) denotes the proper transform of $C \subset \bar{V}$ (of $D \subset \bar{V}$, resp.) then $(C'^2) = (C^2) - 1$, $(D'^2) = (D^2) - r$, $(E_r^2) = -1$, and $(E_i^2) = -2$ for $1 \leq i \leq r-1$. For the dual graph of the total transform of $C \cup D$ in $\bar{W}$, we use the following notation:



In 2.6–2.9 we establish some useful properties of affine surfaces completable by a zigzag.

LEMMA 2.6. *Let $\bar{V}$ be a normal projective surface. Let $B \subset \bar{V}$ be a zigzag such that $\bar{V}$ is nonsingular along B and $V := \bar{V} \setminus B$ is affine. If B is irreducible then $(B^2) > 0$. If B is reducible then it contains an irreducible component C with $(C^2) \geq -1$.*

Proof. Since $V = \bar{V} \setminus B$ is affine, by a theorem of Goodman [8] there exists an ample divisor D on $\bar{V}$ such that $\mathrm{Supp}(D) = B$. Hence the first assertion follows. Now let B be reducible: $B = \bigcup_{i=1}^{n} C_i$ with C_i irreducible and $n \geq 2$; and let $D = \sum_{i=1}^{n} m_i C_i$ with $m_i > 0$ for all $1 \leq i \leq n$. Since B is a zigzag, we have $\big(C_i \cdot \sum_{j \neq i} C_j\big) \leq 2$. From

$$(D \cdot B) = \sum_{i=1}^{n} m_i (C_i \cdot B) = \sum_{i=1}^{n} m_i \bigg((C_i^2) + \Big(C_i \cdot \sum_{j \neq i} C_j \Big) \bigg) > 0$$

we conclude that there exists an i_0 with $(C_{i_0}^2) > -\big(C_{i_0} \cdot \sum_{j \neq i_0} C_j\big) \geq -2$, whence $(C_{i_0}^2) \geq -1$. □

LEMMA 2.7. *Given a normal affine surface V completable by a zigzag, there exists a minimal completion $\bar{V}$ of V by an oriented zigzag B such that its left boundary C_1 has nonnegative self-intersection.*

Proof. If B is irreducible then the assertion follows from Lemma 2.6. Thus we may assume that $B = \bigcup_{i=1}^{n} C_i$ with $n \geq 2$. By Lemma 2.6, $(C_{i_0}^2) \geq -1$ for some i_0, $1 \leq i_0 \leq n$. In fact, $(C_{i_0}^2) \geq 0$ because B is minimal. If $i_0 = 1$ or $i_0 = n$ then, up to reversing the ordering, we are done. If not, we let i_0 be the minimal index such that $(C_i^2) \geq 0$, and we denote $C(B) := C_{i_0}$ and $d(B) = d(C_1, C(B)) = i_0 - 1$. Thus $(C_i^2) \leq -2$ for every component C_i to the left of $C(B)$.

Since $C(B)$ is not a boundary of B, the successor $C(B)^+$ of $C(B)$ exists; hence we can perform the iterative modification $\sigma\colon \bar{W} \to \bar{V}$ of $\bar{V}$ with center $(C(B), C(B)^+)$, length $c+1$, and divisors $E_1, \ldots, E_c, E_{c+1}$ with $c := (C(B)^2)$. This yields $(C(B)'^2) = (E_{c+1}^2) = -1$. If $\tau\colon \bar{W} \to \bar{W}_1$ is the contraction of $C(B)'$ then $(\tau(C(B)'^-)^2) = ((C(B)^-)^2) + 1$ and $(\tau(E_{c+1})^2) = 0$. By iterating this procedure, we obtain that $((C(B)^-)^2) = -1$ and $(C(B)^2) = 0$. Contracting $C(B)^-$ and all (-1)-curves that arise successively to the left of $C(B)$, we arrive at a new completion $\bar{V}_1$ of V by a zigzag B_1 with $d(B_1) < d(B)$. Since no (-1)-curve has been created on the right of $C(B_1)$ under this procedure, it follows that $\bar{V}_1$ is a minimal completion of V. Now the proof can be completed by induction. □

COROLLARY 2.8. *A normal affine surface completable by a zigzag is rational.*

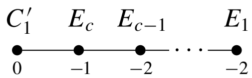
Proof. It is enough to show that there exists a completion $\bar{W}$ of V and a nonsingular rational curve $C \subset \bar{W}_{\mathrm{reg}}$ with $(C^2) > 0$. Let $\bar{V}$, B, and C_1 be as in Lemma 2.7. If $(C_1^2) > 0$ then we are done. If not, then B is reducible because it is the support of an ample divisor. By assumption, $(C_1^2) = 0$ and $(C_2^2) \le 0$. After blowing-up with center in $C_1 \setminus C_2$, the proper transform of C_1 becomes a (-1)-curve; we then contract it, obtaining a completion of V with (C_2^2) increased by one. By iterating this procedure we derive a completion $\bar{W}$ of V and a nonsingular rational curve $C \subset \bar{W}_{\text{reg}}$ with $(C^2) > 0$. $\square$

LEMMA 2.9. *If V is a normal affine surface completable by a zigzag, then the following assertions hold*:

(1) *if V is completable by a zigzag of type* $(0,0)$, *then* $V \simeq \mathbb{A}^2$;
(2) *if V is completable by a zigzag of type* $(0,0,0)$, *then* $V \simeq \mathbb{C}^* \times \mathbb{A}^1$;
(3) *if $V \not\simeq \mathbb{C}^2$ and $V \not\simeq \mathbb{C}^* \times \mathbb{C}$, then there exists a completion $\bar{V}$ of V by an oriented zigzag of type* $(0,0,k_1,\dots,k_m)$, *where $k_i \le -2$ for $1 \le i \le m$.*

Proof. We let $\bar{W}$ be a minimal completion of V by an oriented zigzag $B = \bigcup_{i=1}^n C_i$ such that its left boundary C_1 is a curve with nonnegative self-intersection.

