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C^α-Compactness and the Calabi Flow on Kähler Surfaces with Negative Scalar Curvature

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1. Introduction

Let M be a compact Kähler m-manifold that has a Kähler metric $ds^2 = g_{\alpha\bar\beta}dz^\alpha \otimes d\bar z^\beta$. Then it is known that, for the Ricci curvature tensor $R_{\alpha\bar\beta} = -(\partial^2/\partial z^\alpha \partial \bar z^\beta)\log\det(g_{\lambda\bar\mu})$, $\frac{\sqrt{-1}}{2\pi}R_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$ is a closed $(1,1)$-form and its cohomology class is equal to the first Chern class $C_1(M)$. Conversely, it was Calabi who asked if, for any closed $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\tilde R_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$ that is cohomologous to $C_1(M)$, can one find a Kähler metric $\tilde g_{\alpha\bar\beta}$ on M such that $\tilde R_{\alpha\bar\beta}$ is the Ricci curvature tensor of $\tilde g_{\alpha\bar\beta}$? As a consequence of Aubin and Yau's results, one can find a Kähler–Einstein metric on M with $C_1(M) = 0$ or $C_1(M) < 0$. When $C_1(M) > 0$, the space of Kähler–Einstein metrics are invariant under automorphism group. However, the existence does not always hold in general [F; M; T; TY].

Instead of the Kähler–Einstein metric, we consider the notion of extremal metrics due to Calabi [C1]. Namely, fix a Kähler class $\Omega_0 = [\omega_0]$ on a compact Kähler manifold M and denote by H_{Ω_0} the space of all Kähler metrics with the same fixed Kähler class Ω_0. Now consider the functional $\Phi\colon H_{\Omega_0} \to \mathbf{R}$,

$$\Phi(g) = \int_M R^2\, d\mu_g,$$

where R denotes the scalar curvature of g. A critical point of Φ is called an *extremal metric*. In particular, any Kähler–Einstein metric is an extremal metric that also minimizes $\int_M R^2\, d\mu_g$ in H_{Ω_0}. Furthermore, if $\Omega_0 = C_1(M) > 0$ and if there exist no nonzero holomorphic vector fields on M, then an extremal metric is a Kähler–Einstein metric. On the other hand, there exist some obstructions to the existence of extremal metrics due to Calabi [C2], LeBrun [L], Levine [Le], and Burns and deBartolomeis [BB]. However, so far there is no known example of a compact Kähler manifold M with $C_1(M) > 0$ and no nonzero holomorphic tangent vector field that does not carry any extremal metric. Concerning the existence of extremal metrics, Calabi has asked whether one can always minimize the Φ in H_{Ω_0} on M if there exist no nonzero holomorphic tangent vector fields and if the tangent bundle of M is stable (see [C1; D; SY; UY]).

Throughout this note, we consider a compact Kähler surface M (see Remark 2.2) with a fixed Kähler class $\Omega_0 = [\omega_0]$ for $\omega_0 = \frac{\sqrt{-1}}{2\pi}\overset{0}{g}_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$. For any metric

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$g \in H_{\Omega_0}$, there exists a real-valued scalar function φ, globally defined on M, such that

$$g_{\alpha\bar{\beta}} = \overset{0}{g}_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}},$$

where $\varphi_{\alpha\bar{\beta}} = \partial^2\varphi/\partial z^\alpha \partial \bar{z}^\beta$.

Now consider the following Calabi flow on $(M, [\omega_0])$:

$$\begin{cases} \frac{\partial g_{\alpha\bar{\beta}}(z,\bar{z},t)}{\partial t} = \frac{\partial^2 R(g(t))}{\partial z^\alpha \partial \bar{z}^\beta}, \\ g_{\alpha\bar{\beta}}(z,\bar{z},t) = \overset{0}{g}_{\alpha\bar{\beta}}(z,\bar{z}) + \varphi_{\alpha\bar{\beta}}(z,\bar{z},t), \quad t \geq 0, \end{cases} \tag{1.1}$$

where

$$R = R(g_{\alpha\bar{\beta}}) = -\Delta \log\det(\overset{0}{g}_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}}), \quad R_0 = R(\overset{0}{g}_{\alpha\bar{\beta}}) = -\Delta \log\det(\overset{0}{g}_{\alpha\bar{\beta}}),$$

and $g_{\alpha\bar{\beta}}(z,\bar{z},0) = \overset{0}{g}_{\alpha\bar{\beta}}(z,\bar{z}) + \varphi_{\alpha\bar{\beta}}(z,\bar{z},0)$ is positive definite. An interesting observation is that, if there exist no nonzero holomorphic tangent vector fields on M, then the functional Φ decays along the Calabi flow on $(M, [\omega_0])$ (Lemma 2.1). If we let

$$F(z,\bar{z},t) = \log\det(\overset{0}{g}_{\alpha\bar{\beta}}(z,\bar{z}) + \varphi_{\alpha\bar{\beta}}(z,\bar{z},t)) - \log\det(\overset{0}{g}_{\alpha\bar{\beta}}(z,\bar{z})),$$

then

$$R_{\alpha\bar{\beta}} = \overset{0}{R}_{\alpha\bar{\beta}} - F_{\alpha\bar{\beta}}$$

and

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial}{\partial t}(\log\det g) \\ &= g^{\alpha\bar{\beta}} \frac{\partial g_{\alpha\bar{\beta}}}{\partial t} \\ &= g^{\alpha\bar{\beta}} \frac{\partial^2 R(g(t))}{\partial z^\alpha \partial \bar{z}^\beta} \\ &= \Delta R. \end{aligned}$$

We can then reformulate the Calabi flow as follows to yield the so-called modified Calabi flow on Kähler manifolds:

$$\begin{cases} \frac{\partial F}{\partial t} = \Delta R = -\Delta^2 F + \Delta(g^{\alpha\bar{\beta}} \overset{0}{R}_{\alpha\bar{\beta}}), \\ g_{\alpha\bar{\beta}}(z,\bar{z},t) = \overset{0}{g}_{\alpha\bar{\beta}}(z,\bar{z}) + \varphi_{\alpha\bar{\beta}}(z,\bar{z},t), \quad t \geq 0, \\ \int_M e^{F_0}\, d\mu_0 = \int_M d\mu_0, \quad F_0(z,z) = F(z,\bar{z},0). \end{cases} \tag{1.2}$$

Here $F\colon M \times [0,\infty) \to \mathbf{R}$ is a smooth function, $\Delta = \Delta_{g_{\alpha\bar{\beta}}}$, $d\mu_0 = d\mu_{\overset{0}{g}}$, and $d\mu = e^F d\mu_0$.

