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Some Real and Unreal Enumerative Geometry for Flag Manifolds

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To Bill Fulton on the occasion of his 60th birthday

Introduction

For us, enumerative geometry is concerned with counting the geometric figures of some kind that have specified position with respect to some fixed, but general, figures. For instance, how many lines in space are incident on four general (fixed) lines? (Answer: 2.) Of the figures having specified positions with respect to fixed *real* figures, some will be real while the rest occur in complex conjugate pairs, and the distribution between these two types depends subtly upon the configuration of the fixed figures. Fulton [12] asked how many solutions to such a problem of enumerative geometry can be real and later with Pragacz [14] reiterated this question in the context of flag manifolds.

It is interesting that, in every known case, *all* solutions may be real. These include the classical problem of 3264 plane conics tangent to 5 plane conics [30], the 40 positions of the Stewart platform of robotics [5], the 12 lines mutually tangent to 4 spheres [24], the 12 rational plane cubics meeting 8 points in the plane [19], all problems of enumerating linear subspaces of a vector space satisfying special Schubert conditions [34], and certain problems of enumerating rational curves in Grassmannians [36]. These last two examples give infinitely many families of nontrivial enumerative problems for which all solutions may be real. They were motivated by recent, spectacular computations [9; 40] and a very interesting conjecture of Shapiro and Shapiro [35], and were proved using an idea from a homotopy continuation algorithm [16; 17].

We first formalize the method of constructing real solutions introduced in [34; 36], which will help extend these reality results to other enumerative problems. This method gives lower bounds on the maximum number of real solutions to some enumerative problems, in the spirit of [18; 38]. We then apply this theory to two families of enumerative problems, one on classical (SL_n) flag manifolds and the other on Grassmannians of maximal isotropic subspaces in an orthogonal vector space, showing that all solutions may be real. These techniques allow us to prove the opposite result—that we may have no real solutions—for a family of enumerative problems on the Lagrangian Grassmannian. Finally, we suggest a further problem to study concerning this method.

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1. Schubert Induction

Let $\mathbb{K}$ be a field and let $\mathbb{A}^1$ be an affine 1-space over $\mathbb{K}$. A *Bruhat decomposition* of an irreducible algebraic variety X defined over $\mathbb{K}$ is a finite decomposition

$$X = \coprod_{w \in I} X_w^\circ$$

satisfying the following conditions.

(1) Each stratum X_w° is a (Zariski) locally closed irreducible subvariety defined over $\mathbb{K}$ whose closure $\overline{X_w^\circ}$ is a union of some strata X_v°.

(2) There is a unique 0-dimensional stratum $X_{\hat{0}}^\circ$.

(3) For any $w, v \in I$, the intersection $\overline{X_w^\circ} \cap \overline{X_v^\circ}$ is a union of some strata X_u°.

Since X is irreducible, there is a unique largest stratum $X_{\hat{1}}^\circ$. Such spaces X include flag manifolds, where the X_w° are the Schubert cells in the Bruhat decomposition defined with respect to a fixed flag, as well as the quantum Grassmannian [29; 36; 37]. These are the only examples to which the theory developed here presently applies, but we expect it (or a variant) will apply to other varieties that have such a Bruhat decomposition, particularly some spherical varieties [21] and analogs of the quantum Grassmannian for other flag manifolds. The key to applying this theory is to find certain geometrically interesting families $\mathcal{Y} \to \mathbb{A}^1$ of subvarieties having special properties with respect to the Bruhat decomposition (which we describe below).

Suppose X has a Bruhat decomposition. Define the *Schubert variety* X_w to be the closure of the stratum X_w°. The *Bruhat order* on I is the order induced by inclusion of Schubert varieties: $u \le v$ if $X_u \subset X_v$. For flag manifolds G/P, these are the Schubert varieties and the Bruhat order on W/W_P; for the quantum Grassmannian, its quantum Schubert varieties and quantum Bruhat order. Set $|w| := \dim X_w$. For flag manifolds G/P, if $\tau \in W$ is a minimal representative of the coset $w \in W/W_P$ then $|w| = \ell(\tau)$, its length in the Coxeter group W.

Let $\mathcal{Y} \to \mathbb{A}^1$ be a flat family of codimension-c subvarieties of X. For $s \in \mathbb{A}^1$, let $Y(s)$ be the fiber of $\mathcal{Y}$ over the point s. We say that $\mathcal{Y}$ *respects* the Bruhat decomposition if, for every $w \in I$, the (scheme-theoretic) limit $\lim_{s\to 0}(Y(s) \cap X_w)$ is supported on a union of Schubert subvarieties X_v of codimension c in X_w. This implies that the intersection $Y(s) \cap X_w$ is proper for generic $s \in \mathbb{A}^1$. That is, the intersection is proper when s is the generic point of the scheme $\mathbb{A}^1$.

Given such a family, we have the cycle-theoretic equality

$$\lim_{s\to 0}(Y(s) \cap X_w) = \sum_{v \prec_{\mathcal{Y}} w} m^v_{\mathcal{Y},w} X_v .$$

Here $v \prec_{\mathcal{Y}} w$ if X_v is a component of the support of $\lim_{s\to 0}(Y(s) \cap X_w)$, and the multiplicity $m^v_{\mathcal{Y},w}$ is the length of the local ring of the limit scheme $\lim_{s\to 0}(Y(s) \cap X_w)$ at the generic point of X_v. Thus, if X is smooth then we have the formula

$$[X_w] \cdot [Y] = \sum_{w \prec_{\mathcal{Y}} v} m^v_{\mathcal{Y},w}[X_v] \tag{1}$$

in the Chow [10; 12] or cohomology ring of X. Here $[Z]$ denotes the cycle class of a subvariety Z, and Y is any fiber of the family $\mathcal{Y}$. When these multiplicities $m^v_{\mathcal{Y},w}$ are all 1 (or 0), we call $\mathcal{Y}$ a *multiplicity-free family.*

A collection of families $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ respecting the Bruhat decomposition of X is in *general position* (with respect to the Bruhat decomposition) if, for all $w \in I$, general $s_1, \ldots, s_r \in \mathbb{A}^1$, and $1 \le k \le r$, the intersection

$$Y_1(s_1) \cap Y_2(s_2) \cap \cdots \cap Y_k(s_k) \cap X_w \qquad (2)$$

is *proper* in that either it is empty or else it has dimension $|w| - \sum_{i=1}^k c_i$, where c_i is the codimension in X of the fibers of $\mathcal{Y}_i$. Note that, more generally (and intuitively), we could require that the intersection

$$Y_{i_1}(s_{i_1}) \cap Y_{i_2}(s_{i_2}) \cap \cdots \cap Y_{i_k}(s_{i_k}) \cap X_w$$

be proper for any k-subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$. We do not use this added generality, although it does hold for every application we have of this theory. By general points $s_1, \ldots, s_k \in \mathbb{A}^1$, we mean general in the sense of algebraic geometry: there is a nonempty open subset of the scheme $\mathbb{A}^k$ consisting of points $(s_1, \ldots, s_k)$ for which the intersection (2) is proper. When $c_1 + \cdots + c_k = |w|$, the intersection (2) is 0-dimensional. Determining its degree is a problem in enumerative geometry.

We model this problem with combinatorics. Given a collection of families $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ in general position respecting the Bruhat decomposition with $|\hat{1}| = \dim X = c_1 + \cdots + c_r$, we construct the *multiplicity poset* of this enumerative problem. Write $\lessdot_i$ for $\lessdot_{\mathcal{Y}_i}$. The elements of rank k in the multiplicity poset are those $w \in I$ for which there is a chain

$$\hat{0} \lessdot_1 w_1 \lessdot_2 w_2 \lessdot_3 \cdots \lessdot_{k-1} w_{k-1} \lessdot_k w_k = w. \qquad (3)$$

The cover relation between the $(i-1)$th and ith ranks is $\lessdot_i$. The *multiplicity* of a chain (3) is the product of the multiplicities $m^{w_{i-1}}_{\mathcal{Y}_i, w_i}$ of the covers in that chain. Let $\deg(w)$ be the sum of the multiplicities of all chains (3) from $\hat{0}$ to w. If X is smooth and $|w| = c_1 + \cdots + c_k$ then $\deg(w)$ is the degree of the intersection (2), since it is proper, and so we have the formula (1).

