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On the Boundary Accumulation Points for the Holomorphic Automorphism Groups

Jisoo Byun

1. Introduction

For a domain Ω in $\mathbb{C}^n$, we denote by $\mathrm{Aut}(\Omega)$ the group of holomorphic automorphisms of Ω. It is obvious that $\mathrm{Aut}(\Omega)$ is a topological group with respect to the law of composition and the compact-open topology. In particular, it is a theorem of H. Cartan that $\mathrm{Aut}(\Omega)$ is in fact a Lie group if Ω is bounded.

In light of the outstanding question "Which domains possess noncompact automorphism group?" there is much interest focused upon the existence and nonexistence of orbits of the automorphism group action accumulating at a given boundary point. The well-known Greene–Krantz conjecture belongs to such a line of research. In this paper, we discuss the *finite*-type boundary points that repel automorphism orbits.

Denote by $\tau_\Sigma(q)$ the D'Angelo type (see [10]) at q of the real hypersurface Σ in $\mathbb{C}^n$. Now we present our main theorem.

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^2$. Assume that there exists a point $p \in \partial\Omega$ admitting an open neighborhood U in $\mathbb{C}^2$ satisfying the conditions*

(1) *the boundary $\partial\Omega$ is C^∞-smooth pseudoconvex in U, and*
(2) *$\tau_{\partial\Omega}(q) < \tau_{\partial\Omega}(p) < \infty$ for every $q \in U \cap \partial\Omega \setminus \{p\}$.*

Then there are no automorphism orbits in Ω accumulating at p.

In particular, this implies the following theorem of Byun.

Theorem 1.2 [8]. *In the Kohn–Nirenberg domain defined by the inequality*

$$\operatorname{Re} w + |zw|^2 + |z|^8 + \frac{15}{7}|z|^2 \operatorname{Re} z^6 < 0,$$

there does not exist any automorphism group orbit accumulating at the origin.

Although several experts commented that the nonconvexifiability of the boundary at the origin should be the reason for the conclusion of Byun's theorem, it is now apparent by our main theorem that the essential reason in fact lies elsewhere: any isolated maximum finite-type boundary point repels automorphism orbits.

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Another important theme in the study of pseudoconvex domains concerns the set $S(\Omega)$ of all orbit accumulation boundary points of the given domain Ω. Fu, Isaev, and Krantz [11] analyzed the structure of $S(\Omega)$ for the case when Ω is Reinhardt, showing that $S(\Omega)$ forms a manifold of odd dimension between 1 and $2n-1$ inclusive. Isaev and Krantz [15] showed that it is a perfect set if Ω is a bounded pseudoconvex domain with finite type boundary and if $S(\Omega)$ contains at least three points. Huang [14] analyzed the rank of Levi forms at the boundary accumulation point. We present in this article a resonant result in a more general situation, without assuming the boundedness or the Rienhardtness condition.

THEOREM 1.3. *Let Ω be a domain in $\mathbb{C}^2$ with a boundary point $p \in \partial\Omega$ admitting an open neighborhood U in which $\partial\Omega$ is C^∞-smooth pseudoconvex of finite type in the sense of D'Angelo. If p is an automorphism orbit accumulation point, then p is also an accumulation point of the set $S(\Omega)$.*

COROLLARY 1.4. *If Ω is a pseudoconvex domain with C^∞-smooth boundary of D'Angelo finite type, then $S(\Omega)$ is a perfect set.*

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TERMINOLOGY AND NOTATION. Throughout this paper, (z, w) denotes the standard Euclidean coordinate system of $\mathbb{C}^2$.

From now on, $P(z)$ will be understood as a real-valued polynomial. Then we define the following concepts.