(1) If $B = C_1$ then $c := (B^2) > 0$ because B is the support of an ample divisor. Let $D \subset \bar{W}$ be a nonsingular curve germ meeting C_1 transversally in a single point, and consider the iterative modification $\sigma : \bar{W}_1 \to \bar{W}$ of $\bar{W}$ with center (D, C_1), length c, and divisors $E_1, \dots, E_c$ (see Definition 2.5). Then the total transform B_1 of B is a zigzag whose left boundary is the proper transform C_1' of C_1. Moreover, $(C_1'^2) = 0$, $(E_c^2) = -1$, and $(E_i^2) = -2$ for $1 \le i \le c-1$. Thus B is now replaced by a zigzag with the following dual graph:



Let $\pi : \bar{W}_2 \to \bar{W}_1$ be the blow-up of a point $v \in C_1' \setminus E_c$ with exceptional component $E \subset \bar{W}_2$. Then the proper transform of C_1' in $\bar{W}_2$ is a (-1)-curve that can be contracted to obtain a completion $\bar{V}$ of V by a zigzag of type $(0,0,-2,\dots,-2)$.

(2) If $B \neq C_1$ and $c = (C_1^2) > 0$, then by applying the same procedure as in (1) we obtain a new minimal completion $\bar{W}_1$ of V by a reducible zigzag such that $(C_1^2) = 0$. Performing (if necessary) elementary transformations, we obtain a minimal completion by a zigzag with $(C_1^2) = (C_2^2) = 0$. We must distinguish then the following three cases.

Case 1: $B = C_1 \cup C_2$. Since $\bar{W}_1$ is rational, the linear system $|C_1|$ defines a $\mathbb{P}^1$-fibration $\bar{q} : \bar{W}_1 \to \bar{Z} = \mathbb{P}^1$ whose restriction to V is an $\mathbb{A}^1$-fibration $q : V \to Z = \bar{Z} \setminus \{\bar{q}(C_1)\} \simeq \mathbb{A}^1$. Thus $\bar{W}_1$ is a good completion of V with respect to q. Moreover, every fiber $\bar{q}^{-1}(z)$, $z \in Z$, coincides with the closure of $q^{-1}(z)$ in $\bar{V}$ and, being connected, is irreducible. Therefore, by virtue of Lemma 1.2(2), $\bar{q}$ has no degenerate fiber and hence $\bar{W}_1$ is nonsingular. From $(C_1^2) = (C_2^2) = 0$ we finally deduce $\bar{W}_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, so that $V = \bar{W}_1 \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{A}^2$.

Case 2: If $B = C_1 \cup C_2 \cup C_3$ and $(C_3^2) = 0$, then the linear system $|C_1|$ defines a $\mathbb{P}^1$-fibration $\bar{q} : \bar{W}_1 \to \bar{Z} = \mathbb{P}^1$ whose restriction to V is an $\mathbb{A}^1$-fibration $q : V \to Z = \bar{Z} \setminus \{\bar{q}(C_1), \bar{q}(C_3)\} \simeq \mathbb{C}^*$. Thus $\bar{W}_1$ is a good completion of V with respect to

q, and we can again conclude that $\bar{q}$ has no degenerate fiber. Hence $\bar{W}_1$ is a nonsingular surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Finally we have $V = \bar{W}_1 \setminus (C_1 \cup C_2 \cup C_3) \simeq \mathbb{C}^* \times \mathbb{A}^1$.

Case 3: It remains to consider the case $B = C_1 \cup C_2 \cup G$, where either $G = C_3$ with $(C_3^2) \neq 0$ or $G = \bigcup_{i=3}^n C_i$ with $n > 3$. The linear system $|C_1|$ defines a $\mathbb{P}^1$-fibration $\bar{q}: \bar{W}_1 \to \mathbb{P}^1$ having C_2 as a cross-section. Since G is connected and does not intersect C_1, it must be contained in a fiber F of $\bar{q}$. Moreover, F must be a singular fiber of $\bar{q}$, for otherwise we would have $F = C_3$ and hence $0 = (F^2) = (C_3^2) \neq 0$, a contradiction. By Lemma 1.2, every C_i with $3 \leq i \leq n$ has negative self-intersection. Since the initial completion $\bar{W}$ has been assumed minimal and since our transformations do not affect the curves C_i for $3 \leq i \leq n$, we conclude that $(C_i^2) \leq -2$ for all $3 \leq i \leq n$. □

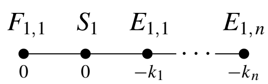
The next proposition proves one of the two implications of Theorem 2.1.

PROPOSITION 2.10. *If V is a normal affine surface nonisomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ and completable by a zigzag, then V has a trivial Makar-Limanov invariant.*

Proof. If V admits a completion $\bar{V}$ by a zigzag of type $(0,0)$, then Lemma 2.9(1) shows that $V \simeq \mathbb{C}^2$, which has a trivial Makar-Limanov invariant. We may thus assume from now on that Lemma 2.9(3) holds—that is, V has a completion $\bar{V}_1$ by a zigzag B_1 of type $(0, 0, -k_1, \dots, -k_n)$ with $k_i \geq 2$, $1 \leq i \leq n$. As in 1.6, we write

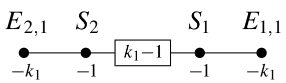
$$B_1 = F_{1,1} \cup S_1 \cup \left(\bigcup_{i=1}^n E_{1,i} \right),$$

where $(F_{1,1}^2) = (S_1^2) = 0$ and $(E_{1,i}^2) = -k_i$ for $1 \leq i \leq n$. The dual graph $\Gamma(B_1)$ is as follows:



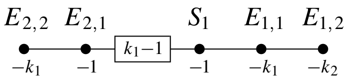
The linear system $|F_{1,1}|$ defines a $\mathbb{P}^1$-fibration $\bar{q}_1: \bar{V}_1 \to \mathbb{P}^1$ with S_1 as a cross-section, so the restriction $q_1: V \to \mathbb{A}^1$ of $\bar{q}_1$ to V is an $\mathbb{A}^1$-fibration. Thus it remains to find a second $\mathbb{A}^1$-fibration $q_2: V \to \mathbb{A}^1$ such that the general fibers of q_1 and q_2 do not coincide. In order to do this we construct a completion $\bar{W}$ of V together with a birational morphism $\sigma_1: \bar{W} \to \bar{V}_1$, which will also dominate a good completion $\bar{V}_2$ of V with respect to this $\mathbb{A}^1$-fibration q_2. It will be convenient in the sequel to denote the component $F_{1,1}$ of B by $E_{2,n}$.