THEOREM 1.1. *Suppose X has a Bruhat decomposition, $\mathcal{Y}_1, \ldots, \mathcal{Y}_r$ are a collection of multiplicity-free families of subvarieties over $\mathbb{A}^1$ in general position, and each family respects this Bruhat decomposition. Let c_i be the codimension of the fibers of $\mathcal{Y}_i$.*

(1) *For every k and every $w \in I$ with $|w| = c_1 + \cdots + c_k$, the intersection* (2) *is transverse for general $s_1, \ldots, s_k \in \mathbb{A}^1$ and has degree* $\deg(w)$. *In particular, when $\mathbb{K}$ is algebraically closed, such an intersection consists of* $\deg(w)$ *reduced points.*

(2) *When $\mathbb{K} = \mathbb{R}$, there exist real numbers $s_1, \ldots, s_r$ such that, for every k and every $w \in I$ with $|w| = c_1 + \cdots + c_k$, the intersection* (2) *is transverse with all points real.*

Proof. For the first statement, we work in the algebraic closure of $\mathbb{K}$, so that the degree of a transverse, 0-dimensional intersection is simply the number of points in that intersection. We argue by induction on k.

When $k = 1$, suppose $|w| = c_1$. Since $\mathcal{Y}_1$ is a multiplicity-free family that respects the Bruhat decomposition, we have

$$\lim_{s\to 0}(Y_1(s) \cap X_w) = m^{\hat{0}}_{\mathcal{Y}_1,w} X_{\hat{0}},$$

with $m^{\hat{0}}_{\mathcal{Y}_1,w}$ either 0 or 1. Thus, for generic $s \in \mathbb{A}^1$, either $Y_1(s) \cap X_w$ is empty or it is a single reduced point and hence transverse. Note here that $\deg(w) = m^{\hat{0}}_{\mathcal{Y}_1,w}$.

Suppose we have proven statement (1) of the theorem for $k < l$. Let $|w| = c_1 + \cdots + c_l$. We claim that, for generic $s_1, \ldots, s_{l-1}$, the intersection

$$Y_1(s_1) \cap \cdots \cap Y_{l-1}(s_{l-1}) \cap \sum_{v \prec_l w} X_v \tag{4}$$

is transverse and consists of $\deg(w)$ points. Its degree is $\deg(w)$, because $\deg(w)$ satisfies the recursion $\deg(w) = \sum_{v\prec_l w} \deg(v)$. Transversality will follow if no two summands have a point in common. Consider the intersection of two summands

$$Y_1(s_1) \cap \cdots \cap Y_{l-1}(s_{l-1}) \cap (X_u \cap X_v). \tag{5}$$

Since $X_u \cap X_v$ is a union of Schubert varieties of dimensions less than $|w| - c_l$ and since the collection of families $\mathcal{Y}_1, \ldots, \mathcal{Y}_{l-1}$ is in general position, it follows that (5) is empty for generic $s_1, \ldots, s_{l-1}$, which proves transversality. Consider now the family defined by $Y_l(s) \cap X_w$ for s generic. Since $\sum_{v\prec_l w} X_v$ is the fiber of this family at $s = 0$ and since the intersection (4) is transverse and consists of $\deg(w)$ points, for generic $s_l \in \mathbb{A}^1$ the intersection

$$Y_1(s_1) \cap Y_2(s_2) \cap \cdots \cap Y_{l-1}(s_{l-1}) \cap Y_l(s_l) \cap X_w \tag{6}$$

is transverse and consists of $\deg(w)$ points.

For statement (2) of the theorem, we inductively construct real numbers $s_1, \ldots, s_r$ having the properties that: (a) for any $w \in I$ and k with $|w| = c_1 + \cdots + c_k$, the intersection (2) is transverse with all points real; and (b) if $|w| < c_1 + \cdots + c_k$, then (2) is empty. Suppose $|w| = c_1$. Since for general $s \in \mathbb{R}$ the intersection $X_w \cap Y_1(s)$ is either empty or consists of a single reduced point, we may select a general $s \in \mathbb{R}$ with the additional property that if $|v| < c_1$ then $Y_1(s) \cap X_v$ is empty.

Suppose now that we have constructed $s_1, \ldots, s_{l-1} \in \mathbb{R}$ such that (a) if $|v| = c_1 + \cdots + c_{l-1}$ then the intersection $Y_1(s_1) \cap \cdots \cap Y_{l-1}(s_{l-1}) \cap X_v$ is transverse with all points real, and (b) if $|v| < c_1 + \cdots + c_{l-1}$ then this intersection is empty. Let $|w| = c_1 + \cdots + c_l$. Then the intersection (4) is transverse with all points real. Thus there exists $\varepsilon_w > 0$ such that, if $0 < s_l \le \varepsilon_w$, then the intersection (6) is transverse with all points real. Set $s_l = \min\{\varepsilon_w : |w| = c_1 + \cdots + c_l\}$. Since it is an open condition (in the usual topology) on the l-tuple $(s_1, \ldots, s_l) \in \mathbb{R}^l$ for the intersection (6) to be transverse with all points real and since there are finitely many $w \in I$, we may (if necessary) choose a nearby l-tuple of points such that, if $|w| < c_1 + \cdots + c_l$, then the intersection (6) is empty. □

REMARK 1.2. The statement and proof of Theorem 1.1 are a generalization of the main results of [34, Thm. 1] and [36, Thms. 3.1 and 3.2], and they constitute a stronger version of the theory presented in [33]. (Part (1) generalizes [6, Thm. 8.3].) We call this method of proof *Schubert induction*. The proof of the second statement is based upon the fact that small (real) perturbations of a transverse intersection preserve transversality as well as the number of real and complex points in that intersection. In principle, this leads to an optimal numerical homotopy continuation algorithm for finding all complex points in the intersection (2). A construction and correctness proof of such an algorithm could be modeled on the Pieri homotopy algorithm of [16; 17].

REMARK 1.3. The first statement of Theorem 1.1 gives an elementary proof of generic transversality for some enumerative problems involving multiplicity-free families. In characteristic 0, it is an alternative to Kleiman's transversality theorem [20] and could provide a basis to prove generic transversality in arbitrary characteristic, extending the result in [32] that the intersection of general Schubert varieties in a Grassmannian of 2-planes is generically transverse in any characteristic. It also provides a proof that $\deg(w)$ is the intersection number—without using Chow or cohomology rings, the traditional tool in enumerative geometry.

REMARK 1.4. If the families $\mathcal{Y}_i$ are not multiplicity-free, then we can prove a lower bound on the maximum number of real solutions. A (saturated) chain (3) in the multiplicity poset is *odd* if it has odd multiplicity. Let $\mathrm{odd}(w)$ count the odd chains from $\hat{0}$ to w in the multiplicity poset.

THEOREM 1.5. *Suppose X has a Bruhat decomposition, $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ are a collection of families of subvarieties over $\mathbb{A}^1$ in general position, and each family respects this Bruhat decomposition. Let c_i be the codimension of the fibers of $\mathcal{Y}_i$.*

(1) *Suppose $\mathbb{K}$ is algebraically closed. For every k, every $w \in I$ with $|w| = c_1 + \cdots + c_k$, and general $s_1, \dots, s_k \in \mathbb{A}^1$, the* 0*-dimensional intersection* (2) *has degree* $\deg(w)$.
(2) *When $\mathbb{K} = \mathbb{R}$, there exist real numbers $s_1, \dots, s_r$ such that, for every k and every $w \in I$ with $|w| = c_1 + \cdots + c_k$, the intersection* (2) *is* 0*-dimensional and has at least* $\mathrm{odd}(w)$ *real points.*

Sketch of Proof. For the first statement, the same arguments as in the proof of Theorem 1.1 suffice if we replace the phrase "transverse and consists of $\deg(w)$ points" throughout by "proper and has degree $\deg(w)$". For statement (2) of the theorem, observe that a point in the intersection $Y_1(s_1) \cap \cdots \cap Y_{l-1}(s_{l-1}) \cap X_v$ becomes $m^v_{\mathcal{Y}_l,w}$ points counted with multiplicity in (6), when s_l is a small real number. If this multiplicity $m^v_{\mathcal{Y}_l,w}$ is odd and the original point was real, then at least one of these $m^v_{\mathcal{Y}_l,w}$ points are real. □

The lower bound of Theorem 1.5 is the analog of the bound for sparse polynomial systems in terms of alternating mixed cells [18; 26; 39]. Like that bound, it is not sharp [23; 39]. We give an example using the notation of Section 2. The Grassmannian of 3-planes in $\mathbb{C}^7$ has a Bruhat decomposition indexed by triples $1 \leq \alpha_1 < \alpha_2 < \alpha_3 \leq 7$ of integers. Let $r = 4$ and suppose that each family $\mathcal{Y}_i$ is the family of Schubert varieties $X_{357}F_\cdot(s)$, where $F_\cdot(s)$ is the flag of subspaces osculating a real rational normal curve. In [35, Thm. 3.9(iii)] it is proven that if s, t, u, v are distinct real points then

$$Y(s) \cap Y(t) \cap Y(u) \cap Y(v)$$

is transverse and consists of eight real points. However, there are five chains in the multiplicity poset; four of them odd and one of multiplicity 4. In Figure 1, we show the Hasse diagram of this multiplicity poset, indicating multiplicities greater than 1.