(1) $a_j \preceq b_j$ if and only if there is a $C > 0$ such that $a_j \leq Cb_j$ for all j, where a_j, b_j are positive real numbers.
(2) $\mathcal{P}_{2k}$ denotes the set of all real-valued polynomials with degree less than $2k+1$ that has no harmonic terms. This is a finite-dimensional $\mathbb{R}$-vector space. $\|P(z)\|$ represents the maximum of absolute values of all coefficients of the polynomial $P(z)$. Naturally, $\|\cdot\|$ defines a norm on $\mathcal{P}_{2k}$.
(3) $\mathcal{H}_{2k}$ is a set of all homogeneous subharmonic polynomials of degree $2k$ without harmonic terms.
(4) $P(z) \sim Q(z)$ (i.e., P is equivalent to Q) if and only if there exist a real number $\gamma > 0$ as well as a holomorphic polynomial $r(z)$ and an automorphism $g(z)$ of $\mathbb{C}$ such that

$$P(z) = \gamma \operatorname{Re} r(z) + \gamma Q(g(z)).$$

(5) $P^*(z)$ denotes a polynomial $Q(z)$ whose terms consist of all terms in $P(z)$ *except* harmonic terms.
(6) $P^h(z) = P(z) - P^*(z)$.
(7) $M_Q = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Re} w + Q(z) < 0\}$, where $Q \in \mathcal{P}_{2k}$ is called the *model domain* of Q.

(8) Let D and D_j be domains in $\mathbb{C}^2$. We say that D_j *converges* to D if the following two properties hold:
  (a) for every compact set $K \subset D$, there is an integer N such that $K \subset D_j$ if $j \geq N$;
  (b) if K is a compact set that is contained in D_j for all sufficiently large j, then $K \subset D$.

  This is, in effect, equivalent to the local Hausdorff set convergence of D_j to D in $\mathbb{C}^2$.

2. The Scaling Method and Its Convergence

The content of this section follows the treatment in [6]. However, for the sake of a smooth exposition, we choose to include some details.

Let D be a domain in $\mathbb{C}^2$ and let $p_0 \in \partial D$. Assume that ∂D is smooth and pseudoconvex of finite type in a neighborhood of p_0. Let $2k$ be the type of ∂D at p_0 in the sense of D'Angelo [10]. We further assume that $p_0 = (0, 0)$ and that $\operatorname{Re} \frac{\partial}{\partial w}$ is an outward normal vector to ∂D at p_0.

Let $\{p_j\}$ be a sequence of points in D that converges to p_0. For every j sufficiently large, there exists a unique point $q_j \in \partial D$ such that

$$p_j + (0, \varepsilon_j) = q_j, \quad \varepsilon_j > 0.$$

Write $q_j = (a_j, b_j) \in \partial D$. According to [9, Prop. 1.1], there is a neighborhood U of p_0 such that

$$(z, w) \in D \cap U \iff \operatorname{Re} w + H(z) + R(\operatorname{Im} w, z) < 0,$$

where $H \in \mathcal{H}_{2k}$ and $R(\operatorname{Im} w, z) \sim o(|z|^{2k} + |\operatorname{Im} w|)$. Consider a sequence of maps Φ_j defined by

$$(\Phi_j : \mathbb{C}^2 \to \mathbb{C}^2) : (z, w) \mapsto (z - a_j, w - b_j + c_j(z - a_j)),$$

where $c_j \in \mathbb{C}$ is chosen so that the complex tangent line of $\partial \Phi_j(D)$ at p_0 is $\{(z, w) \mid w = 0\}$. Then we have

$$\Phi_j(q_j) = (0, 0), \qquad \Phi_j(p_j) = (0, -\varepsilon_j),$$

and

$$(z, w) \in \Phi_j(D \cap U) \iff \operatorname{Re} w + \sum_{l=2}^{2k} P_{l,j}(z) + R_j(\operatorname{Im} w, z) < 0,$$

where the $P_{l,j}(z)$ are homogeneous polynomials of degree l and where $R_j = O(|z|^{2k+1} + |\operatorname{Im} w|)$.

We may let $P_{l,j}(z) = P^*_{l,j}(z) + P^h_{l,j}(z)$. Since $P^h_{l,j}$ is harmonic, there exist $\alpha_{l,j} \in \mathbb{C}$ such that

$$P^h_{l,j}(z) = \alpha_{l,j} z^l + \overline{\alpha_{l,j} z^l}.$$

For each j, we define a map Ψ_j by

$$(\Psi_j : \mathbb{C}^2 \to \mathbb{C}^2) : (z, w) \mapsto \left(z, w + 2\sum_{l=2}^{2k} \alpha_{l,j} z^l(z)\right).$$