If $n = 1$, then $\sigma_1: \bar{W} \to \bar{V}_1$ is the iterative modification of $\bar{V}_1$ with center $(S_1, E_{2,1})$, length k_1, and divisors $D_1, \dots, D_{k_1-1}, S_2$. For the total transform B of B_1 we obtain the following symmetrical dual graph:

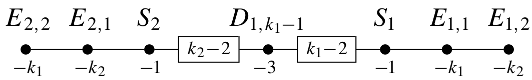


For $n = 2$, we obtain $\sigma_1: \bar{W} \to \bar{V}_1$ by the following procedure.

Step 1 is the iterative modification $\pi_1 \colon \bar{W}_1 \to \bar{V}_1$ with center $(S_1, E_{2,2})$, length k_1, and divisors $D_{1,1}, D_{1,k_1-1}, E_{2,1}$. The dual graph of the total transform of B_1 is as follows:



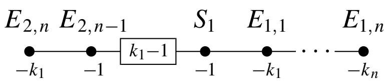
Step 2 is the iterative modification $\pi_2 \colon \bar{W}_2 \to \bar{W}_1$ of $\bar{W}_1$ with center $(E^+_{2,1} = D_{1,k_1-1}, E_{2,1})$, length $k_2 - 1$, and divisors $D_{2,1}, \ldots, D_{2,k_2-2}, S_2$ if $k_2 > 2$ or just S_2 if $k_2 = 2$. We then let $\bar{W} := \bar{W}_2$ and $\sigma_1 = \pi_1 \circ \pi_2 \colon \bar{W} \to \bar{V}_1$. The dual graph of the total transform $B = \sigma_1^{-1}(B_1)$ of B_1 has the following structure:



We observe that the same dual graph can be obtained from a zigzag of type $(0, 0, -k_2, -k_1)$ by reversing the ordering and the blow-up procedure.

For $n \geq 3$, $\bar{W}$ is obtained from $\bar{V}_1$ by the following procedure.

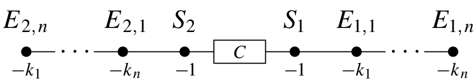
Step 1 is the iterative modification $\pi_1 \colon \bar{W}_1 \to \bar{V}_1$ with center $(S_1, E_{2,n})$, length k_1, and divisors $D_{1,1}, \ldots, D_{1,k_1-1}, E_{2,1}$. Then the dual graph of the total transform of B_1 is



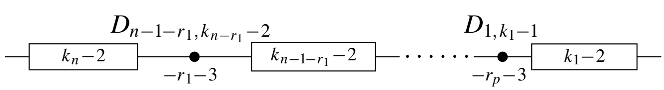
Step m, where $2 \leq m \leq n-1$, is the iterative modification $\pi_m \colon \bar{W}_m \to \bar{W}_{m-1}$ of $\bar{W}_{m-1}$ with center $(E^+_{2,n-m}, E_{2,n-m})$, length $k_m - 1$, and divisors $D_{m,1}, \ldots, D_{m,k_m-2}$, $E_{2,n-m-1}$ if $k_m > 2$ or just $E_{2,n-m-1}$ if $k_m = 2$.

Step n is the last step, consisting of the iterative modification $\pi_n \colon \bar{W}_n \to \bar{W}_{n-1}$ of $\bar{W}_{n-1}$ with center $(E^+_{2,1}, E_{2,1})$, length $k_n - 1$, and divisors $D_{n,1}, \ldots, D_{n,k_n-2}, S_2$ if $k_n > 2$ or just S_2 if $k_n = 2$.

Then we let $\bar{W} := \bar{W}_n$ and $\sigma_1 := \pi_1 \circ \cdots \circ \pi_n \colon \bar{W} \to \bar{V}_1$. For the total transform $B := \sigma_1^{-1}(B_1)$ of B_1, we obtain the following dual graph:



The dual graph of C looks like



where the $r_j \geq 0$ $(0 \leq j \leq p)$ depend on the number of (-2)-curves among the $E_{1,i}$ $(1 \leq i \leq n)$. Obviously, $V = \bar{W} \setminus B$. We observe as before that the same dual graph can be obtained from a zigzag of type $(0, 0, -k_n, \ldots, -k_1)$ by a symmetric blow-up procedure.

Henceforth, the sub-zigzag

$$D := C \cup S_1 \cup \bigcup_{i=1}^{n-1} E_{1,i}$$

of B can be contracted to a nonsingular point. We denote this contraction by $\sigma_2 \colon \bar{W} \to \bar{V}_2$ and let

$$B_2 = F_{2,1} \cup S_2 \cup \left(\bigcup_{i=1}^{n} E_{2,n-i+1} \right)$$

be the image of B by σ_2, where $F_{2,1} := E_{1,n}$. Then $V = \bar{V}_2 \setminus B_2$, where B_2 is a zigzag of type $(0, 0, -k_n, \ldots, -k_1)$.

The linear system $|F_{2,1}|$ then defines a $\mathbb{P}^1$-fibration $\bar{q}_2 \colon \bar{V}_2 \to \mathbb{P}^1$ whose restriction to V is a second $\mathbb{A}^1$-fibration $q_2 \colon V \to \mathbb{A}^1$. Moreover, since

$$\sigma_2(\sigma_1^*(F_{1,1})) = \alpha S_2 + \sum_{i=1}^{n} \beta_i E_{2,i}$$

with $\alpha > 0$ and $\beta_i \geq 0$ $(1 \leq i \leq n)$, it follows that $\left(F_{2,1} \cdot \sigma_2(\sigma_1^*(F_{1,1}))\right) \geq 1$. Thus the general fibers of q_1 and q_2 do not coincide, whence V has a trivial Makar-Limanov invariant. □

Finally, we have the following proposition.