For $m = 1$, the Riemann surface, (1.1) and (1.2) are equivalent. More precisely, in this case let $(\Sigma, \overset{0}{g})$ be a Riemann surface with a given conformal class $[\overset{0}{g}]$ on Σ. In the author's previous paper [Ch3] we considered the Calabi flow on $(\Sigma, [\overset{0}{g}])$:

$$\begin{cases} \frac{\partial\lambda}{\partial t} = \frac{1}{2}\Delta R, \\ \lambda(p,0) = \lambda_0(p), \\ g_{ij} = e^{2\lambda}\overset{0}{g}_{ij}, \\ \int_\Sigma e^{2\lambda_0}\, d\mu_0 = \int_\Sigma d\mu_0, \end{cases} \tag{1.3}$$

where $\lambda\colon \Sigma \times [0,\infty) \to \mathbf{R}$ is a smooth function and $\Delta = \Delta_{g_{ij}}$. Then we have the following.

PROPOSITION 1.1. *Let $(\Sigma, \overset{0}{g})$ be a closed Riemann surface. For any given smooth initial value λ_0, there exists a smooth solution $\lambda(t)$ of* (1.3) *on $\Sigma \times [0,\infty)$. Furthermore, there exists a subsequence of solutions that converges to one of constant curvature metric.*

REMARK 1.1. If Σ is a 2-sphere then Proposition 1.1 holds up to the conformal group. The crucial points for the C^0-estimate of the conformal factor λ (and then the higher-order estimates as well) are the bounds on the $\int_\Sigma R^2\, d\mu_g$ and Bondi mass $\int_\Sigma e^{3\lambda}\, d\mu_0$. For details, see [Chr] and [Ch3].

In this paper, we will investigate similar properties of the flow (1.1) for $m = 2$ as we did in the flow (1.3). First, we will show some C^α-compactness properties for F (Theorem 1.2, Corollary 1.3); then we will apply those results to the Calabi flow. In fact, we prove some kind of Harnack estimate for the Calabi flow (1.1). As consequences, under condition $(**)$ we show the long-time existence and asymptotic convergence of solutions of (1.1) on compact Kähler surfaces with no nonzero holomorphic tangent vector fields and $R_0 < 0$ on $(M, [\omega_0])$ (Theorem 1.4, Theorem 1.5). Finally, combining with the results of [L], we show the blow-up behavior of the Calabi flow on the ruled surface (Corollary 1.6).

We shall follow the notation of [G] and [Ch3] (see also Section 4). First, for a sequence of smooth Kähler metrics $\{g^i\}$ on a compact Kähler surface $(M, [\omega_0])$ in a fixed Kähler class H_{Ω_0}, let $g^i = \overset{0}{g}_{\alpha\bar\beta}(z,\bar z) + \varphi^i_{\alpha\bar\beta}(z,\bar z)$ and consider

$$F^i(z,\bar z) = \log\frac{\det g^i}{\det \overset{0}{g}} = \log\frac{\det(\overset{0}{g}_{\alpha\bar\beta}(z,\bar z) + \varphi^i_{\alpha\bar\beta}(z,\bar z))}{\det(\overset{0}{g}_{\alpha\bar\beta}(z,\bar z))}.$$

DEFINITION 1.1. We say that F^i satisfies the property $(*)$ if there is a point $x \in M$ and if there exist positive constants ρ, ε, H independent of F^i such that

$$\int_{B(x,\rho)} e^{-\varepsilon F^i}\, d\mu_0 \le H. \tag{$*$}$$

Using results of [G] for the case of Riemannian manifolds with fixed conformal class then yields the following theorem.

THEOREM 1.2. *Let $\{g^i\}$ be a sequence of smooth Kähler metrics on a compact Kähler surface $(M, [\omega_0])$ in a fixed Kähler class H_{Ω_0} such that*:

(i) *$F^i(z, \bar{z})$ satisfies the property* $(*)$; *and*
(ii) *for fixed positive constants K and p,*

$$\int_M |\mathrm{Ric}(g^i)|^p \, d\mu_{g^i} \le K, \quad p > 2. \tag{$**$}$$

Here $\mathrm{Ric}(g^i)$ *is the Ricci curvature tensor with respect to g^i.*
Then there is a constant C such that

$$\|F^i\|_{W^{2,p}} \le C.$$

Hence a subsequence of $\{F^i\}$ converges in C^α with $0 < \alpha < \frac{2p-4}{p}$.

REMARK 1.2. It is worthy to note that the role of F in a fixed Kähler class corresponds to the role of the conformal factor λ in a fixed conformal class. We refer to [Ch3; ChW; G] for details.

As a consequence, under condition $(**)$, property $(*)$ holds for a compact Kähler surface with $R_0 < 0$ (Lemma 4.2) and so we have the following.

COROLLARY 1.3. *Let $\{g^i\}$ be a sequence of smooth Kähler metrics on a compact Kähler surface $(M, [\omega_0])$ with $R_0 < 0$ in a fixed Kähler class H_{Ω_0} such that $(**)$ holds. Then there is a constant C such that*

$$\|F^i\|_{W^{2,p}} \le C.$$

Hence a subsequence of $\{F^i\}$ converges in C^α with $0 < \alpha < \frac{2p-4}{p}$.

With applications, $F^i(z, \bar{z})$ will be replaced by $F(z, \bar{z}, t)$ as in (1.2). Then we have our next theorem as follows.