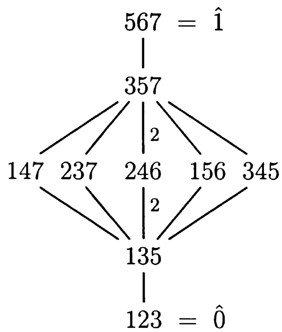

Figure 1 The multiplicity poset

Despite this lack of sharpness, Theorem 1.5 gives new results for the Grassmannian. In [7], Eisenbud and Harris show that families of Schubert subvarieties of a Grassmannian defined by flags of subspaces osculating a rational normal curve respect the Bruhat decomposition given by any such osculating flag, and any collection is in general position. Consequently, given a collection of these families with $\text{odd}(w) > 0$, it follows that $\text{odd}(w)$ is a nontrivial lower bound (new if the Schubert varieties are not special Schubert varieties) on the number of real points in such a 0-dimensional intersection of these Schubert varieties.

For example, in the Grassmannian of 3-planes in $\mathbb{C}^{r+3}$, let $Y(s)$ be the Schubert variety consisting of 3-planes having nontrivial intersection with $F_{r-1}(s)$ and whose linear span with $F_{r+1}(s)$ is not all of $\mathbb{C}^{r+3}$. (Here, $F_i(s)$ is the i-dimensional subspace osculating a real rational normal curve γ at the point $\gamma(s)$.) This Schubert variety has codimension 3. Consider the enumerative problem given by intersecting r of these Schubert varieties. Table 1 gives both the number of solutions $(\deg(\hat{1}))$ and the number of odd chains $(\text{odd}(\hat{1}))$ in the multiplicity poset for $r = 2, 3, \ldots, 11$. The case $r = 4$ we have already described. The conjecture of Shapiro

Table 1 Number of solutions and odd chains

r	2	3	4	5	6	7	8	9	10	11
$\deg(\hat{1})$	1	2	8	32	145	702	3,598	19,280	107,160	614,000
$\mathrm{odd}(\hat{1})$	1	0	4	6	37	116	534	2,128	9,512	41,656

and Shapiro [35] asserts that all solutions for any r-tuple of distinct real points will be real, which is stronger than the consequence of Theorem 1.5 that there is some r-tuple of real points for which there will be at least as many real solutions as odd chains.

REMARK 1.6. The requirement that there be a unique 0-dimensional stratum in a Bruhat decomposition may be relaxed. We could allow several 0-dimensional strata X_z for $z \in Z$, each consisting of a single $\mathbb{K}$-rational point. This is the case for toric varieties [11] and more generally for spherical varieties [21].

If we define the multiplicity poset as before, then Z indexes its minimal elements. We define the intersection number $\deg(w)$ and the bound $\mathrm{odd}(w)$ using chains

$$z \prec_1 w_1 \prec_2 w_2 \prec_3 \cdots \prec_x w_k = w \quad \text{with} \quad z \in Z.$$

Then almost the same proof as we gave for Theorem 1.1 proves the same statement in this new context. We do not yet know of any applications of this extension of Theorem 1.1, but we expect that some will be found.

2. The Classical Flag Manifolds

Fix integers $n \geq m > 0$ and a sequence $\mathbf{d}\colon 0 < d_1 < \cdots < d_m < n$ of integers. A *partial flag of type* $\mathbf{d}$ is a sequence of linear subspaces

$$E_{d_1} \subset E_{d_2} \subset \cdots \subset E_{d_m} \subset \mathbb{C}^n$$

with $\dim E_i = d_i$ for each $i = 1, \ldots, m$. The *flag manifold* $\mathrm{Fl}_{\mathbf{d}}$ is the collection of all partial flags of type $\mathbf{d}$. This manifold is the homogeneous space $\mathrm{SL}(n, \mathbb{C})/P_{\mathbf{d}}$, where $P_{\mathbf{d}}$ is the parabolic subgroup of $\mathrm{SL}(n, \mathbb{C})$ defined by the simple roots *not* indexed by $\{d_1, \ldots, d_m\}$. See [3] or [13] for further information on partial flag varieties.

A fixed *complete flag* $F_{\bullet}$ ($F_1 \subset \cdots \subset F_n = \mathbb{C}^n$ with $\dim F_i = i$) induces a Bruhat decomposition of $\mathrm{Fl}_{\mathbf{d}}$

$$\mathrm{Fl}_{\mathbf{d}} = \coprod X_w^\circ F_{\bullet} \tag{7}$$

indexed by those permutations $w = w_1 \ldots w_n$ in the symmetric group $\mathcal{S}_n$ whose descent set $\{i \mid w_i > w_{i+1}\}$ is a subset of $\{d_1, \ldots, d_m\}$. Write $I_{\mathbf{d}}$ for this set of permutations. Then $|w| = \ell(w)$, as $I_{\mathbf{d}}$ is the set of minimal coset representatives for $W_{P_{\mathbf{d}}}$. The Schubert variety $X_w F_{\bullet}$ is the closure of the Schubert cell $X_w^\circ F_{\bullet}$.

Fix any real rational normal curve $\gamma : \mathbb{C} \to \mathbb{C}^n$, which is a map given by $\gamma : s \mapsto (p_1(s), \ldots, p_n(s))$ and where $p_1, \ldots, p_n$ are a basis for the space of real polynomials of degree less than n. All real rational normal curves are isomorphic by a real linear transformation. For any $s \in \mathbb{C}$, let $F_\bullet(s)$ be the complete flag of subspaces osculating the curve γ at the point $\gamma(s)$. The dimension-i subspace $F_i(s)$ of $F_\bullet(s)$ is the linear span of the vectors $\gamma(s)$ and $\gamma'(s) := \frac{d}{ds}\gamma(s), \ldots, \gamma^{(i-1)}(s)$.

For each $i = 1, \ldots, m$, we have *simple Schubert variety* $X_i F_\bullet$ of $\mathrm{Fl}_{\mathbf{d}}$. Geometrically,

$$X_i F_\bullet := \{E_\bullet \in \mathrm{Fl}_{\mathbf{d}} \mid E_{d_i} \cap F_{n-d_i} \neq \{0\}\}.$$

We call these "simple" Schubert varieties, for they give simple (codimension-1) conditions on partial flags in $\mathrm{Fl}_{\mathbf{d}}$. Let $\mathcal{X}_i \to \mathbb{A}^1$ be the family whose fiber over $s \in \mathbb{A}^1$ is $X_i F_\bullet(s)$. We study these families.

THEOREM 2.1. *Let* $\mathbf{d} = 0 < d_1 < \cdots < d_m < n$ *be a sequence of integers. For any* $i = 1, \ldots, m$, *the family* $\mathcal{X}_i \to \mathbb{A}^1$ *of simple Schubert varieties is a multiplicity-free family that respects the Bruhat decomposition of* $\mathrm{Fl}_{\mathbf{d}}$ *given by the flag* $F_\bullet(0)$.

Any collection of these families of simple Schubert varieties is in general position.

We shall prove Theorem 2.1 shortly. First, by Theorem 1.1, we deduce the following corollary.

COROLLARY 2.2. *Let* $w \in I_{\mathbf{d}}$ *and set* $r := |w| = \dim X_w$. *Then, for any list of numbers* $i_1, \ldots, i_r \in \{1, \ldots, m\}$, *there exist real numbers* $s_1, \ldots, s_r$ *such that*

$$X_w F_\bullet(0) \cap X_{i_1} F_\bullet(s_1) \cap \cdots \cap X_{i_r} F_\bullet(s_r) \tag{8}$$

is transverse and consists only of real points.