PROPOSITION 2.11. *Every normal affine toric surface except for $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{C}^* \times \mathbb{A}^1$ has a trivial Makar-Limanov invariant. Consequently, every cyclic quotient singularity appears as a singular point of an ML-surface.*

Proof. Recall that, given a 2-dimensional lattice N, an affine toric surface corresponds to a strictly convex rational polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. If V is a normal affine toric surface nonisomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}$, then there exists a basis of N such that V is given by the cone $\sigma_{12} = \langle e_1, e_2 \rangle$ with $e_1 = (1, 0)$ and $e_2 = (n, q)$, where n and q are coprime integers; see Figure 1.

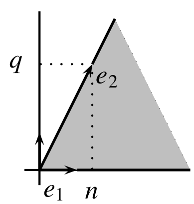


Figure 1

In order to construct a completion of V, we need to include σ_{12} into a complete fan Δ in $N_{\mathbb{R}}$. This can be done, for example, as shown in Figure 2. We let $\sigma_{ij} = \langle e_i, e_j \rangle$ with $e_3 = (0, 1)$, $e_4 = (-1, 0)$, and $e_5 = (0, -1)$. In Δ, the only possibly

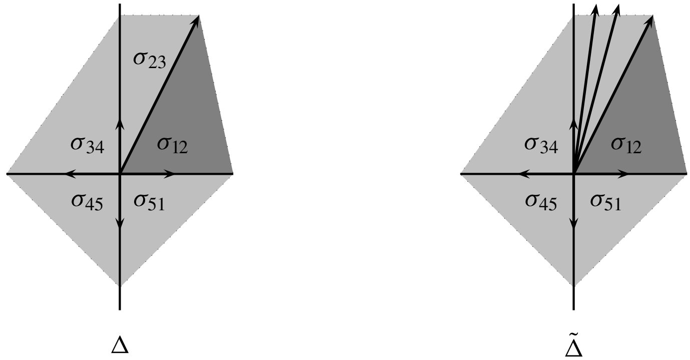


Figure 2

singular cones (i.e., cones whose generators do not form a basis of N) are σ_{12} and σ_{23}. We can subdivide the cone σ_{23} if necessary to obtain a new fan $\tilde{\Delta}$ such that σ_{12} is the only possibly singular cone in $\tilde{\Delta}$. We denote by e_i for $6 \le i \le r$ the new generators introduced in this subdivision procedure. Then $\bar{V} := V(\tilde{\Delta})$ is a completion of $V := V(\sigma_{12})$. We let $D_i = V(\tau_i)$ be the divisor on $\bar{V}$ corresponding to the cone $\tau_i = \langle e_i \rangle$ for $3 \le i \le r$. Then $B := \bar{V} \setminus V = D_3 \cup D_4 \cup \cdots \cup D_r$ is a zigzag, whence V has a trivial Makar-Limanov invariant by Proposition 2.10. $\square$

Completion of a Normal Affine Surface with a Trivial Makar-Limanov Invariant

In this section we prove that, conversely, every ML-surface V is completable by a zigzag.

2.12. By Corollary 2.3 there exist two $\mathbb{A}^1$-fibrations, $q_1 \colon V \to Z_1 \simeq \mathbb{A}^1$ and $q_2 \colon V \to Z_2 \simeq \mathbb{A}^1$, whose general fibers do not coincide. We denote by $\bar{V}_1$ a good completion of V with respect to q_1, with a boundary divisor $B = H \cup S \cup G \subset (\bar{V}_1)_{\mathrm{reg}}$ as in 1.6. Thus q_1 extends to a $\mathbb{P}^1$-fibration $\bar{q}_1 \colon \bar{V}_1 \to \bar{Z}_1 = \mathbb{P}^1$, so that $H = \bar{q}_1^{-1}(\infty) =: F_\infty$ is a nondegenerate fiber of $\bar{q}_1$ over the point $\infty := \bar{Z}_1 \setminus Z_1$, and $S \simeq \mathbb{P}^1$ is a section.

We let $\bar{q}_2 \colon \bar{V}_1 \dashrightarrow \bar{Z}_2 \simeq \mathbb{P}^1$ be the rational map that extends $q_2 \colon V \to Z_2$. We let $\bar{T}_2$ be the closure in $\bar{V}_1$ of a general fiber T_2 of q_2. The point $\bar{T}_2 \setminus T_2$ belongs to F_∞, for otherwise the restriction of q_1 to a general fiber of q_2 would be constant and then the general fibers of these two $\mathbb{A}^1$-fibration would coincide, contrary to our assumption. Because G is disjoint from F_∞, the map $\bar{q}_2$ has no base point on G and so $\bar{q}_2|_G$ must be locally constant. Moreover, $\bar{q}_2|_{S\setminus\{P_0\}} = \infty$, for otherwise q_2 would be bounded and thus constant along a general fiber of q_1. Since $S \cup G$ is connected, it follows that $\bar{q}_2|_{(S\cup G)\setminus\{P_0\}} = \infty$.