THEOREM 1.4. *Let $(M, [\omega_0])$ be a compact Kähler surface admitting no nonzero holomorphic tangent vector fields. Under the flow* (1.1), *let F satisfy* (1.2) *on* $[0, T)$ *with the property* $(*)$ *and condition* $(**)$; *that is,*

$$\int_M |\mathrm{Ric}|^p \, d\mu \le K, \quad p > 2$$

for the positive constants H, K, p that are independent of t. Then the solution of (1.1) *exists on $M \times [0, \infty)$. Moreover, there exists a subsequence of solutions $\{g(t)\}$ of* (1.1) *on $M \times [0, \infty)$ that converges smoothly to one of constant scalar curvature metric.*

As a consequence of Theorem 1.4, under condition $(**)$, property $(*)$ holds when (M, g_0) is a Kähler surface with $R_0 < 0$ (Lemma 4.2). Hence we have the following theorem.

THEOREM 1.5. *Let $(M, [\omega_0])$ be a compact Kähler surface admitting no nonzero holomorphic tangent vector fields and with $R_0 < 0$. Given the Calabi flow with condition $(**)$ on $[0, T)$, the solution of* (1.1) *exists on $M \times [0, \infty)$ and there* *exists a subsequence of solutions* $\{g(t)\}$ *of* (1.1) *on* $M \times [0, \infty)$ *that converges smoothly to a negative constant scalar curvature metric* g_∞.

Remark 1.3. (i) If $m = 1$ then, under the flow (1.3), condition $(**)$ holds for $p = 2 > 1$. Then we have Proposition 1.1 without condition $(**)$.

(ii) If $m = 2$ then, under the flow (1.1) on M admitting no nonzero holomorphic tangent vector fields, we have bounds on the L^2-norm of scalar curvature, Ricci curvature, and Riemannian curvature in the fixed Kähler class (Corollary 2.2).

(iii) LeBrun [L] proved the following. Let $E \to \Sigma$ be a rank-2 holomorphic vector bundle over a compact Riemann surface Σ, and let $(M, J) = P(E)$ be the total space of the associated CP^1-bundle. Let $[\omega]$ be a Kähler class on M with $C_1(M) \cdot [\omega] < 0$ (e.g., genus$(\Sigma) \geq 2$). Then $[\omega]$ contains a Kähler metric of negative constant scalar curvature if and only if E is a semistable vector bundle; that is, $E = \bigoplus E_j$, where the E_j are stable vector bundles in the sense of Mumford–Takemoto [K; Lü; UY]. On the other hand, in [BB] the authors constructed a rank-2 non-semistable holomorphic vector bundle E' over Σ of genus ≥ 2 such that $P(E')$ admits no nonzero holomorphic tangent vector fields and does not admit an extremal Kähler metric in the fixed Kähler class (and then does not admit a constant scalar curvature either).

In view of these results of [L] and [BB], we have the following corollary.

Corollary 1.6. *Let* $E \to \Sigma$ *be a rank*-2 *holomorphic vector bundle over a compact Riemann surface* Σ *of genus* ≥ 2, *where* E *is not a semistable vector bundle and* $P(E)$ *admits no nonzero holomorphic tangent vector fields. Let* $[\omega]$ *be a Kähler class on* $P(E)$ *admitting a metric of negative scalar curvature* (*and then* $C_1(P(E)) \cdot [\omega] < 0$). *Then the Calabi flow does blow up on* $P(E) \times [0, T)$ *in the sense that, for any* $p > 2$,

$$\int_{P(E)} |\mathrm{Ric}|^p \, d\mu \to \infty \quad \textit{as } t \to T.$$

Remark 1.4. (i) We may easily modify the example of [BB] so that the ruled surfaces $P(E)$ as in Corollary 1.6 do exist.

(ii) It may be possible for $T = \infty$.

It is difficult to estimate the C^0-bound of the fourth-order parabolic equation (1.1) owing to a lack of the maximum principle.

We briefly describe the methods used in our proofs. In Section 2 we derive the key estimate of equation (1.1) (Theorem 2.3). In Section 3—based on the Green formula, [Cao] and [Y]—we have the Harnack estimate for the equation (1.1). Then we obtain the C^0-bound for solution $F(t)$ of (1.2) and the higher-order $W_{k,p}$-estimates of the solution for (1.1). Furthermore, if we assume the condition $(**)$ and the uniformly lower bound for solution $F(t)$ of (1.2), then we have the long-time existence and convergence of solutions of (1.1).

In view of Section 3, we reduce the proof of our main theorems to finding a uniformly lower bound of $F(t)$ in Section 4. In fact, following Theorem 3.2 and the condition $(**)$, we have the uniformly lower bound for the solution F of (1.2) if $R_0 < 0$. Finally, in Section 5 we study the asymptotic behavior of solutions of (1.1).

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2. The Mass Decay Estimates

In this section we consider the Calabi flow (1.1) on M admitting no nonzero holomorphic tangent vector fields. Under this flow, we have the following lemma.

LEMMA 2.1. *For $t \geq t_0$,*

$$\int_M R^2 \, d\mu\Big|_t \leq \int_M R^2 \, d\mu\Big|_{t_0}.$$

Proof. The lemma follows easily from [C1]. □

COROLLARY 2.2. *There exists a positive constant C that is independent of t such that*

$$\int_M R^2 \, d\mu \leq C$$

for $0 \leq T \leq \infty$.

REMARK 2.1. From [C1], we also have bounds on the L^2-norm of Ricci curvature and Riemannian curvature in the fixed Kähler class.

Next, we will derive the following integral bound on e^F. From now on, C denotes a generic constant that may vary from line to line.

THEOREM 2.3. *Under the Calabi flow on a compact Kähler surface $(M, [\omega_0])$,*

$$\int_M e^{\alpha F} \, d\mu_0 \leq C(\alpha)$$

for $0 \leq \alpha < \infty$.