This corollary generalizes the intersection of the main results of [34] and [36], which is the case of Corollary 2.2 for Grassmannians ($\mathbf{d} = d_1$ has only a single part). This result also extends (part of) Theorem 13 in [33], which states that, if $\mathbf{d} = 2 < n-2$ and $i_1, \ldots, i_r$ are any numbers from $\{2, n-2\}$ ($r = \dim \mathrm{Fl}_{\mathbf{d}} = 4n-12$), then there exist real flags $F_\bullet^1, \ldots, F_\bullet^r$ such that

$$X_{i_1} F_\bullet^1 \cap \cdots \cap X_{i_r} F_\bullet^r$$

is transverse and consists only of real points.

We recall some additional facts about the cohomology of the partial flag manifolds $\mathrm{Fl}_{\mathbf{d}}$. Each stratum $X_w^\circ F_\bullet$ is isomorphic to $\mathbb{C}^{|w|}$ and the Bruhat decomposition (7) is a cellular decomposition of $\mathrm{Fl}_{\mathbf{d}}$ into even- (real) dimensional cells. Let σ_w be the cohomology class Poincaré dual to the fundamental (homology) cycle of the Schubert variety $X_w F_\bullet$. Then these Schubert classes σ_w provide a basis for the integral cohomology ring $H^*(\mathrm{Fl}_{\mathbf{d}}, \mathbb{Z})$ with $\sigma_w \in H^{2c(w)}(\mathrm{Fl}_{\mathbf{d}}, \mathbb{Z})$, where $c(w)$ is the complex codimension of $X_w F_\bullet$ in $\mathrm{Fl}_{\mathbf{d}}$.

Let τ_i be the class of the simple Schubert variety $X_i F_\bullet$. There is a simple formula due to Monk [25] and Chevalley [4] expressing the product $\sigma_w \cdot \tau_i$ in terms of the basis of Schubert classes. Let $w \in I_{\mathbf{d}}$. Then

$$\sigma_w \cdot \tau_i = \sum \sigma_{w(j,k)},$$

where (j,k) is a transposition; the sum is over all $j \le d_i < k$, where

(1) $w_j > w_k$ and
(2) if $j < l < k$ then either $w_l > w_j$ or else $w_k > w_l$.

Write $w(j,k) \lessdot_i w$ for such $w(j,k)$. Note that, if $w \in I_{\mathbf{d}}$, then so is any $v \in \mathcal{S}_n$ with $v \lessdot_i w$ for any $i = 1, \ldots, m$.

Let $\mathrm{Gr}(d_i)$ be the Grassmannian of d_i-dimensional subspaces of $\mathbb{C}^n$. The association $E_\bullet \mapsto E_{d_i}$ induces a projection $\pi_i : \mathrm{Fl}_{\mathbf{d}} \to \mathrm{Gr}(d_i)$. The Grassmannian has a Bruhat decomposition

$$\mathrm{Gr}(d_i) = \coprod \Omega^\circ_\alpha F_\bullet$$

indexed by increasing sequences α of length d_i, $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{d_i} \le n$, with the Bruhat order given by componentwise comparison. Such an increasing sequence can be uniquely completed to a permutation $w(\alpha)$ whose only descent is at d_i. The map π_i respects the two Bruhat decompositions in that $\pi_i^{-1}(\Omega_\alpha) = X_{w(\alpha)} F_\bullet$ and $\pi_i(X_w F_\bullet) = \Omega_{\alpha(w)} F_\bullet$, where $\alpha(w)$ is the sequence obtained by writing $w_1, \ldots, w_{d_i}$ in increasing order. Thus, if $\beta < \alpha(w)$, then $X_w F_\bullet \cap \pi_i^{-1} \Omega_\beta F_\bullet$ is a union of proper Schubert subvarieties of $X_w F_\bullet$.

The Grassmannian has a distinguished simple Schubert variety

$$\Upsilon F_\bullet = \{E \in \mathrm{Gr}(d_i) \mid E \cap F_{n-d_i} \neq \{0\}\}.$$

This shows $X_i F_\bullet = \pi_i^{-1}(\Upsilon F_\bullet)$. We have $\Upsilon F_\bullet = \Omega_{(n-d_i, n-d_i+2, \ldots, n)} F_\bullet$.

We need the following useful fact about the families $\mathcal{X}_w \to \mathbb{A}^1$.

LEMMA 2.3. *For any $w \in I_{\mathbf{d}}$, we have $\bigcap_{s \in \mathbb{A}^1} X_w F_\bullet(s) = \emptyset$.*

Proof. Any Schubert variety $X_w F_\bullet$ is a subset of some simple Schubert variety $X_i F_\bullet = \pi_i^{-1} \Upsilon F_\bullet$. Thus it suffices to prove the lemma for the simple Schubert varieties $\Upsilon F_\bullet(s)$ of a Grassmannian. But this is simply a consequence of [6, Thm. 2.3]. □

Proof of Theorem 2.1. For any $w \in I_{\mathbf{d}}$, we consider the scheme-theoretic limit $\lim_{s \to 0}(X_w F_\bullet(0) \cap X_i F_\bullet(s))$. Since $X_i F_\bullet = \pi_i^{-1}(\Upsilon F_\bullet)$, for any $s \in \mathbb{C}$ we have

$$X_w F_\bullet(0) \cap X_i F_\bullet(s) = X_w F_\bullet(0) \cap \pi_i^{-1}(\Omega_{\alpha(w)} F_\bullet(0) \cap \Upsilon F_\bullet(s)),$$

since $\pi_i X_w F_\bullet(0) = \Omega_{\alpha(w)} F_\bullet(0)$. Thus, set-theoretically we have

$$\lim_{s \to 0}(X_w F_\bullet(0) \cap X_i F_\bullet(s)) \subset X_w F_\bullet(0) \cap \pi_i^{-1}\Big(\lim_{s \to 0}(\Omega_{\alpha(w)} F_\bullet(0) \cap \Upsilon F_\bullet(s))\Big).$$

But this second limit is $\bigcup_{\beta < \alpha(w)} \Omega_\beta F_\bullet$ by [6, Thm. 8.3]. Thus

$$\lim_{s\to 0}(X_w F_\bullet(0) \cap X_i F_\bullet(s)) \subset X_w F_\bullet(0) \cap \pi_i^{-1}\Bigg(\bigcup_{\beta<\alpha(w)} \Omega_\beta F_\bullet(0)\Bigg)$$
$$\subset \bigcup_{v\lessdot w} X_v F_\bullet(0),$$

set-theoretically.

Since the limit scheme $\lim_{s\to 0}(X_w F_\bullet(0) \cap X_i F_\bullet(s))$ is supported on this union of proper Schubert subvarieties of $X_w F_\bullet(0)$ and has dimension at least $\dim X_w F_\bullet(0) - 1$, its support must be a union of codimension-1 Schubert subvarieties of $X_w F_\bullet(0)$. Hence the family $\mathcal{X}_i \to \mathbb{A}^1$ respects the Bruhat decomposition, and we have

$$\lim_{s\to 0}(X_w F_\bullet(0) \cap X_i F_\bullet(s)) = \sum_{v\lessdot w} m_{i,w}^v X_v F_\bullet(0);$$

thus $\sigma_w \cdot \tau_i = \sum_{v\lessdot w} m_{i,w}^v \sigma_v$ in the Chow ring. Since the Schubert classes σ_v are linearly independent in the Chow ring, these multiplicities are either 0 or 1 by Monk's formula, and they are 1 precisely when $v \lessdot_i w$. Thus the family $\mathcal{X}_i \to \mathbb{A}^1$ is multiplicity-free, and we have proven the first statement of Theorem 2.1.

To complete the proof, let $\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_r}$ be a collection of families of simple Schubert varieties defined by the flags $F_\bullet(s)$. We show that this collection is in general position with respect to the Bruhat decomposition defined by the flag $F_\bullet(0)$. If not, then there is some index w and integer k with k minimal such that, for general $s_1, \dots, s_k \in \mathbb{C}$,

$$X_w F_\bullet(0) \cap X_{i_1} F_\bullet(s_1) \cap \cdots \cap X_{i_{k-1}} F_\bullet(s_{k-1}) \tag{9}$$

has dimension $|w| - k + 1$, but

$$X_w F_\bullet(0) \cap X_{i_1} F_\bullet(s_1) \cap \cdots \cap X_{i_k} F_\bullet(s_k)$$

has dimension exceeding $|w| - k$. Hence its dimension is $|w| - k + 1$. But then, for general $s \in \mathbb{C}$, some component of (9) lies in $X_{i_k} F_\bullet(s)$, which implies that this component lies in $X_{i_k} F_\bullet(s)$ for all $s \in \mathbb{C}$, contradicting Lemma 2.3. □

The previous paragraph provides a proof of the following useful lemma.