LEMMA 2.13. *If $\bar{q}_2 \colon \bar{V}_1 \dashrightarrow \bar{Z}_2$ is a morphism, then $G = \emptyset$, $B = F_\infty \cup S$ is a zigzag, and $V \simeq \mathbb{A}^2$.*

Proof. If $\bar{q}_2 \colon \bar{V}_1 \to \bar{Z}_2$ is a morphism, then it is a $\mathbb{P}^1$-fibration and its general fiber meets F_∞ at one point. It follows that F_∞ is a section of $\bar{q}_2$ and that $S \cup G$ is contained in the fiber $\bar{q}_2^{-1}(\infty) \subset (\bar{V}_1)_{\mathrm{reg}}$. Moreover, $\bar{q}_2^{-1}(\infty) = S \cup G$, as $\bar{q}_2^{-1}(\infty) \subset \bar{V}_1 \setminus V$. Since $\bar{V}_1$ is a minimal completion of V, it follows that $S \cup G$ contains no (-1)-curve and thus is a nondegenerate fiber of $\bar{q}_2$ (see (5) of Lemma 1.2). Hence $(S^2) = 0$, $G = \emptyset$, and $\bar{q}_2^{-1}(\infty) = S$, so the zigzag $B = F_\infty \cup S$ is of type $(0, 0)$ and $V \simeq \mathbb{A}^2$ by Lemma 2.9. $\square$

2.14. If $\bar{q}_2$ is not a morphism then it defines a linear pencil with a unique base point $P \in F_\infty$. Suppose that $P = P_0 := S \cap F_\infty$. If we blow up the point P_0 into an exceptional component E, the proper transform F'_∞ of F_∞ is a (-1)-curve. By contracting F'_∞, we obtain a new completion of V in which (S^2) has decreased by one. By applying these transformations with center P_0 several times, we arrive at the situation that the linear pencil $\bar{q}_2 \colon \bar{V}_1 \dashrightarrow \bar{Z}_2 \simeq \mathbb{P}^1$ has no base point on the proper transform of S. So we may assume from the very beginning that $\bar{V}_1$ is a good completion of V with respect to q_1 such that $\bar{q}_2$ has a unique base point $P \in F_\infty \setminus S$. Note that this new completion $\bar{V}_1$ of V is not necessarily minimal, but in any event the only possible (-1)-curve in the boundary B is a section S of $\bar{q}_1$. Observe also that, since $\bar{V}_1$ is obtained from a given good completion $\bar{V}$ of V with respect to $\bar{q}_1$ by means of elementary transformations with centers in F_∞, it follows that $\bar{V}_1 \setminus V$ is a zigzag if and only if $\bar{V} \setminus V$ is.

The following proposition proves the second implication of Theorem 2.1.

PROPOSITION 2.15. *Let V be a ML-surface with an $\mathbb{A}^1$-fibration $q \colon V \to Z \simeq \mathbb{A}^1$. Then, for any good completion $\bar{V}$ of V with respect to q as in 1.6, the divisor $B = \bar{V} \setminus V$ is a zigzag. Moreover, the $\mathbb{A}^1$-fibration q has at most one degenerate fiber.*

Proof. If $\bar{q}_2 \colon \bar{V} \to \bar{Z}_2$ is a morphism then, by Lemma 2.13, B is a zigzag and we are done. We now suppose that $\bar{q}_2$ is not a morphism. By 2.14 we can also suppose that the unique base point P of the linear pencil $\bar{q}_2$ belongs to $F_\infty \setminus S$. We let $\pi \colon \bar{W} \to \bar{V}_1$ be a minimal resolution of the base points of $\bar{q}_2$ and denote by $\tilde{q}_2 \colon \bar{W} \to \bar{Z}_2$ the $\mathbb{P}^1$-fibration that lifts $\bar{q}_2$. The last (-1)-curve arising from this elimination procedure gives rise to a section S_2 of $\tilde{q}_2$, and it is a unique (-1)-curve in $\pi^{-1}(P)$. Since $\bar{q}_2|_{S \cup G} = \infty$, the proper transform of $S \cup G$ in $\bar{W}$ is contained in the fiber $\tilde{q}_2^{-1}(\infty)$. If $\tilde{T}_2$ is a general fiber of $\tilde{q}_2$, then the point $\tilde{T}_2 \setminus T_2$ belongs to $\pi^{-1}(P)$. It follows that the proper transform of F_∞ in $\bar{W}$ is disjoint from $\tilde{T}_2$ and thus is contained in a fiber of $\tilde{q}_2$. Since $P \in F_\infty \setminus S$, we know that the proper transform of $B = F_\infty \cup S \cup G$ is connected and so is contained in $\tilde{q}_2^{-1}(\infty) \subset \bar{W}_{\mathrm{reg}}$. As $\tilde{q}_2^{-1}(\infty) \subset \bar{W} \setminus V$ is then degenerate, by Lemma 1.2(8) it must contain a (-1)-curve. Since no such curve can be contained in $G \cup (\pi^{-1}(P) \cap \tilde{q}_2^{-1}(\infty))$, it follows that the proper transform of S or F_∞ is a (-1)-curve. Since these two curves meet and are contained in a maximal simple zigzag of $\tilde{q}_2^{-1}(\infty)$ that intersects the section S_2, we deduce from Lemma 1.5 that $\tilde{q}_2^{-1}(\infty) \cup S_2$ is a zigzag.

Therefore G is connected and is a zigzag, whence q has a unique degenerate fiber. It follows that $B = F_\infty \cup S \cup G$ is a zigzag. □

More generally, we have the following theorem.

THEOREM 2.16. *If V is an ML-surface, then the boundary divisor $C := \bar{V} \setminus V$ of any minimal completion $\bar{V}$ of V is a zigzag.*

The proof is worked out in 2.17–2.20. Recall (see Definition 1.1) that $\bar{V}$ is a minimal completion of V if and only if C is an SNC-divisor containing no (-1)-curve that meets at most two other irreducible components transversally in a single point. Since V is affine, C is connected. The $\mathbb{A}^1$-fibration $q_1 : V \to \mathbb{A}^1$ extends to a rational map $\bar{q}_1 : \bar{V} \dashrightarrow \mathbb{P}^1$ with at most one base point P on C.

LEMMA 2.17. *If $\bar{q}_1 : \bar{V} \to \mathbb{P}^1$ is a morphism, then C is a zigzag.*

Proof. Since the closure $\bar{T}_1$ of a general fiber T_1 of q_1 intersects C in a single point, it follows that there exists a unique irreducible component S of C that is a section of $\bar{q}_1$. If $C = S$ we are done.