REMARK 2.2. (i) In the proof of Theorem 2.3, we need only the bound on the L^2-norm of the Ricci curvature tensor that is satisfied owing to the flow.

(ii) Under this flow,

$$\frac{d}{dt} \int_M d\mu = \frac{d}{dt} \int_M e^F \, d\mu_0 = \int_M (\Delta R) \, d\mu = 0;$$

then the volume is preserved and

$$\int_M e^F \, d\mu_0 < \infty.$$

(iii) For $m > 2$, the inequality (2.3) does not hold. This is one of the reasons why the theorem works for $m = 2$ only.

Proof of Theorem 2.3. First, with respect to g_0, we have the Sobolev constant A_0. That is, for $N = \frac{2n}{n-2}$, $n = 2m$, and $\varphi \in C^\infty(M)$,

$$\left(\int_M |\varphi|^N \, d\mu_0\right)^{2/N} \leq A_0\left(\int_M |\overset{0}{\nabla}\varphi|^2 \, d\mu_0 + \int_M \varphi^2 \, d\mu_0\right).$$

But for $\varphi = e^{F/N} f$,

$$\begin{aligned}\left(\int_M |f|^N \, d\mu\right)^{2/N} &= \left(\int_M |\varphi|^N e^{-F} \, d\mu\right)^{2/N} \\ &= \left(\int_M |\varphi|^N \, d\mu_0\right)^{2/N} \\ &\leq A_0\left(\int_M |\overset{0}{\nabla}\varphi|^2 \, d\mu_0 + \int_M \varphi^2 \, d\mu_0\right).\end{aligned} \tag{2.1}$$

Now, for $f = e^{\alpha F}$ with $\alpha > 0$ we have

$$\Delta_0 e^{(2/N+2\alpha)F} = e^{(2/N+2\alpha)F}\left[\left(\frac{2}{N} + 2\alpha\right)\Delta_0 F + \left(\frac{2}{N} + 2\alpha\right)^2 |\overset{0}{\nabla}F|^2\right]$$

and so

$$\begin{aligned}&\int_M |\overset{0}{\nabla}\varphi|^2 \, d\mu_0 + \int_M \varphi^2 \, d\mu_0 \\ &\quad= \int_M |\overset{0}{\nabla}e^{(1/N+\alpha)F}|^2 \, d\mu_0 + \int_M e^{(2/N+2\alpha)F} \, d\mu_0 \\ &\quad= \left(\frac{1}{N} + \alpha\right)^2 \int_M e^{(2/N+2\alpha)F}|\overset{0}{\nabla}F|^2 \, d\mu_0 + \int_M e^{(2/N+2\alpha)F} \, d\mu_0 \\ &\quad= -\left(\frac{1}{N} + \alpha\right)\int_M e^{(2/N+2\alpha)F}\Delta_0 F \, d\mu_0 + \int_M e^{(2/N+2\alpha)F} \, d\mu_0.\end{aligned}$$

But

$$-\Delta_0 F = \overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta} - R_0,$$

and this implies that

$$\begin{aligned}&\int_M |\overset{0}{\nabla}\varphi|^2 \, d\mu_0 + \int_M \varphi^2 \, d\mu_0 \\ &\quad= \left(\frac{1}{N} + \alpha\right)\int_M e^{(2/N+2\alpha)F}(\overset{0}{g}_{\alpha\bar\beta} R_{\alpha\bar\beta} - R_0) \, d\mu_0 + \int_M e^{(2/N+2\alpha)F} \, d\mu_0 \\ &\quad\leq C_1 \int_M e^{(4/N-1+4\alpha)F} \, d\mu_0 + C_2 \int_M |\overset{0}{g}_{\alpha\bar\beta} R_{\alpha\bar\beta}|^2 \, d\mu + C_3 \int_M e^{(2/N+2\alpha)F} \, d\mu_0 \\ &\quad\leq C_1 \int_M e^{(4/N-1+4\alpha)F} \, d\mu_0 + C_3 \int_M e^{(2/N+2\alpha)F} \, d\mu_0 + C_4.\end{aligned} \tag{2.2}$$

Together, (2.1) and (2.2) imply

$$\left(\int_M e^{(1+N\alpha)F}\,d\mu_0\right)^{2/N} \le C_5\left(\int_M e^{(4/N-1+4\alpha)F}\,d\mu_0+\int_M e^{(2/N+2\alpha)F}\,d\mu_0\right)+C_6.$$

In particular, for $m=2$,

$$\left(\int_M e^{(1+4\alpha)F}\,d\mu_0\right)^{1/2} \le C_5\left(\int_M e^{4\alpha F}\,d\mu_0+\int_M e^{(1/2+2\alpha)F}\,d\mu_0\right)+C_6. \quad (2.3)$$

Now, for $\alpha>0$ we have $(1+4\alpha)>4\alpha$ and $(1+4\alpha)>\left(\frac{1}{2}+2\alpha\right)$. On the other hand, $\int_M e^F\,d\mu_0$ is bounded. Thus, for (say) $\alpha=\frac{1}{4}$ we have

$$\left(\int_M e^{2F}\,d\mu_0\right)^{1/2} \le C_5\left(\int_M e^{F}\,d\mu_0+\int_M e^{F}\,d\mu_0\right)+C_6 \le C;$$

$\alpha=\frac{1}{2}$ yields Young's inequality,

$$\begin{aligned}\left(\int_M e^{3F}\,d\mu_0\right)^{1/2} &\le C_5\left(\int_M e^{2F}\,d\mu_0+\int_M e^{(3/2)F}\,d\mu_0\right)+C_6\\ &\le C\left(\int_M e^{2F}\,d\mu_0+\int_M e^{F}\,d\mu_0\right)+C_6\\ &\le C.\end{aligned}$$

Repeating this iteration, one then obtains

$$\int_M e^{\alpha F}\,d\mu_0 \le C(\alpha)$$

for $0\le\alpha<\infty$. □

3. A Priori Estimates

In this section (following [Cao; Ch1; Chr; Y]), under $(*)$ and $(**)$, we will derive all $W_{k,p}$-norm bounds on the solution F of (1.2); in particular, we have the $W_{2,p}$-norm bounds on $\{F^i\}$. This, together with the local existence result, will then show the long-time existence of solutions of (1.1) and (1.2).