LEMMA 2.4. *Suppose a variety X has a Bruhat decomposition. Let $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ be a collection of codimension-1 families in X, each of which respects this Bruhat decomposition. If each family $\mathcal{Y}_i \to \mathbb{A}^1$ satisfies*

$$\bigcap_{s\in\mathbb{A}^1} Y_i(s) = \emptyset,$$

then the collection of families $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ is in general position.

A fruitful question is to ask how much freedom we have to select the real numbers $s_1, \dots, s_r$ of Corollary 2.2 so that all the points of the intersection (8) are real. In 1995, Boris Shapiro and Michael Shapiro conjectured that we have almost complete freedom: For generic real numbers $s_1, \dots, s_r$, all points of (8) are real. This remarkable conjecture is false in a very interesting way.

EXAMPLE 2.5. Let $n = 5$ and $\mathbf{d} : 2 < 3$ so that $\mathrm{Fl}_{\mathbf{d}}$ is the manifold of flags $E_2 \subset E_3 \subset \mathbb{C}^5$. This 8-dimensional flag manifold has two types of simple Schubert varieties $X_i F_{\bullet}$ for $i = 2, 3$, where $X_i F_{\bullet}$ consists of those flags $E_2 \subset E_3$ with $E_i \cap F_{5-i} \neq \{0\}$. Write $X_i(s)$ for $X_i F_{\bullet}(s)$. A calculation (using Maple and Singular [15]) shows that

$$X_2(-8) \cap X_3(-4) \cap X_2(-2) \cap X_3(-1) \cap X_2(1) \cap X_3(2) \cap X_2(4) \cap X_3(8)$$

is transverse and consists of twelve points, none of which are real.

Despite this counterexample, quite a lot may be salvaged from the conjecture of Shapiro and Shapiro. When the partial flag manifold $\mathrm{Fl}_{\mathbf{d}}$ is a Grassmannian, there are no known counterexamples, many enumerative problems, and choices of real numbers $s_1, \ldots, s_r$ for which all solutions are real [35]; in [8], the conjecture is proven for any Grassmannian of 2-planes. The general situation seems much subtler. In our counterexample, the points $\{-8, -2, 1, 4\}$ at which we evaluate X_2 alternate with the points $\{-4, -1, 2, 8\}$ at which we evaluate X_3. If, however, we evaluate X_2 at points $s_1, \ldots, s_4$ and X_3 at points $s_5, \ldots, s_8$ with $s_1 < s_2 < \cdots < s_8$, then we know of no examples with any points of intersection not real. We have checked this for all 24,310 subsets of eight numbers from

$$\{-6, -5, -4, -3, -2, -1, 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}.$$

On the other hand, if we evaluate X_2 at any four of the eight numbers

$$\{1, 2, 3^2, 4^3, 5^4, 6^5, 7^6, 8^7\}$$

and X_3 at the other four numbers, then all twelve points of intersection are real.

3. The Orthogonal Grassmannian

Let V be a vector space equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. A subspace $H \subset V$ is *isotropic* if the restriction of the form to H is identically zero. Isotropic subspaces have dimension at most half that of V. The *orthogonal Grassmannian* is the collection of all isotropic subspaces of V with maximal dimension. If the dimension of V is even then the orthogonal Grassmannian has two connected components, and each is isomorphic to the orthogonal Grassmannian for a generic hyperplane section of V; the isomorphism is given by intersecting with that hyperplane. Thus, it suffices to consider only the case when the dimension of V is odd.

When V has dimension $2n + 1$, a maximal isotropic subspace H of V has dimension n, and we write $\mathrm{OG}(n)$ for this orthogonal Grassmannian. To ensure that $\mathrm{OG}(n)$ has $\mathbb{K}$-rational points, we assume that V has a $\mathbb{K}$-basis $e_1, \ldots, e_{2n+1}$, for which our form is

$$\left\langle \sum x_i e_i, \sum y_j e_j \right\rangle = \sum x_i y_{2n+2-i}. \tag{10}$$

Then $\mathrm{OG}(n)$ is a homogeneous space of the (split) special orthogonal group $\mathrm{SO}(2n + 1, \mathbb{K}) = \mathrm{Aut}(V, \langle \cdot, \cdot \rangle)$. This algebraic manifold has dimension $\binom{n+1}{2}$.

An *isotropic flag* is a complete flag $F_{\bullet}$ of V such that (a) F_n is isotropic and (b) for every $i > n$, F_i is the annihilator of F_{2n+1-i}, that is, $\langle F_{2n+1-i}, F_i \rangle \equiv 0$. An isotropic flag induces a Bruhat decomposition

$$\mathrm{OG}(n) = \coprod X_\lambda^\circ F_{\bullet}$$

indexed by decreasing sequences λ of positive integers $n \geq \lambda_1 > \cdots > \lambda_l > 0$, called *strict partitions.* Let $\mathrm{SP}(n)$ denote this set of strict partitions. The Schubert variety $X_\lambda F_{\bullet}$ is the closure of $X_\lambda^\circ F_{\bullet}$ and has dimension $|\lambda| := \lambda_1 + \cdots + \lambda_l$. The Bruhat order is given by componentwise comparison: $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all i with both $\lambda_i, \mu_i > 0$. Figure 2 illustrates this Bruhat order when $n = 3$.

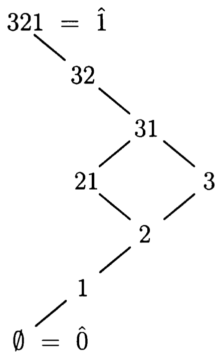

Figure 2 The Bruhat order for OG(3)

The unique simple Schubert variety of $\mathrm{OG}(n)$ is (set-theoretically)

$$YF_{\bullet} := \{H \in \mathrm{OG}(n) \mid H \cap F_{n+1} \neq \{0\}\}.$$

Thus $YF_{\bullet}$ is the set-theoretic intersection of $\mathrm{OG}(n)$ with the simple Schubert variety $\Upsilon F_{\bullet}$ of the ordinary Grassmannian $\mathrm{Gr}(n)$ of n-dimensional subspaces of V. The multiplicity of this intersection is 2 (see [14, p. 68]). We have $YF_{\bullet} = X_{(n,n-1,\ldots,2)}F_{\bullet}$. The Bruhat orders of these two Grassmannians ($\mathrm{OG}(n)$ and $\mathrm{Gr}(n)$) are related.

LEMMA 3.1. *Let $F_{\bullet}$ be a fixed isotropic flag in V. Then every Schubert cell $X_\lambda^\circ F_{\bullet}$ of* $\mathrm{OG}(n)$ *lies in a unique Schubert cell $\Omega_{\alpha(\lambda)}^\circ F_{\bullet}$ of* $\mathrm{Gr}(n)$. *Moreover, for any strict partition λ, we have the set-theoretic equality*

$$X_\lambda F_{\bullet} \cap \bigcup_{\beta \lessdot \alpha(\lambda)} \Omega_\beta F_{\bullet} = \bigcup_{\mu \lessdot \lambda} X_\mu F_{\bullet}.$$

Let τ be the cohomology class dual to the fundamental cycle of $YF_{\bullet}$, and let σ_λ be the class dual to the fundamental cycle of $X_\lambda F_{\bullet}$. The Chevalley formula for $\mathrm{OG}(n)$ is

$$\sigma_\lambda \cdot \tau = \sum_{\mu \lessdot \lambda} \sigma_\mu,$$

which is free of multiplicities.