If S is a terminal component of C, then $C \setminus S$ is connected and thus contained in a unique fiber F of $\bar{q}_1$. Moreover, since $\bar{q}_1^{-1}(\infty) \subset C$, we have $F = F_\infty = \bar{q}_1^{-1}(\infty)$ and $F_\infty = \overline{C \setminus S} \subset \bar{V}_{\mathrm{reg}}$. By the minimality of C, it then follows from Remark 1.3 that $\overline{C \setminus S}$ cannot contain a (-1)-curve. Hence the fiber F_∞ of $\bar{q}_1$ is nondegenerate and so $C = S \cup F_\infty$ is a zigzag with two components.

If S is not a terminal component of C, we denote by $G_1, \dots, G_n$ the connected components of $\overline{C \setminus S}$. Then every G_i is contained in a fiber F_i of $\bar{q}_1$, whence (using the same argument as before) G_i cannot contain a (-1)-curve. Since one of the F_i (say, F_n) is the fiber $F_\infty = \bar{q}_1^{-1}(\infty) \subset \bar{V}_{\mathrm{reg}}$, it follows that $G_n = F_\infty \simeq \mathbb{P}^1$. Hence $\bar{V}$ is a minimal good completion of V with respect to q_1 (see 1.6). Thus, according to Proposition 2.15, $n = 2$ and $C = F_\infty \cup S \cup G_1$ is a zigzag. □

2.18. We may therefore suppose in the sequel that that neither q_1 nor q_2 extends to a morphism on $\bar{V}$. Let $P \in C$ be the unique base point of the rational map $\bar{q}_1 : \bar{V} \dashrightarrow \bar{Z}_1$ and let $\pi : \bar{W} \to \bar{V}$ be a minimal resolution of P. That is, $\bar{q}_1$ lifts to a $\mathbb{P}^1$-fibration $\tilde{q}_1 : \bar{W} \to \bar{Z}_1$ and $\pi^{-1}(P)$ contains a unique (-1)-curve S that is a section of $\tilde{q}_1$. Since the closure $\tilde{T}_1$ in $\bar{W}$ of a general fiber of q_1 meets $\pi^{-1}(C)$ in a single point, it follows that every connected component of the proper transform C' of C in $\bar{W}$ is contained in a fiber of $\tilde{q}_1$.

LEMMA 2.19. *If P belongs to just one irreducible component D of C, then C is a zigzag.*

Proof. In this case C' is connected and so is contained in the fiber F_∞ of $\tilde{q}_1$. Thus $F_\infty \subset \bar{W}_{\mathrm{reg}}$ does not contain (-1)-curves except perhaps for the proper transform D' of D. Indeed, by the minimality of C and of P's resolution, such a (-1)-curve in F_∞ (different from D') must be a ramification point of C', which is excluded by 1.3. If the fiber F_∞ does not contain a (-1)-curve then it is nondegenerate, so $F_\infty = D'$ with $(D'^2) = 0$ and $C = D$ is a zigzag.

We now suppose that $C \neq D$. Then F_∞ is degenerate, and D' is a unique (-1)-curve in F_∞. Therefore D is a terminal component of C, for otherwise D' is a ramification vertex of $\Gamma(F_\infty \cup S)$, which contradicts Remark 1.3. If C is not a zigzag then $F_\infty \cup S$ is not a zigzag either, since it contains C'. To eliminate this possibility, we note that if $\pi^{-1}(P)$ is a zigzag then D' is contained in a maximal simple zigzag of F_∞ that meets S; otherwise, D' is contained in a maximal double zigzag of F_∞. But both these possibilities are excluded by Lemma 1.5. Hence C is a zigzag. □

The following lemma completes the proof of Theorem 2.16.

LEMMA 2.20. *In the situation of 2.18, if P belongs to two irreducible components (say, D_1 and D_2) of C, then C is a zigzag.*

Proof. In this case the proper transform C' of C has two connected components C'_1 and C'_2, where D'_i is a terminal component of C'_i, $i = 1, 2$. Therefore, either C' is entirely contained in the fiber F_∞ of $\tilde{q}_1$ or there exists another fiber F_1 of $\tilde{q}_1$ such that say $C'_1 \subset F_1$ and $C'_2 \subset F_\infty$. The latter happens if and only if $D'_1 \cup \pi^{-1}(P) \cup D'_2$ is a zigzag. Indeed, otherwise—at some step $k \geq 2$ of the resolution procedure—we must have blown up a simple point $P_k \in \pi_{k-1}^{-1}(D_1 \cup D_2)$ into an exceptional component E_k. Because E_k is terminal in the dual graph of $D'_1 \cup \pi_k^{-1}(P) \cup D'_2$, we then conclude that $\pi_k^{-1}(C) \setminus E_k$ is connected. Since all further blow-ups have their centers over E_k, it follows that the proper transform of $\pi_{k-1}^{-1}(C)$ in $\bar{W}$ contains C' and is connected. This implies that C' is entirely contained in a fiber of $\tilde{q}_1$.

1. We first suppose that C' is contained in the fiber $F_\infty \subset \bar{W}_{\mathrm{reg}}$ of $\tilde{q}_1$. For $i = 1, 2$ we consider the shortest paths joining D'_i to S in the tree $\Gamma(F_\infty \cup S)$, and we denote by D_0 the vertex where they meet. Since C' is not connected, it follows that D_0 is contained in $\overline{F_\infty \setminus C'}$ and is a ramification vertex of $\Gamma(F_\infty \cup S)$. Moreover, F_∞ is degenerate, and the only possible (-1)-curves in F_∞ are D'_1 and D'_2. Hence, by Lemma 1.5(1) at least one of the D'_i (say, D'_1) is a (-1)-curve contained in a maximal terminal zigzag of F_∞. Clearly, this zigzag also contains C'_1. This implies that D_1 is not a ramification vertex of $\Gamma(C)$, for otherwise D'_1 is a ramification vertex of $\Gamma(F_\infty \cup S)$, which contradicts Remark 1.3.