First set

$$\phi=\varphi-\frac{1}{\mathrm{Vol}(M)}\int_M \varphi\,d\mu_0.$$

Then, as shown in [Y], we have the following result.

LEMMA 3.1. *There exist constant C_8 and C_9, independent of t, such that*

$$\sup_{M\times[0,T)}\phi\le C_8,\qquad \sup_{M\times[0,T)}\int_M|\phi|\,d\mu_0\le C_9.$$

Based on the Green formula and Theorem 2.3, we have the following uniformly upper bound on $F(t)$ (or $\{F^i\}$).

THEOREM 3.2. *Under the flow* (1.2), *if* $(**)$ *is satisfied then*

$$F \le C_{10}$$

for all $t \in [0, \infty]$.

Proof. From the Green formula, let $G(X, \cdot)$ denote the Green's function for Δ_0 with singularity $X \in M$. Then $G(X, \cdot) \ge 0$ and $\|G(X, \cdot)\|_{L^q} \le C$ for $q < 2$ $(m = 2)$. Now

$$\begin{aligned}
e^F(X) &= \frac{\int_M e^F\, d\mu_0}{\int_M d\mu_0} - \int_M G(X, \cdot)\Delta_0 e^F\, d\mu_0 \\
&= \frac{\int_M e^F\, d\mu_0}{\int_M d\mu_0} - \int_M G(X, \cdot) e^F(\Delta_0 F + |\overset{0}{\nabla} F|^2)\, d\mu_0 \\
&\le C + \int_M G(X, \cdot)|\overset{0}{g}_{\alpha\bar\beta} R_{\alpha\bar\beta} - R_0|\, d\mu \\
&\le C + \left(\int_M G(X, \cdot)^q\, d\mu\right)^{1/q} \left(\int_M |\overset{0}{g}_{\alpha\bar\beta} R_{\alpha\bar\beta}|^p\, d\mu\right)^{1/p} \\
&\le C + C\left(\int_M G(X, \cdot)^q\, d\mu\right)^{1/q}
\end{aligned}$$

for $q < 2$ (since $p > 2$). But from Theorem 2.3, for $1/l_1 + 1/l_2 = 1$ and $ql_1 < 2$ we have

$$\begin{aligned}
\left(\int_M G(X, \cdot)^q\, d\mu\right)^{1/q} &= \left(\int_M G(X, \cdot)^q e^F\, d\mu_0\right)^{1/q} \\
&\le \left[\left(\int_M G(X, \cdot)^{ql_1}\, d\mu_0\right)^{1/l_1} \left(\int_M e^{l_2 F}\, d\mu_0\right)^{1/l_2}\right]^{1/q} \\
&\le C.
\end{aligned}$$

Then

$$e^F(X) \le C,$$

which completes the proof of the theorem. □

COROLLARY 3.3. *Under the flow* (1.2), *if* $(**)$ *is satisfied then*

$$\|\phi\|_{L_\infty} \le C_{11}.$$

Proof. From Lemma 3.1, we can renormalize ϕ such that $\phi \le -1$. Then use [Cao, Lemma 3] except replace (1.14) in the proof [Cao, p. 364] by

$$\int_M (-\phi)^{p-2} |\overset{0}{\nabla}\phi|^2\, d\mu_0 \le n! \int_M \frac{(-\phi)^{p-1}}{p-1}(e^F - 1)\, d\mu_0 \le \frac{C}{p-1}\int_M (-\phi)^{p-1}\, d\mu_0.$$

Finally, use a Nash–Moser iteration argument to obtain the C^0-estimate for ϕ. □

THEOREM 3.4. *Let $F(t)$ satisfy* (1.2), *and suppose that* $(*)$ *and* $(**)$ *hold. Then there exists a constant $C = C(l) > 0$ $(l \geq 1)$ such that*

$$\|\overset{0}{\nabla}{}^l F(z, \bar{z}, t)\|_{L_p} \leq C \quad \forall t \in [0, \infty].$$

Moreover, if $p > 2$ then

$$\|F(t)\|_{W_{2,p}} \leq C \quad \forall t \in [0, \infty].$$

REMARK 3.1. For the sequence $\{F^i\}$, under $(*)$ and $(**)$ we have only the $W_{2,p}$-norm bounds; that is,

$$\|F^i\|_{W_{2,p}} \leq C, \quad p > 2.$$

Proof of Theorem 3.4. From Lemma 4.1, under condition $(**)$, the property $(*)$ implies

$$F \geq -C_0.$$

This and Theorem 3.2 plus $(**)$ imply

$$\int_M (\Delta_0 F)^p \, d\mu_0 \leq C, \quad p > 2,$$

and so

$$\|F\|_{W_{2,p}} \leq C. \tag{3.1}$$

On the other hand, by [Chr, Sec. 4]—in particular, by the interpolation inequality and Sobolev imbedding theorem for 4-manifolds—it follows that

$$\frac{d}{dt}\|\overset{0}{\nabla}{}^l F\|_{L_p} \leq -2\|\overset{0}{\nabla}{}^{l+2} F\|_{L_p} + C\|F - \langle F\rangle\|_{W_{2,p}}$$

where $\langle F\rangle = \left(\int_M F \, d\mu_0\right) / \left(\int_M d\mu_0\right)$. Together, these results imply that

$$\|\overset{0}{\nabla}{}^l F\|_{L_p} \leq C.$$

We refer to [Ch1; Chr] for details. □

Applying the results as in [Y] then yields the following.

PROPOSITION 3.5. *There exist constants C_{12}, C_{13} such that*

$$0 < m + \Delta_0 \varphi \leq C_{13} \exp\Big(C_{12}\Big(\varphi - \inf_{M \times [0,T)} \varphi\Big)\Big)$$

for all $t \in [0, T)$.