Let $\mathbb{K} = \mathbb{C}$. As in Section 2, we study families of Schubert varieties defined by flags $F_\bullet(s)$ of isotropic subspaces osculating a real rational normal curve $\gamma : \mathbb{C} \to V$ at $\gamma(s)$. With our given form $\langle \cdot, \cdot \rangle$ and basis $e_1, \dots, e_{2n+1}$, one choice for a real rational normal curve γ whose flags of osculating subspaces are isotropic is

$$\gamma(s) = \left(1, s, \frac{s^2}{2}, \dots, \frac{s^n}{n!}, -\frac{s^{n+1}}{(n+1)!}, \frac{s^{n+2}}{(n+2)!}, \dots, (-1)^n \frac{s^{2n}}{(2n)!}\right).$$

THEOREM 3.2. *The family $\mathcal{Y} \to \mathbb{A}^1$ of simple Schubert varieties $YF_\bullet(s)$ is multiplicity-free and respects the Bruhat decomposition of* $\mathrm{OG}(n)$ *induced by the flag $F_\bullet(0)$.*

Any collection of these families of simple Schubert varieties is in general position.

We omit the proof of this theorem, which is nearly identical to the proof of Theorem 2.1. By Theorem 1.1, we deduce the following corollary.

COROLLARY 3.3. *Let $\lambda \in \mathrm{SP}(n)$. Then there exist real numbers $s_1, \dots, s_{|\lambda|}$ such that*

$$X_\lambda F_\bullet(0) \cap YF_\bullet(s_1) \cap \cdots \cap YF_\bullet(s_{|\lambda|}) \tag{11}$$

is transverse and consists only of real points.

By Theorem 3.2 and the Chevalley formula, for a strict partition λ and general complex numbers $s_1, \dots, s_{|\lambda|}$, the intersection (11) is transverse and consists of $\deg(\lambda)$ points, where $\deg(\lambda)$ is the number of chains in the Bruhat order from $0 = \hat{0}$ to λ.

As in Section 2, we may ask how much freedom we have to select the real numbers $s_1, \dots, s_{|\lambda|}$ of Corollary 3.3 so that all the points of the intersection (11) are real. When $n = 3$ and $\lambda = \hat{1}$ (Figure 2 shows that $|\hat{1}| = 6$ and $\deg(\hat{1}) = 2$), the discriminant of a polynomial formulation of this problem is

$$\sum_{w \in \mathcal{S}_6} (s_{w_1} - s_{w_2})^2 (s_{w_3} - s_{w_4})^2 (s_{w_5} - s_{w_6})^2,$$

which vanishes only when four of the s_i coincide. In particular, this implies that the number of real solutions does not depend upon the choice of the s_i (when the s_i are distinct). Hence both solutions are always real. When $n = 4$ and $\lambda = \hat{1}$ we have checked that, for each of the 1,001 choices of $s_1, \dots, s_{10}$ chosen from

$$\{1, 2, 3, 5, 7, 10, 11, 13, 15, 16, 17, 23, 29, 31\},$$

there are twelve ($= \deg(\hat{1})$) solutions, and all are real.

4. The Lagrangian Grassmannian

The *Lagrangian Grassmannian* $\mathrm{LG}(n)$ is the space of all Lagrangian (maximal isotropic) subspaces in a $2n$-dimensional vector space V equipped with a nondegenerate alternating form $\langle\cdot,\cdot\rangle$. Such Lagrangian subspaces have dimension n. In contrast to the flag manifolds $\mathrm{Fl}_{\mathbf{d}}$ and orthogonal Grassmannian $\mathrm{OG}(n)$, we show that there may be no real solutions for the enumerative problems we consider. We may assume that V has a $\mathbb{K}$-basis $e_1,\ldots,e_{2n}$, for which our form is

$$\Big\langle \sum x_i e_i, \sum y_j e_j \Big\rangle = \sum_{i=1}^{n} x_i y_{2n+1-i} - y_i x_{2n+1-i}.$$

An isotropic flag is a complete flag $F_{\bullet}$ of V such that F_n is Lagrangian and, for every $i>n$, F_i is the annihilator of F_{2n-i}; that is, $\langle F_{2n-i}, F_i\rangle \equiv 0$. An isotropic flag induces a Bruhat decomposition

$$\mathrm{LG}(n) = \coprod X^{\circ}_{\lambda} F_{\bullet}$$

indexed by strict partitions $\lambda \in \mathrm{SP}(n)$. The Schubert variety $X_\lambda F_{\bullet}$ is the closure of the Schubert cell $X^{\circ}_{\lambda}F_{\bullet}$ and has dimension $|\lambda|$. The Bruhat order is given (as for $\mathrm{OG}(n)$) by componentwise comparison of sequences. Although the $\mathrm{OG}(n)$ and $\mathrm{LG}(n)$ have identical Bruhat decompositions, they are very different spaces.

The unique simple Schubert variety of $\mathrm{LG}(n)$ is

$$YF_{\bullet} := \{H \in \mathrm{LG}(n) \mid H \cap F_n \neq \{0\}\}.$$

Thus $YF_{\bullet}$ is the set-theoretic intersection of $\mathrm{LG}(n)$ with the simple Schubert variety $\Upsilon F_{\bullet}$ of the ordinary Grassmannian $\mathrm{Gr}(n)$ of n-dimensional subspaces of V. This is generically transverse. As with $\mathrm{OG}(n)$, the strict partition indexing $YF_{\bullet}$ is $n, n-1, \ldots, 2$. The Bruhat decomposition of the Lagrangian Grassmannian is related to that of the ordinary Grassmannian in the same way as that of the orthogonal Grassmannian (see Lemma 3.1).

Let $\mathbb{K}=\mathbb{C}$. We study families of Schubert varieties defined by isotropic flags $F_{\bullet}(s)$ osculating a real rational normal curve $\gamma\colon \mathbb{C} \to V$ at $\gamma(s)$. With our given form $\langle\cdot,\cdot\rangle$ and basis $e_1,\ldots,e_{2n}$, one choice for γ whose osculating flags are isotropic is

$$\gamma(s) = \left(1, s, \frac{s^2}{2}, \ldots, \frac{s^n}{n!}, -\frac{s^{n+1}}{(n+1)!}, \frac{s^{n+2}}{(n+2)!}, \ldots, (-1)^{n-1}\frac{s^{2n-1}}{(2n-1)!}\right). \tag{12}$$

Let τ be the cohomology class dual to the fundamental cycle of $YF_{\bullet}$, and let σ_λ be the class dual to the fundamental cycle of $X_\lambda F_{\bullet}$. The Chevalley formula for $\mathrm{LG}(n)$ is

$$\sigma_\lambda \cdot \tau = \sum_{\mu \lessdot \lambda} m^{\mu}_{\lambda} \sigma_\mu,$$

where the multiplicity m_λ^μ is either 2 or 1, depending (respectively) upon whether or not the sequences λ and μ have the same length. Figure 3 shows the multiplicity posets for the enumerative problem in $\mathrm{LG}(2)$ and $\mathrm{LG}(3)$ given by the simple Schubert varieties $YF_\bullet(s)$.

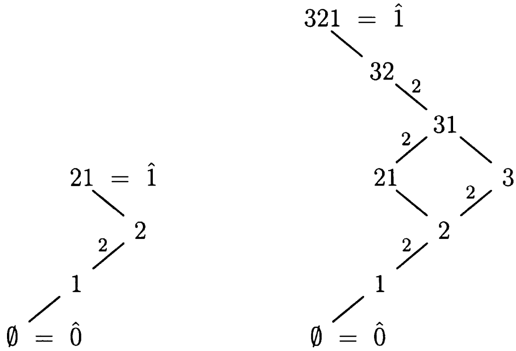

Figure 3 The multiplicity posets $\mathrm{LG}(2)$ and $\mathrm{LG}(3)$

As in Sections 2 and 3, the family $\mathcal{Y} \to \mathbb{A}^1$ whose fibers are the simple Schubert varieties $YF_\bullet(s)$ respects the Bruhat decomposition of $\mathrm{LG}(n)$, and any collection is in general position. From the Chevalley formula, we see that it is not multiplicity-free.

THEOREM 4.1. *The family $\mathcal{Y} \to \mathbb{A}^1$ of simple Schubert varieties $YF_\bullet(s)$ respects the Bruhat decomposition of $\mathrm{LG}(n)$ induced by the flag $F_\bullet(0)$.*

Any collection of families of simple Schubert varieties is in general position.

The proof of Theorem 4.1, like that of Theorem 3.2, is virtually identical to that of Theorem 2.1; hence, we omit it.