If either C'_2 is not a zigzag or D_2 is a ramification vertex of $\Gamma(C)$, then D'_1 is a unique (-1)-curve contained in a maximal terminal zigzag of F_∞ and there exists a ramification vertex H' of $\Gamma(F_\infty \cup S)$ that is not contained in the shortest path joining D'_1 to S in $\Gamma(F_\infty \cup S)$. Indeed, in the first case C'_2 is not a zigzag, whence it contains such a ramification vertex H'; in the second case, we can choose $H' = D'_2$. This contradicts Lemma 1.5(2) and so C'_2 is a zigzag and D_2 is not a ramification vertex of $\Gamma(C)$. Thus, $C = C_1 \cup C_2$ is a zigzag, too.

2. We now suppose that C' is not entirely contained in a fiber of $\tilde{q}_1$. Thus $D'_1 \cup \pi^{-1}(P) \cup D'_2$ is a zigzag. Moreover, there exist two connected components (say, G_1 and G_2) of $\overline{\pi^{-1}(C) \setminus S}$ as well as two different fibers F_1 and $F_2 = F_\infty$

of $\tilde{q}_1$ such that $C_i' \subset G_i \subset F_i$ for $i = 1, 2$. Since $F_\infty \subset \bar{W}_{\mathrm{reg}}$, we can deduce (similarly as in Lemma 2.19) that $F_\infty \cup S$ is a zigzag. This implies that C_2 is a zigzag and that D_2 is not a ramification vertex of $\Gamma(C)$. We let $\tau_\infty \colon \bar{W} \to \bar{W}_1$ be the contraction of F_∞ to a nondegenerate fiber of a $\mathbb{P}^1$-fibration. That is, $\tau_\infty(F_\infty) \simeq \mathbb{P}^1$ is a nondegenerate fiber $\hat{F}_\infty = \hat{q}_1^{-1}(\infty)$ of the resulting $\mathbb{P}^1$-fibration $\hat{q}_1 \colon \bar{W}_1 \to \bar{Z}_1$. Since the components of F_1 are not affected by this contraction, $\tau_\infty(D_1')$ is the only possible (-1)-curve in $\tau_\infty(G_1) \subset \hat{F}_1 = \tau_\infty(F_1)$. Moreover, since $D_1' \cup \pi^{-1}(P) \cup D_2'$ is a zigzag, $\tau_\infty(D_1')$ is contained in the maximal simple zigzag of $\tau_\infty(G_1)$ that meets the section $\hat{S}$ of $\hat{q}_1$.

If $\tau_\infty(G_1)$ contains no (-1)-curve, then $\bar{W}_1$ is a good completion of V with respect to q_1 and it follows (from Proposition 2.15) that $\tau_\infty(G_1 \cup \hat{S})$ is a zigzag. Thus C_1 also is a zigzag and D_1 is not a ramification vertex of $\Gamma(C)$.

Otherwise $\tau_\infty(D_1')$ is a unique (-1)-curve of $\tau_\infty(G_1)$. Starting with $\tau_\infty(D_1')$, we can successively contract the (-1)-curves that arise in $\tau_\infty(G_1)$ to obtain a minimal good completion $\bar{W}_2$ of V with respect to q_1. Hence the image of $\tau_\infty(G_1 \cup \hat{S})$ in $\bar{W}_2$ is a zigzag, by Proposition 2.15. Because $\tau_\infty(D_1')$ is contained in a maximal simple zigzag of $\tau_\infty(G_1)$ that meets $\hat{S}$, none of the possible ramification vertices of $\tau_\infty(G_1 \cup \hat{S})$ has been eliminated by the foregoing contractions. This means that $\tau_\infty(G_1 \cup \hat{S})$ is also a zigzag. Thus, C_1 is a zigzag and D_1 is not a ramification vertex of $\Gamma(C)$. Hence $C = C_1 \cup C_2$ is a zigzag, too. □

We complete our discussion with a characterization of the affine plane. We need the following lemma.

LEMMA 2.21 (see also [1]). *Let V be an ML-surface, and let $q_i \colon V \to Z_i \simeq \mathbb{A}^1$ $(i = 1, 2)$ be two $\mathbb{A}^1$-fibrations whose general fibers do not coincide. Then $\phi_{12} := q_1 \times q_2 \colon V \to \mathbb{A}^2$ is a surjective, quasifinite morphism.*

Proof. We let $\bar{V}$ be a good completion of V with respect to q_1 by a zigzag $B = G \cup S \cup F_\infty$ as in 2.12, and we denote by $\bar{q}_1 \colon \bar{V} \to \bar{Z}_1 \simeq \mathbb{P}^1$ the $\mathbb{P}^1$-fibration that extends q_1. If the $\mathbb{A}^1$-fibration $q_2 \colon V \to Z_2 \simeq \mathbb{A}^1$ extends to a $\mathbb{P}^1$-fibration $\bar{q}_2 \colon \bar{V} \to \bar{Z}_2 \simeq \mathbb{P}^1$, then $V \simeq \mathbb{A}^2$ by Lemma 2.13 and q_1 and q_2 are coordinates on V, which proves the assertion. So we may assume from now on that $\bar{q}_2 \colon \bar{V} \dashrightarrow \bar{Z}_2 \simeq \mathbb{P}^1$ is a linear pencil with a unique base point $P \in F_\infty \setminus S$ (see 2.14). Therefore, $\bar{q}_2|_{S \cup G} = \infty$ and $\bar{T}_2 \setminus T_2 = P$ for the closure $\bar{T}_2$ of a general fiber T_2 of q_2.

To prove that ϕ_{12} is quasifinite, it is sufficient to show that none of the irreducible components of a fiber of q_2 is contained in a fiber of q_1. Suppose on the contrary that there exists an irreducible component C of a fiber F_1 of q_1 that is contained in a fiber F_2 of q_2. If F_1 were a nondegenerate fiber of q_1, then its closure $\bar{F}_1 = \bar{C}$ in $\bar{V}$ would meet S in a single point P_1. Since $\bar{q}_2|_C$ is constant and finite and since $\bar{q}_2(\bar{C} \cap S) = \infty$, it follows that P_1 would be a base point of $\bar{q}_2$, which is impossible. Thus, by Proposition 2.15, F_1 is a unique degenerate fiber of q_1 and hence $\bar{C}$ meets G (see 2.12). Since $q_2|_C$ is constant and finite and since $\bar{q}_2(G \cap \bar{C}) = \infty$, it follows that $Q = G \cap \bar{C}$ is a base point of $\bar{q}_2$, which again is impossible. Hence there is no such curve C on V and so ϕ_{12} is quasifinite.