Then the higher-order estimates for (1.1) will be established. For details, see [Cao; Y].

4. Find a Lower Bound

In view of Section 3, we reduce the proof of our main theorems to finding a uniformly lower bound on $F(t)$ (or F^i).

DEFINITION 4.1. We say that $F(t)$ (or F^i) satisfies the property $(*)$ if there is a point $x \in M$ as well as positive constants ρ, ε, H such that

$$\int_{B(x,\rho)} e^{-\varepsilon F}\, d\mu_0 \leq H. \tag{$*$}$$

LEMMA 4.1. *Let F satisfy* (1.2) *on a fixed Kähler class* $(M, [\omega_0])$ *with condition* $(**)$ *and the property* $(*)$*. Then there exist positive constants C_0', δ_0 such that*

$$\int_M e^{-\delta_0 F}\, d\mu_0 \leq C_0'. \tag{4.1}$$

As a consequence, there is a constant C_0 such that

$$F \geq -C_0. \tag{4.2}$$

REMARK 4.1. From Theorem 3.2, we have the upper bound

$$F \leq C_{10}.$$

This is the key point to have a lower bound on F.

Proof of Lemma 4.1. Consider

$$\begin{aligned}
\int_M |\overset{0}{\nabla} e^{-\delta F}|^2\, d\mu_0 &= -\int_M e^{-\delta F} \Delta_0 e^{-\delta F}\, d\mu_0 \\
&= -\int_M e^{-2\delta F}(-\delta \Delta_0 F + \delta^2 |\overset{0}{\nabla} F|^2)\, d\mu_0 \\
&\leq \delta \int_M e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta} + R_0)\, d\mu_0 \\
&\leq \delta \int_M e^{-2\delta F} R_0\, d\mu_0 + \delta \int_M e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu_0.
\end{aligned} \tag{4.3}$$

Now fix t and let $E_b = \{p \in M : \overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta} \leq b\}$. Then

$$\int_M e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu_0 \leq \int_{E_0} e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu_0. \tag{4.4}$$

From the computation of [Y, (2.23)], at the point q where $\exp\{-C_{14}\varphi\}\{2+\Delta_0\varphi\}$ achieves its maximum on E_0, we have

$$e^{-F(q)}(2 + \Delta_0\varphi)^2(q) \leq C_{15}(2 + \Delta_0\varphi)(q) - \Delta_0 F(q) + C_{16}. \tag{4.5}$$

But on E_0 we have $-\Delta_0 F = \overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta} - R_0 \leq -R_0$ and, from Theorem 3.2, $F \leq C_{10}$ over M. This implies

$$(2 + \Delta_0\varphi)^2(q) \leq C_{17}(2 + \Delta_0\varphi)(q) + C_{18}$$

and then

$$(2 + \Delta_0\varphi)(q) \leq C_{19}.$$

These inequalities and Corollary 3.3 imply, as in [Y, (2.24)], that

$$0 < (2+\Delta_0\varphi) \le C_{19}\exp\{C_{14}(\varphi - \inf\varphi)\} \le C_{20} \tag{4.6}$$

on E_0. Moreover, (4.6) and (4.5) imply

$$(2+\Delta_0\varphi)^2(q) \le C_{21} + \overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}(q). \tag{4.7}$$

Next consider $E_s \subseteq E_0$, $s = -C_{21} + C_{22}e^{\inf F} \le 0$ for some constant C_{22}, which is possible because of $F \le C_{10}$. Then use the same method as before, now on E_s, to obtain

$$(2+\Delta_0\varphi)^2(q') \le C_{21} + \overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}(q') \le C_{22}e^{\inf F} \le C_{22}e^F$$

for some point q'; from (4.6),

$$0 < (2+\Delta_0\varphi) \le C_{23}e^{(1/2)F} \tag{4.8}$$

on E_s.

On the other hand, we may choose (as in [Y, (2.8)]) a coordinate system at a point such that $\overset{0}{g}{}^{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$ and $\varphi_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}\varphi_{\alpha\bar\alpha}$. Then, at that point we have

$$g^{\alpha\bar\beta} = \delta_{\alpha\bar\beta}(1+\varphi_{\alpha\bar\alpha})^{-1} \tag{4.9}$$

and, since $e^F = \det g = (1+\varphi_{1\bar1})(1+\varphi_{2\bar2})$,

$$(2+\Delta_0\varphi) = (1+\varphi_{1\bar1}) + (1+\varphi_{2\bar2}) \ge 2e^{F/2}. \tag{4.10}$$

Then, from (4.10) and (4.8) we have

$$C_{24}e^{-F/2} \le g^{\alpha\bar\alpha} \le C_{25}e^{-F/2}$$

and so

$$|R| \ge C_{26}e^{-F/2}|\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}| \tag{4.11}$$

on E_s.

From (4.4) and (4.11) it now follows that, for small δ with $\left(\frac12 + 2\delta\right)q \le 1$, $1/q + 1/p = 1$, and $p > 2$,

$$\begin{aligned}
&\int_M e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta})\,d\mu_0 \\
&\quad\le \int_{E_0} e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta})\,d\mu_0 \\
&\quad\le \int_{E_s} e^{-F/2-2\delta F}(-e^{-F/2}\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta})\,d\mu + C_{27}\int_M e^{-2\delta F}\,d\mu_0 \\
&\quad\le \left(\int_{E_s} e^{(-1/2-2\delta)qF}\,d\mu\right)^{1/q}\left(\int_{E_s}(-e^{-F/2}\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta})^p\,d\mu\right)^{1/p} \\
&\qquad + C_{27}\int_M e^{-2\delta F}\,d\mu_0 \\
&\quad\le C_{28} + C_{27}\int_M e^{-2\delta F}\,d\mu_0.
\end{aligned}$$

From (4.3) we may then conclude that

$$\int_M |\overset{0}{\nabla} e^{-\delta F}|^2\,d\mu_0 \le C_{29} + \delta C_{30}\int_M e^{-2\delta F}\,d\mu_0.$$