Since the family $\mathcal{Y}$ is not multiplicity-free, we do not have analogs of Corollaries 2.2 and 3.3 showing that all solutions may be real. When $|\lambda| > 1$, every chain (3) in the multiplicity poset contains the cover $1 < 2$, which has multiplicity 2 and so is even. Thus the refined statement of Theorem 1.5 does not guarantee any real solutions. We show that there may be no real solutions.

THEOREM 4.2. *Let λ be a strict partition with $|\lambda| = r > 1$. Then there exist real numbers $s_1, \ldots, s_r$ such that the intersection*

$$X_\lambda F_\bullet(0) \cap YF_\bullet(s_1) \cap \cdots \cap YF_\bullet(s_r) \tag{13}$$

is 0-dimensional and has no real points.

When $|\lambda|$ is 0 or 1, the degree $(\deg(\lambda))$ of the intersection (13) is 1 and so its only point is real. For all other λ, $\deg(\lambda)$ is even. Thus we cannot deduce that the intersection is transverse even for generic complex numbers $s_1, \ldots, s_{|\lambda|}$. However, the intersection has been transverse in every case we have computed.

Proof. We induct on the dimension $|\lambda|$ of $X_\lambda F_\cdot(0)$ with the initial case of $|\lambda| = 2$ proven in Example 4.3 (to follow). Suppose we have constructed $s_1, \ldots, s_{r-1} \in \mathbb{R}$ having the properties that: (a) for any μ, the intersection

$$YF_\cdot(s_1) \cap \cdots \cap YF_\cdot(s_{r-1}) \cap X_\mu F_\cdot(0)$$

is proper; and (b) when $|\mu| = r - 1$, it is (necessarily) 0-dimensional, has degree $\deg(\mu)$, and no real points.

Let λ be a strict partition with $|\lambda| = r$. Then the cycle

$$YF_\cdot(s_1) \cap \cdots \cap YF_\cdot(s_{r-1}) \cap \sum_{\mu \lessdot \lambda} m_\lambda^\mu X_\mu F_\cdot(0)$$

is 0-dimensional, has degree $\deg(\lambda)$, and no real points. Since the family $YF_\cdot(s)$ respects the Bruhat decomposition given by the flag $F_\cdot(0)$, we have

$$\lim_{s \to 0} (YF_\cdot(s) \cap X_\lambda F_\cdot(0)) = \sum_{\mu \lessdot \lambda} m_\lambda^\mu X_\mu F_\cdot(0).$$

Hence there is some $\varepsilon_\lambda > 0$ such that, if $0 < s_r \le \varepsilon_\lambda$, then the intersection (13) has dimension 0, degree $\deg(\lambda)$, and no real points.

Set $s_r = \min\{\varepsilon_\lambda : |\lambda| = r\}$. Since it is an open condition (in the usual topology) on $(s_1, \ldots, s_r) \in \mathbb{R}^r$ for the intersection (13) to be proper with no real points and since there are finitely many strict partitions, we may (if necessary) choose a nearby r-tuple of points such that the intersection (13) is proper for every strict partition λ. □

EXAMPLE 4.3. When $|\lambda| = 2$, we necessarily have $\lambda = 2$ and

$$X_2 F_\cdot = \{H \in \mathrm{LG}(n) \mid F_{n-2} \subset H \subset F_{n+2} \text{ and } \dim(H \cap F_n) \ge n - 1\},$$

which is the image of a simple Schubert variety $YG_\cdot = X_2 G_\cdot$ of $\mathrm{LG}(2)$ under an inclusion $\mathrm{LG}(2) \hookrightarrow \mathrm{LG}(n)$. Since F_{n+2} annihilates F_{n-2}, the alternating form $\langle\cdot,\cdot\rangle$ induces an alternating form on the 4-dimensional space $W := F_{n+2}/F_{n-2}$, and the flag $F_\cdot$ likewise induces an isotropic flag $G_\cdot$ in W. The inverse image in F_{n+2} of a Lagrangian subspace of W is a Lagrangian subspace of V contained in F_{n+2}. If we let $\varphi\colon \mathrm{LG}(2) \hookrightarrow \mathrm{LG}(n)$ be the induced map, then $X_2 F_\cdot = \varphi(X_2 G_\cdot)$.

Consider this map for the isotropic flag $F_\cdot(\infty)$ of subspaces osculating the point at infinity of γ. Then $(f_1, f_2, f_3, f_4) := (e_{n-1}, e_n, e_{n+1}, e_{n+2})$ provide a basis for W. An explicit calculation using the rational curve γ (12) shows that the flag induced on W is $G_\cdot(\infty)$, where $G_\cdot(s)$ is the flag of subspaces osculating the rational normal curve γ in W and where $\varphi^{-1}(YF_\cdot(s)) = YG_\cdot(s)$ for $s \in \mathbb{R}$. We describe the intersection

$$X_2 F_\cdot(\infty) \cap YF_\cdot(s) \cap YF_\cdot(t) = \varphi(X_2 G_\cdot(\infty) \cap YG_\cdot(s) \cap YG_\cdot(t))$$

when s and t are distinct real numbers.

The Lagrangian subspace $G_2(s)$ is the row space of the matrix

$$\begin{bmatrix} 1 & s & s^2/2 & -s^3/6 \\ 0 & 1 & s & -s^2/2 \end{bmatrix}.$$

The flag $G_\bullet(\infty)$ is $\langle f_4\rangle \subset \langle f_4, f_3\rangle \subset \langle f_4, f_3, f_2\rangle \subset W$. A Lagrangian subspace in the Schubert cell $X_2^\circ G_\bullet(\infty)$ is the row space of the matrix

$$\begin{bmatrix} 1 & x & 0 & y \\ 0 & 0 & 1 & -x \end{bmatrix},$$

where x and y are in $\mathbb{C}$. In this way, $\mathbb{C}^2$ gives coordinates for the Schubert cell. The condition for a Lagrangian subspace $H \in X_2^\circ G_\bullet(\infty)$ to meet $G_2(s)$, which locally defines the intersection $X_2 G_\bullet(\infty) \cap YG_\bullet(s)$, is

$$\det \begin{bmatrix} 1 & s & s^2/2 & -s^3/6 \\ 0 & 1 & s & -s^2/2 \\ 1 & x & 0 & y \\ 0 & 0 & 1 & -x \end{bmatrix} = -y + sx^2 - xs^2 + s^3/3 = 0.$$

If we call this polynomial $g(s)$, then the polynomial system $g(s) = g(t) = 0$ describes the intersection $X_2 G_\bullet(\infty) \cap YG_\bullet(s) \cap YG_\bullet(t)$. When $s \neq t$, the solutions are

$$x = \frac{s+t}{2} \pm (s-t)\frac{\sqrt{-3}}{6}$$
$$y = \frac{s^2t + st^2}{6} \pm (s^2t - st^2)\frac{\sqrt{-3}}{6},$$

which are not real for $s, t \in \mathbb{R}$.

To see that this gives the initial case of Theorem 4.2 we observe that, by reparameterizing the rational normal curve, we may move any three points to any other three points; thus it is no loss to use $X_2 F_\bullet(\infty)$ in place of $X_2 F_\bullet(0)$.

As before, we ask how much freedom we have to select the real numbers $s_1, \ldots, s_r$ of Theorem 4.2 so that no points in the intersection (11) are real. When $n = 2$ and s_1, s_2, s_3 are distinct and real, no point in (11) is real. This is a consequence of Example 4.3 because, when $n = 2$, we have $X_2 F_\bullet = YF_\bullet$. When $n = 3$ and $\lambda = \hat{1}$ we have checked that, for each of the 924 choices of $s_1, \ldots, s_6$ chosen from

$$\{1, 2, 3, 4, 5, 6, 11, 12, 13, 17, 19, 23\},$$

there are 16 ($= \deg(\hat{1})$) solutions and none are real.

5. Schubert Induction for General Schubert Varieties?

The results in Sections 2, 3, and 4 involve only codimension-1 Schubert varieties because we cannot show that families of general Schubert varieties given by flags osculating a rational normal curve respect the Bruhat decomposition or that any collection is in general position. Eisenbud and Harris [6, Thm. 8.1; 7] proved this for families $\Omega_\alpha F_\bullet(s)$ of arbitrary Schubert varieties on Grassmannians. Their result should extend to all flag manifolds. We make a precise conjecture for flag varieties of the classical groups.