The normalization of every irreducible component C of a fiber of q_2 is isomorphic to $\mathbb{A}^1$ by Lemma 1.7. Hence the restriction of q_1 to C is nonconstant and surjective and so $\phi\colon V \to \mathbb{A}^2$ is a surjection, as required. □

COROLLARY 2.22. *A normal affine surface V is isomorphic to $\mathbb{A}^2$ if and only if it admits two $\mathbb{A}^1$-fibrations whose general fibers meet in a single point.*

Proof. We let $q_i\colon V \to Z_i \simeq \mathbb{A}^1$ $(i = 1, 2)$ be two $\mathbb{A}^1$-fibrations as before. The morphism $\phi := q_1 \times q_2\colon V \to \mathbb{A}^2$ is surjective and quasifinite by Lemma 2.21. Since the general fibers of q_1 and q_2 meet in a single point, ϕ must be birational. By the Zariski main theorem (see e.g. [9]) there exists a factorization

$$\phi\colon V \xrightarrow{\phi'} X \xrightarrow{u} \mathbb{A}^2,$$

where ϕ' is an open immersion and $u\colon X \to \mathbb{A}^2$ is finite and birational, whence an isomorphism. Then $\phi' = \phi$ is also an isomorphism since ϕ is surjective. □

To conclude, we provide a series of examples of nonsingular affine surfaces in $\mathbb{A}^3$ with easily computable completions, distinguishing ML-surfaces among these.

EXAMPLE 2.23. Consider the hypersurface $V := V_{P,n}$ of $\mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ with equation $x^n z = P(y)$, where $P = \prod_{i=1}^{r}(y - y_i)$ is a polynomial with r simple roots. Given $n > 1$, let us show that if V has a nontrivial Makar-Limanov invariant then $r \geq 2$ (see [11] and [12] for a purely algebraic proof of this result). By Theorem 2.16 it is sufficient to find a minimal completion $\bar{V}$ of V such that $B = \bar{V} \setminus V$ is not a zigzag. We proceed as follows.

Consider the birational morphism

$$\begin{aligned} V &\xrightarrow{\phi_0} V_0 := \mathbb{A}^2 \subset \bar{V}_0 := \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y, z) &\longmapsto (x, y), \end{aligned}$$

and let $S = \mathbb{P}^1 \times \{\infty\} \subset \bar{V}_0$, $F_\infty = \{\infty\} \times \mathbb{P}^1 \subset \bar{V}_0$, and $F_0 = \{0\} \times \mathbb{P}^1 \subset \bar{V}_0$. We denote by $C_i \subset V$ the curve $x = 0$, $y = y_i$ for $1 \leq i \leq r$; these are the irreducible components of the degenerate fiber of the $\mathbb{A}^1$-fibration $pr_1 \circ \phi_0$ on V. Then $\phi_0(C_i) = (0, y_i) \subset F_0$ is a point. We let $V_i = V \setminus \big(\bigcup_{j \neq i} C_i\big) \simeq \mathbb{A}^2$ with coordinates (x, u_i), where $u_i := x^{-n}(y - y_i) = \prod_{j \neq i}(y - y_j)^{-1} z$. The restriction of ϕ_0 to V_i is given by

$$\begin{aligned} V_i \simeq \mathbb{A}^2 &\xrightarrow{\phi_0|_{V_i}} \mathbb{A}^2, \\ (x, u_i) &\longmapsto (x, x^n u_i + y_i). \end{aligned}$$

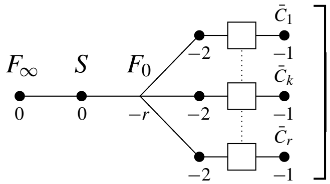
Now let $\pi_1\colon \bar{V}_1 \to \bar{V}_0$ be the blow-up of $\bar{V}_0$ in the points $\phi_0(C_i)$ with exceptional divisors $E_{1,i}$ for $1 \leq i \leq r$. Clearly $\phi_0\colon V \to V_0$ lifts to a morphism $\phi_1\colon V \to V_1 \subset \bar{V}_1 \setminus (F'_\infty \cup S' \cup F'_0)$. Moreover,

$$\phi_1(V_i) \subset V_{1,i} := \bar{V}_1 \Bigg\backslash \left(F'_\infty \cup S' \cup F'_0 \cup \left(\bigcup_{j \neq i} E_{1,j}\right)\right) \simeq \mathbb{A}^2,$$

and ϕ_1 is given by

$$V_i \simeq \mathbb{A}^2 \xrightarrow{\phi_1|_{V_i}} \mathbb{A}^2,$$
$$(x, u_i) \longmapsto (x, x^{n-1}u_i + y_{1,i}),$$

for some $y_{1,i} \in \mathbb{C}$. Iterating the construction, after n blow-ups as before we arrive at an open embedding $\phi_n \colon V \hookrightarrow \bar{V}$ of V in a nonsingular projective surface $\bar{V}$. Let $B = \bar{V} \setminus V$. If $\bar{C}_i$ denotes the closure of $\phi_n(C_i)$ in $\bar{V}$, then the dual graph of $B \cup \bar{C}_1 \cup \cdots \cup \bar{C}_r$ has the following structure:



where $\square$ stands for a linear chain of (-2)-curves of length $n-3$ (provided $n \geq 3$).

Thus $\bar{V}$ is a minimal completion of V by an SNC-divisor B, which is a zigzag iff $r = 1$. Hence by Propositions 2.10 and 2.15, V has a trivial Makar-Limanov invariant iff $n = 1$ or $n > 1$ and $r = 1$. The interested reader is referred to [1; 4] for a more systematic study of these surfaces and to [2] for more explicit examples of surfaces with $\mathbb{C}_+$-actions.

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