Then

$$(z, w) \in \Psi_j \circ \Phi_j(D \cap U) \iff \operatorname{Re} w + \sum_{l=2}^{2k} P^*_{l,j}(z) + R'_j(\operatorname{Im} w, z) < 0.$$

We choose $\delta_j > 0$ such that

$$\left\| \varepsilon_j^{-1} \sum_{l=2}^{2k} P^*_{l,j}(\delta_j z) \right\| = 1. \tag{2.1}$$

Since $\lim P^*_{l,j} = 0$ $(l < 2k)$ and $\lim P^*_{2k,j} = H$, we obtain $\delta_j \preceq \varepsilon_j^{1/2k}$.

Finally, the scaling map Λ_j is defined by

$$\Lambda_j(z, w) = \left(\frac{z}{\delta_j}, \frac{w}{\varepsilon_j}\right)$$

for every $(z, w) \in \mathbb{C}^2$. Then

$$\begin{aligned}(z, w) &\in \Lambda_j \circ \Psi_j \circ \Phi_j(D \cap U)\\ &\iff \operatorname{Re} w + \frac{1}{\varepsilon_j} \sum_{l=2}^{2k} P^*_{l,j}(\delta_j z) + \frac{1}{\varepsilon_j} R'_j(\varepsilon_j \operatorname{Im} w, \delta_j z) < 0.\end{aligned}$$

Set $T_j = \Lambda_j \circ \Psi_j \circ \Phi_j$. Since the norm is fixed independently of j, the sequence of polynomials $\left\{\varepsilon_j^{-1} \sum_{l=2}^{2k} P^*_{l,j}(\delta_j z)\right\}$ converges, choosing a subsequence if necessary, to some polynomial Q of degree at most $2k$ (we call Q the *limit polynomial* with respect to p_j). Since the remainder term of the defining function tends to zero as $j \to \infty$, we see that the sequence of domains $T_j(D \cap U)$ converges to a model domain

$$\{(z, w) \mid \operatorname{Re} w + Q(z) < 0\}$$

and that $\|Q\| = 1$. The following theorem by Berteloot guarantees that the sequence of such scaling maps forms a normal family and that the limit polynomial becomes a homogeneous polynomial.

THEOREM 2.1 [6]. *Let Ω be a domain in $\mathbb{C}^2$ and let p_0 be a point on $\partial\Omega$. Suppose that $\partial\Omega$ is of class C^∞, pseudoconvex, and of finite type in a neighborhood of p_0. Let $\phi_j \in \operatorname{Aut}(\Omega)$ satisfy*

$$\lim_{j\to\infty} \phi_j(z_0) = p_0$$

for a point $z_0 \in \Omega$. Then Ω is biholomorphically equivalent to the model domain M_H, where $H \in \mathcal{H}_{\tau_{\partial\Omega}(p_0)}$.

3. Proofs

Let Δ denote the open unit disc in $\mathbb{C}$.

LEMMA 3.1. *Let Ω be a domain in $\mathbb{C}^n$ and let p be a boundary point of Ω. Assume that there exist an open neighborhood U of p in $\mathbb{C}^n$ and a sequence of injective proper holomorphic maps $g_j \colon \Delta \to \Omega$ satisfying the following conditions*:

(1) *$U \cap \Omega$ is pseudoconvex*; *and*
(2) $\lim_{j\to\infty} g_j(0) = p$.

Let $E = \overline{\bigcup_{j=1}^{\infty} g_j(\Delta)} \cap \partial\Omega$. Then p is not an isolated point of E.

Proof. Expecting a contradiction, we suppose that p is an isolated point of E. So, there exists a $\delta > 0$ such that

$$\|p - q\| \geq \delta \quad \forall q \in E \setminus \{p\}.$$

Choosing a subsequence if necessary, we may assume that

$$g_j(0) \in B\left(p; \frac{\delta}{4}\right) \quad \forall j = 1, 2, \ldots.$$

Now, for each t with $\delta/3 < t < 2\delta/3$, we let

$$S_t = \{z \in \mathbb{C}^n \mid \|p - z\| = t\},$$
$$B_t = \{z \in \mathbb{C}^n \mid \|p - z\| < t\}.$$

Applying the Morse–Sard theorem to the smooth map

$$F \colon g_m(D) \to \mathbb{R}$$

defined by $F(\zeta) = \|\zeta - p\|^2$, we infer that, *for each positive integer m and for almost all values for t, the set $S_t \cap g_m(\Delta)$ is in fact a real one-dimensional manifold without boundary.* We can also conclude that $\overline{B_t} \cap g_m(D)$ is a smooth submanifold with boundary in $S_t \cap g_m(\Delta)$.