Now let λ_1 denote the first nonzero eigenvalue of Δ_0. By Rayleigh inequality, we have

$$\int_M e^{-2\delta F}\,d\mu_0 \le \frac{\left(\int_M e^{-\delta F}\,d\mu_0\right)^2}{\left(\int_M d\mu_0\right)} + \frac{1}{\lambda_1}\int_M |\overset{0}{\nabla} e^{-\delta F}|^2\,d\mu_0$$

$$\le \frac{\left(\int_M e^{-\delta F}\,d\mu_0\right)^2}{\left(\int_M d\mu_0\right)} + \frac{C_{29}}{\lambda_1} + \frac{\delta}{\lambda_1}C_{30}\int_\Sigma e^{-2\delta F}\,d\mu_0. \tag{4.12}$$

But for $\delta < \varepsilon$, one has

$$\int_M e^{-\delta F}\,d\mu_0 = \int_{B_\rho} e^{-\delta F}\,d\mu_0 + \int_{B_\rho^c} e^{-\delta F}\,d\mu_0$$

$$\le C + \left(\int_{B_\rho^c} e^{-2\delta F}\,d\mu_0\right)^{1/2}\left(\int_{B_\rho^c} d\mu_0\right)^{1/2}$$

and then, for any $\eta > 0$,

$$\frac{\left(\int_M e^{-\delta F}\,d\mu_0\right)^2}{\left(\int_M d\mu_0\right)} \le C(\eta) + (1+\eta)\frac{\left(\int_{B_\rho^c} d\mu_0\right)}{\left(\int_M d\mu_0\right)}\int_M e^{-2\delta F}\,d\mu_0.$$

This implies that

$$\int_M e^{-2\delta F}\,d\mu_0 \le C(\eta) + (1+\eta)\frac{\left(\int_{B_\rho^c} d\mu_0\right)}{\left(\int_M d\mu_0\right)}\int_M e^{-2\delta F}\,d\mu_0 + \frac{\delta}{\lambda_1}C_{30}\int_M e^{-2\delta F}\,d\mu_0.$$

Then choose η, δ small enough and take $\delta_0 = 2\delta$, which gives us (4.1).

To see that (4.2) follows from (4.1), apply the Green formula again. Now take $\delta_1 \ll \delta_0$ with $\delta_1 q_2 \le \delta_0$, $1/q_1 + 1/q_2 = 1$, and $q_1 < 2$; this yields

$$\begin{aligned}
e^{-\delta_1 F}(x) &= \frac{\int_M e^{-\delta_1 F}\,d\mu_0}{\int_M d\mu_0} - \int_M G(x,\cdot)\Delta_0(e^{-\delta_1 F})\,d\mu_0\\
&\le C - \int_M G(x,\cdot)e^{-\delta_1 F}\{-\delta_1\Delta_0 F + \delta_1^2|\overset{0}{\nabla} F|^2\}\,d\mu_0\\
&\le C + \delta_1\int_M e^{-\delta_1 F}G(x,\cdot)[R_0 - \overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}]\,d\mu_0\\
&\le C + C\int_M e^{-\delta_1 F}G(x,\cdot)\,d\mu_0 + \int_M e^{-\delta_1 F}G(x,\cdot)[-\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}]\,d\mu_0\\
&\le C + C\left(\int_M G^{q_1}\,d\mu_0\right)^{1/q_1}\left(\int_M (e^{-\delta_1 F})^{q_2}\,d\mu_0\right)^{1/q_2}\\
&\quad + \int_{E_0} e^{-\delta_1 F}G(x,\cdot)[-\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}]\,d\mu_0\\
&\le C_{31} + \int_{E_0} e^{-\delta_1 F}G(x,\cdot)[-\overset{0}{g}{}^{\alpha\bar\beta}R_{\alpha\bar\beta}]\,d\mu_0.
\end{aligned} \tag{4.13}$$

As before, choose δ_1 small with $\left(\frac{1}{2}+\delta_1\right)q < 1$, $1/q+1/p=1$, $1/q'+1/q''=1$, $qq' < 2$, and $p > 2$:

$$
\begin{aligned}
&\int_{E_0} e^{-\delta_1 F} G(x,\cdot)[-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta}]\, d\mu_0 \\
&\qquad \le \int_{E_s} e^{-F/2-\delta_1 F} G(-e^{-F/2}\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu + C_{27}\int_M e^{-\delta_1 F} G\, d\mu_0 \\
&\qquad \le \left(\int_{E_s} G^q e^{(-1/2-\delta_1)qF}\, d\mu\right)^{1/q}\left(\int_{E_s} (-e^{-F/2}\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})^p\, d\mu\right)^{1/p} + C_{31} \\
&\qquad \le C\left(\int_{E_s} G^q e^{[1-(1/2+\delta_1)q]F}\, d\mu_0\right)^{1/q} + C_{31} \\
&\qquad \le C\left(\int_{E_s} G^{qq'}\, d\mu_0\right)^{1/q'}\left(\int_{E_s} e^{[1-(1/2+\delta_1)q]q''F}\, d\mu_0\right)^{1/q''} + C_{31} \\
&\qquad \le C_{32}.
\end{aligned}
$$

Then

$$e^{-\delta_1 F}(x) \le C_{33},$$

which completes the proof of Lemma 4.1. □

LEMMA 4.2. *Let F satisfy* (1.2) *on a fixed Kähler class* $(M,[\omega_0])$ *with condition* $(**)$ *and $R_0 < 0$. Then $F(t)$ satisfies* $(*)$.