Let V be a vector space and $\langle\cdot,\cdot\rangle$ a bilinear form on V, and set $G := \mathrm{Aut}(V, \langle\cdot,\cdot\rangle)$. We suppose that $\langle\cdot,\cdot\rangle$ is either:

(1) identically zero, so that G is a general linear group;
(2) nondegenerate and symmetric, so that G is an orthogonal group; or
(3) nondegenerate and alternating, so that G is a symplectic group.

For the orthogonal case, we suppose that V has a basis for which $\langle\cdot,\cdot\rangle$ has the form (10) when V has odd dimension and the same form with y_{2n+1-i} replacing y_{2n+2-i} when V has even dimension. This last requirement ensures that the real flag manifolds of G are nonempty. Let γ be a real rational normal curve in V whose flags of osculating subspaces $F_{\bullet}(s)$ for $s \in \gamma$ are isotropic (cases (2) and (3) just listed).

Let P be a parabolic subgroup of G. Given a point $0 \in \gamma$, the isotropic flag $F_{\bullet}(0)$ induces a Bruhat decomposition of the flag manifold G/P indexed by $w \in W/W_P$, where W is the Weyl group of G and W_P is the parabolic subgroup associated to P. For $w \in W/W_P$, let $\mathcal{X}_w \to \gamma$ be the family of Schubert varieties $X_w F_{\bullet}(s)$.

CONJECTURE 5.1. *For any $w \in W/W_P$, the family $\mathcal{X}_w \to \gamma$ respects the Bruhat decomposition of G/P given by the flag $F_{\bullet}(0)$ and any collection of these families is in general position.*

If this conjecture were true then, for any $u, w \in W/W_P$, we would have

$$\lim_{s\to 0}(X_u F_{\bullet}(s) \cap X_w) = \sum_{v \prec w} m^v_{u,w} X_v.$$

These coefficients $m^v_{u,w}$ are the structure constants for the cohomology ring of G/P with respect to its integral basis of Schubert classes. There are few formulas known for these structure constants, and it is an open problem to give a combinatorial formula for these coefficients. Much of what is known may be found in [1; 2; 27; 28; 31]. An explicit proof of Conjecture 5.1 may shed light on this important problem.

One class of coefficients for which a formula is known is when G/P is the partial flag manifold $\mathrm{Fl}_{\mathbf{d}}$ and u is the index of a special Schubert class. For these, the coefficient is either 0 or 1 [22; 31]. A consequence of Conjecture 5.1 would be that any enumerative problem on a partial flag manifold $\mathrm{Fl}_{\mathbf{d}}$ given by these special Schubert classes may have all solutions be real, generalizing the result of [35].

References

[1] N. Bergeron and F. Sottile, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds,* Duke Math. J. 95 (1998), 373–423.

[2] ———, *A Pieri-type formula for isotropic flag manifolds,* www.arXiv.org/math.CO/9810025.

[3] A. Borel, *Linear algebraic groups,* Springer-Verlag, Berlin, 1991.

[4] C. Chevalley, *Sur les décompositions cellulaires des espaces G/B*, Algebraic groups and their generalizations: Classical methods, pp. 1–23, Amer. Math. Soc., Providence, RI, 1994.

[5] P. Dietmaier, *The Stewart–Gough platform of general geometry can have 40 real postures,* Advances in robot kinematics: Analysis and control, pp. 7–16, Kluwer, Dordrecht, 1998.

[6] D. Eisenbud and J. Harris, *Divisors on general curves and cuspidal rational curves*, Invent. Math. 74 (1983), 371–418.
[7] ———, *When ramification points meet,* Invent. Math. 87 (1987), 485–493.
[8] A. Eremenko and A. Gabrielov, *Rational functions with real critical points and B. and M. Shapiro conjecture in real enumerative geometry,* unpublished manuscript, 1999.
[9] J.-C. Faugère, F. Rouillier, and P. Zimmermann, private communication, 1999.
[10] W. Fulton, *Intersection theory,* Ergeb. Math. Grenzgeb. (3), 2, Springer-Verlag, Berlin, 1984.
[11] ———, *Introduction to toric varieties,* Ann. of Math. Stud., 131, Princeton Univ. Press, Princeton, NJ, 1993.
[12] ———, *Introduction to intersection theory in algebraic geometry,* CBMS Regional Conf. Ser. in Math., 54, Amer. Math. Soc., Providence, RI, 1996.
[13] ———, *Young tableaux: With applications to representation theory and geometry,* London Math. Soc. Stud. Texts, 35, Cambridge Univ. Press, Cambridge, U.K., 1997.
[14] W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci,* Lecture Notes in Math., 1689, Springer-Verlag, Berlin, 1998.
[15] G.-M. Greuel, G. Pfister, and H. Schönemann, *Singular version 1.2 user manual,* technical report no. 21, Centre for Computer Algebra, University of Kaiserslautern, 1998.
[16] B. Huber, F. Sottile, and B. Sturmfels, *Numerical Schubert calculus. Symbolic numeric algebra for polynomials,* J. Symbolic Comput. 26 (1998), 767–788.
[17] B. Huber and J. Verschelde, *Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control,* SIAM J. Control Optim. 38 (2000), 1265–1287.
[18] I. Itenberg and M.-F. Roy, *Multivariate Descartes' rule,* Beiträge Algebra Geom. 37 (1996), 337–346.
[19] V. Kharlamov, personal communication, 1999.
[20] S. L. Kleiman, *The transversality of a general translate,* Compositio Math. 28 (1974), 287–297.
[21] F. Knop, *The Luna–Vust theory of spherical embeddings,* Proc. Hyderabad Conf. Alg. Groups, pp. 22–249, Manoj Prakashan, Madras, 1991.
[22] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert,* C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), 447–450.
[23] T. Y. Li and X. Wang, *On multivariate Descartes' rule—a counterexample,* Beiträge Algebra Geom. 39 (1998), 1–5.
[24] J. G. Macdonald, J. Pach, and Th. Theobald, *Common tangents to four unit balls in* $\mathbb{R}^3$, Discrete Comput. Geom. (to appear).
[25] J. D. Monk, *The geometry of flag manifolds,* Proc. London Math. Soc. (3) 9 (1959), 253–286.
[26] P. Pedersen and B. Sturmfels, *Mixed monomial bases,* Progr. Math., 143, pp. 307-316, Birkhäuser, Basel, 1994.
[27] P. Pragacz and J. Ratajski, *A Pieri-type formula for Lagrangian and odd orthogonal Grassmannians,* J. Reine Agnew. Math. 476 (1996), 143–189.
[28] ———, *A Pieri-type theorem for even orthogonal Grassmannians,* preprint, Max-Planck Institut, 1996.
[29] M. S. Ravi, J. Rosenthal, and X. Wang, *Degree of the generalized Plücker embedding of a Quot scheme and quantum cohomology,* Math. Ann. 311 (1998), 11–26.

[30] F. Ronga, A. Tognoli, and Th. Vust, *The number of conics tangent to five given conics: The real case,* Rev. Mat. Univ. Complut. Madrid 10 (1997), 391–421.
[31] F. Sottile, *Pieri's formula for flag manifolds and Schubert polynomials,* Ann. Inst. Fourier (Grenoble) 46 (1996), 89–110.
[32] ———, *Enumerative geometry for the real Grassannian of lines in projective space,* Duke Math. J. 87 (1997), 59–85.
[33] ———, *Real enumerative geometry and effective algebraic equivalence,* J. Pure Appl. Algebra 117/118 (1997), 601–615.
[34] ———, *The special Schubert calculus is real,* Electron. Res. Announc. Amer. Math. Soc. 5 (1999), 35–39.
[35] ———, *Real Schubert calculus: Polynomial systems and a conjecture of Shapiro and Shapiro,* Exper. Math. 9 (2000), 161–182.
[36] ———, *Real rational curves in Grassmannians,* J. Amer. Math. Soc. 13 (2000), 333–341.
[37] F. Sottile and B. Sturmfels, *A sagbi basis for the quantum Grassmannian,* J. Pure Appl. Alg. (to appear).
[38] B. Sturmfels, *On the number of real roots of a sparse polynomial system,* Fields Inst. Commun., 3, pp. 137–143, Amer. Math. Soc., Providence, RI, 1994.
[39] ———, *Polynomial equations and convex polytopes,* Amer. Math. Monthly 105 (1998) 907–922.
[40] J. Verschelde, *Numerical evidence of a conjecture in real algebraic geometry,* Experiment. Math. 9 (2000), 183–196.

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