Moreover, we shall verify that $S_t \cap g_m(\Delta)$ is a compact set. Since it is a bounded subset of $\mathbb{C}^n$, we need only prove that it is closed.

Let $x \in \overline{S_t \cap g_m(\Delta)}$. Then we have a sequence $x_k \in S_t \cap g_m(\Delta)$ such that $x_k \to x$ as $k \to \infty$. Since $x_k \in g_m(\Delta)$, there exists a sequence $\zeta_k \in \Delta$ such that $g_m(\zeta_k) = x_k$ for each k. Because $\bar{\Delta}$ is compact, there exist a point $\zeta \in \bar{\Delta}$ and a subsequence ζ_{k_l} such that $\zeta_{k_l} \to \zeta$ as $l \to \infty$. If $\zeta \in \partial\Delta$ then, by virtue of the properness of g_m, we have

$$\begin{aligned} x &= \lim_{l\to\infty} x_{k_l} \\ &= \lim_{l\to\infty} g_m(\zeta_{k_l}) \\ &\in \partial\Omega. \end{aligned}$$

This leads us to $\|x - p\| = t$ and $x \in E \setminus \{p\}$. Since this is impossible, we must have $\zeta \in \Delta$. Hence $x = g_m(\zeta) \in g_m(\Delta)$. This implies that $S_t \cap g_m(\Delta)$ is closed.

Let X_m denote the connected component of $B_t \cap g_m(\Delta)$ with $g_m(0) \in X_m$, and let $G_m = g_m^{-1}(X_m)$. Then G_m is a domain in Δ.

CLAIM. *There exists a simple closed curve γ_m in Δ satisfying*

(1) $g_m(\gamma_m) \subset S_t$ *and*
(2) 0 *is an interior point of* γ_m.

Proof of Claim. Notice that 0 is an interior point of G_m and that $g_m^{-1}(S_t \cap g_m(\Delta))$ is a finite union of circles. Therefore, the claim follows by the argument principle as soon as we prove that $\partial G_m \subset g_m^{-1}(S_t \cap g_m(\Delta))$.

Step 1: If $x \in \partial G_m \cap \Delta$, then $x \in g_m^{-1}(S_t \cap g_m(\Delta))$. Since $x \in \Delta \cap \partial G_m$, we obtain $g_m(x) \in \overline{X_m} \setminus X_m$. Since $g_m(x) \in \Omega$, we have $g_m(x) \in \Omega \cap \overline{X_m} \setminus X_m \subset S_t \cap g_m(\Delta)$. Therefore, $x \in g_m^{-1}(S_t \cap g_m(\Delta))$.

Step 2: $\partial G_m \subset g_m^{-1}(S_t \cap g_m(\Delta))$. In order to prove this, we suppose that there is a point $x \in \partial\Delta \cap \partial G_m$. We can choose $r_0 < 1$ so that $g_m^{-1}(S_t \cap g_m(\Delta)) \subset \{z \in \mathbb{C} \mid |z| < r_0\}$. Now, we have only to show that $C_r = \{z \in \mathbb{C} \mid |z| = r\}$ is contained in G_m if $r_0 < r < 1$. The existence of $x \in \partial\Delta$ guarantees that $C_r \cap G_m$ is nonempty. If $C_r \not\subset G_m$, then there is a point $q \in C_r \cap \partial G_m$. By step 1, $q \in g_m^{-1}(S_t \cap g_m(\Delta))$ and $|q| = r$. This is a contradiction. □