Proof. Since R_0 is negative, there exists a positive constant $\upsilon > 0$ with $-R_0 \ge \upsilon > 0$. From (4.3), one now obtains

$$
\begin{aligned}
&\delta^{-1}\int_M |\overset{0}{\nabla} e^{-\delta F}|^2\, d\mu_0 \\
&\qquad \le \int_M e^{-2\delta F} R_0\, d\mu_0 + \int_M e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu_0 \\
&\qquad \le \int_M e^{-2\delta F} R_0\, d\mu_0 + \int_{E_0} e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu_0 \\
&\qquad \le -\upsilon\int_M e^{-2\delta F}\, d\mu_0 + \int_{E_0} e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta})\, d\mu_0 \\
&\qquad \le -\upsilon\int_M e^{-2\delta F}\, d\mu_0 - \int_{E_0\setminus E_s} e^{-2\delta F}\overset{0}{g}{}^{\alpha\bar\beta} R_{\alpha\bar\beta}\, d\mu_0 + C_{28}. \qquad (4.14)
\end{aligned}
$$

Again as in [Y, (2.23)], at the maximum point q we have

$$e^{-F(q)}(2+\Delta_0\varphi)^2(q) \le 2C_{34}(2+\Delta_0\varphi)(q) + 4\inf_{i\ne l}\overset{0}{R}_{i\bar i l\bar l} - \Delta_0 F(q).$$

From [Y, (2.21)], we may choose $C_{33}+\inf_{i\ne l}\overset{0}{R}_{i\bar i l\bar l} = 1$. Because $R_0 < 0$ and $\overset{0}{R}_{1\bar 1 2\bar 2} + \overset{0}{R}_{2\bar 2 1\bar 1} = R_0$ at q, it follows that $\inf_{i\ne l}\overset{0}{R}_{i\bar i l\bar l} < 0$ and

$$C(2+\Delta_0\varphi)^2 \le \Big(2 - 2\inf_{i\neq l}\overset{0}{R}_{i\bar{i}l\bar{l}}\Big)(2+\Delta_0\varphi) + \overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} + 4\inf_{i\neq l}\overset{0}{R}_{i\bar{i}l\bar{l}} - R_0$$
$$\le \Big(2 - 2\inf_{i\neq l}\overset{0}{R}_{i\bar{i}l\bar{l}}\Big)(2+\Delta_0\varphi) + \overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}.$$

Now we may assume that

$$(2+\Delta_0\varphi) \le \frac{v}{2(2-2\inf_{i\neq l}\overset{0}{R}_{i\bar{i}l\bar{l}})}$$

on $E_0\backslash E_s$. Otherwise,

$$C_{35} \le (2+\Delta_0\varphi) = (1+\varphi_{1\bar{1}}) + (1+\varphi_{2\bar{2}}) \le C_{20}.$$

But $e^F = \det g = (1+\varphi_{1\bar{1}})(1+\varphi_{2\bar{2}})$, so F has a uniformly lower bound. In this case (say, D and $|\overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}| \le C_{21}$),

$$\int_D e^{-2\delta F}\overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}\,d\mu_0 \le C_{36}$$

and

$$\int_{E_0} e^{-2\delta F}\overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}\,d\mu_0 \le C_{28} + C_{36} + \int_{E_0\backslash(E_s\cup D)} e^{-2\delta F}\overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}\,d\mu_0.$$

On $E_0\backslash(E_s\cup D)$ we have

$$(2+\Delta_0\varphi) \le \tfrac{1}{2}v + \overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$$

and so

$$-\overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}} \le \tfrac{1}{2}v.$$

From (4.14), it follows that

$$\delta^{-1}\int_M |\overset{0}{\nabla} e^{-\delta F}|^2\,d\mu_0 \le -v\int_M e^{-2\delta F}\,d\mu_0 + \int_{E_0} e^{-2\delta F}(-\overset{0}{g}{}^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}})\,d\mu_0$$
$$\le -\frac{1}{2}v\int_M e^{-2\delta F}\,d\mu_0 + C_{28} + C_{36}.$$

Thus

$$\int_M e^{-2\delta F}\,d\mu_0 \le C_{37},$$

and the lemma is proved. □

COROLLARY 4.3. *Given the assumptions of Lemma 4.2, we have*

$$F \ge -C_{38}.$$

5. Asymptotic Convergence to the Metric with Constant Scalar Curvature

In this section, let $(M, [\omega_0])$ be a compact Kähler surface admitting no nonzero holomorphic tangent vector fields. We will show that there exists a subsequence of solutions of (1.1) that converges to the metric with constant Scalar curvature.

Consider the energy functional $\Phi(g)$ together with Simon's general results [S]. Then we have the following theorem.

THEOREM 5.1. *Given the assumptions of Theorem 1.5,*

$$R \xrightarrow{C^\infty} r \quad as\ t_j \to \infty,$$

where r is constant with $r = \left(\int_M R\,d\mu\right)/\left(\int_M d\mu\right)$.

REMARK 5.1. Here $\int_M R\,d\mu$ and $\int_M d\mu$ are all constants in a fixed Kähler class H_{Ω_0}.

Proof of Theorem 5.1. In view of (1.1), (1.2), Lemma 2.1, and the Cauchy–Schwarz inequality,

$$-\frac{d}{dt}\int_M R^2\,d\mu = 2\int_M |R_{\alpha\bar\beta}|^2\,d\mu \geq \int_M (\Delta R)^2\,d\mu$$

and so

$$\int_0^\infty \int_M (\Delta R)^2\,d\mu\,dt < \infty.$$

Then there exists a subsequence $\{t_j\}$ such that

$$\int_M (\Delta R)^2\,d\mu\Big|_{t_j} \to 0 \quad \text{as } t_j \to \infty.$$

We know that $\|F\|_{W^{k,2}} \leq C$ for all $0 \leq t_j \leq \infty$, so the elliptic estimates and interpolation inequalities yield

$$R \xrightarrow{C^\infty} r \quad \text{and} \quad g \xrightarrow{C^\infty} g_\infty$$

as $t_j \to \infty$ such that

$$\Delta_{g_\infty} R_\infty = 0. \qquad \square$$

Now, Theorem 1.2 and Corollary 1.3 follow from Theorem 3.2, Lemma 4.1, Lemma 4.2, and (3.1). Theorem 1.4 and Theorem 1.5 follow from Theorem 3.4, Lemma 4.1, Lemma 4.2, and Theorem 5.1.

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