Now let γ_m be the simply connected curve selected in the preceding claim. Let Γ_m be the set of all interior points of γ_m, which contains the origin by construction. We then choose a Riemann map $f_m \colon \Delta \to \Gamma_m$ with $f_m(0) = 0$. Then the composition $h_m = g_m \circ f_m \colon \Delta \to \Omega$ defines an analytic disc satisfying

$$h_m(0) = g_m(0),$$
$$h_m(\partial\Delta) = g_m(\gamma_m) \subset S_t.$$

Since Ω is pseudoconvex, it follows that $-\log d(x, \partial\Omega) \colon \Omega \to \mathbb{R}$ is a plurisubharmonic function, where $d(x, \partial\Omega)$ denotes the Euclidean distance from x to the boundary $\partial\Omega$. Consequently,

$$\begin{aligned} d(h_m(\partial\Delta), \partial\Omega) &= d(h_m(\Delta), \partial\Omega) \\ &\le d(h_m(0), \partial\Omega) \\ &\le d(g_m(0), p) \\ &\to 0 \ \text{ as } m \to \infty. \end{aligned}$$

In particular, there exist $q_m \in h_m(\partial\Delta) \subset S_t$ such that

$$d(q_m, \partial\Omega) = d(h_m(\partial\Delta), \partial\Omega) \to 0$$

as $m \to \infty$. Thus, there exists a $q \in S_t \cap \partial\Omega$ such that q is a limit point of the sequence q_m. Namely, we have found a point $q \in E \cap S_t$. By the choice of t, we have arrived at the desired contradiction, completing the proof of Lemma 3.1. □

Proof of Theorem 1.3. Suppose there exist a sequence $\{\phi_j\} \subset \operatorname{Aut}\Omega$ and a point $x \in \Omega$ such that

$$\lim_{j\to\infty} \phi_j(x) = p.$$

By [6], there is a biholomorphism Ψ between Ω and the domain $M_H = \{(z, w) \mid \operatorname{Re} w + H(z, \bar{z}) < 0\}$, where H is a homogenous subharmonic polynomial without harmonic terms. Also, we have $\deg H = \tau_{\partial\Omega}(p)$. Since H is homogeneous, the model domain M_H has a two-parameter family of the following automorphisms:

$$l_t(z, w) = (z, w + it) \quad \text{for all } \ t \in \mathbb{R},$$

$$h_\lambda(z, w) = (\lambda z, \lambda^{2k} w) \quad \text{for all } \ \lambda > 0.$$

The plane $P = \{(0, w) \mid \operatorname{Re} w < 0\}$ is contained in the orbit of $(0, -1)$ by the action of $\operatorname{Aut}(M_H)$. Define an injective proper holomorphic map $\mu \colon \Delta \to M_H$ by

$$\mu(z) = \left(0, \frac{z-1}{z+1}\right)$$

for every $z \in \Delta$. We consider a sequence of injective proper holomorphic maps $g_j := \phi_j \circ \Psi^{-1} \circ \mu$ from the unit disc into Ω satisfying

$$\begin{aligned} \phi_j \circ \Psi^{-1} \circ \mu(0) &= \phi_j \circ \Psi^{-1}(0, -1) \\ &= \phi_j(q), \end{aligned}$$

where $q = \Psi^{-1}(0, -1)$. Moreover $g_m(\Delta)$ is contained in the orbit of q by an action of $\operatorname{Aut}(\Omega)$. By Lemma 3.1, p is not an isolated point of $\overline{\bigcup_{j=1}^{\infty} g_j(\Delta)} \cap \partial\Omega$. Since $g_m(\Delta)$ is contained in the orbit of q, it follows that $\overline{\bigcup_{j=1}^{\infty} g_j(\Delta)} \cap \partial\Omega \subset S(\Omega)$. Therefore, Theorem 1.3 is proved. $\square$

Proof of Theorem 1.1. Since p is an accumulation point of $S(\Omega)$, there is a point $\tilde{p} \in U \cap S(\Omega)$ with $\tau(\tilde{p}) < \tau(p)$. Applying the scaling theory to this point $\tilde{p}$, we have another model domain M_Q and a biholomorphism Φ between Ω and M_Q. By [20], we see that $\deg Q = \deg H$ since $M_H \simeq M_Q$. Thus, $\tau(\tilde{p}) = \tau(p)$. This contradiction completes the proof. $\square$

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Department of Mathematics
Seoul National University
Seoul, 151-747
Republic of Korea

jisoo@math.snu.ac.kr