Pluripolar Hulls

Norman Levenberg & Evgeny A. Poletsky

1. Introduction

Let E be a pluripolar set in $\mathbb{C}^N$. That is, for each $z_0 \in E$, there exists a neighborhood U of z_0 and a plurisubharmonic (psh) function $u \not\equiv -\infty$ on U with

$$E \cap U \subset \{ z \in U : u(z) = -\infty \}.$$

From the well-known result of Josefson (cf. [K, Thm. 4.7.4]), there exists a plurisubharmonic function u on $\mathbb{C}^N$, $u \not\equiv -\infty$, with $E \subset \{ z \in D : u(z) = -\infty \}$. For example, if f is holomorphic in an open set D, then

$$E := \{ z \in D : f(z) = 0 \} = \{ z \in D : u(z) := \log|f(z)| = -\infty \}$$

is pluripolar. It can happen that any psh function u that is $-\infty$ on a pluripolar set $E \subset D$ is automatically $-\infty$ on a larger set. As a simple example, if

$$E = \{ z \in \mathbb{C}^N : |z_1| < 1,\ z_2 = \cdots = z_N = 0 \},$$

then any globally defined psh function u that is $-\infty$ on E must be $-\infty$ on

$$E^* = \{ z \in \mathbb{C}^N : z_1 \in \mathbb{C},\ z_2 = \cdots = z_N = 0 \}.$$

This follows since $U(z_1) := u(z_1, 0, \dots, 0)$ is subharmonic on $\mathbb{C}$ and $-\infty$ on the *nonpolar* set $\{ z_1 \in \mathbb{C} : |z_1| < 1 \}$. To describe this phenomenon of "propagation" of pluripolar sets more concretely, given a pluripolar set E in $\mathbb{C}^N$ and a neighborhood D of E, we define two types of *pluripolar hulls* of E relative to D:

$$E_D^* := \bigcap \{ z \in D : u(z) = -\infty \},$$

where the intersection is taken over *all* psh functions in D that are $-\infty$ on E; and

$$E_D^- := \bigcap \{ z \in D : u(z) = -\infty \},$$

where the intersection is taken over all *negative* psh functions in D that are $-\infty$ on E. Clearly, $E_D^* \subset E_D^-$ and if $E \subset D_1 \subset\subset D_2$ then

$$E_{D_1}^- \subset E_{D_2}^* \cap D_1.$$

Received March 24, 1998. Revision received October 19, 1998.
N. Levenberg was partially supported by the Marsden Fund (NZ) and E. A. Poletsky by NSF grant DMS-9804755.
Michigan Math. J. 46 (1999).

In general, a precise description of the pluripolar hulls E_D^* and E_D^- is very difficult. One way of obtaining points in these hulls is if E hits a one-dimensional analytic variety A in a nonpolar set of points of A. Then the points of A lying in D belong to the hull. In the preceding example, the set $E = \{ z \in \mathbb{C}^N : |z_1| < 1, z_2 = \cdots = z_N = 0 \}$ hit the one-dimensional analytic variety $A := \{z_2 = \cdots = z_N = 0\}$ in a *nonpolar* set of points; $E \cap A$ was a disk. However, an example in [L] shows that $E_D^* \setminus E$ can be non-empty even if E hits all such varieties A in *polar* sets (cf. the remark at the end of Section 2).

In this paper we offer two criteria for a point to belong to E_D^-. The first one (Theorem 2.1; see also Corollary 2.2) works for arbitrary pluripolar sets E and claims that $E_D^- = \{ z \in D : \omega(z, E, D) > 0 \}$, where $\omega(z, E, D)$ is the harmonic measure of E relative to D (see Section 2). However, evaluation of the harmonic measure is in general quite difficult; thus, in Corollary 2.6 we present another criterion, which is valid for compact pluripolar sets E and claims that $z \in E_D^-$ if and only if there is a Jensen measure μ on D with barycenter at z such that $\mu(E) > 0$. Note that, by [P2], every Jensen measure is the limit of a sequence of push-forwards of the standard Lebesgue measure on the boundary of the disk under holomorphic mappings f_j $(j = 1, 2, \dots)$ of the disk into D.

Theorems 2.4 and 2.5 allow us to switch to E_D^* from E_D^-. Note that a point $z \in D$ lies outside of E_D^* (E_D^-) precisely when there exists u psh (and negative) in D with $u = -\infty$ on E but with $u(z) > -\infty$; that is, u "separates" E and z. The question as to whether one could find a psh u in $\mathbb{C}^2$ that separates the origin from the set $\{w = z^\alpha, z \neq 0\}$, where $\alpha > 0$ is an irrational number, is related to a problem of Sadullaev (see [S] and [B]). We solve this problem in Section 3 by using our techniques to determine the pluripolar hull of this set (Theorem 3.5).

To motivate our results, recall that in [P1] the second author gave a characterization of the polynomial hull $\hat{X}$ of a compact set X in $\mathbb{C}^N$; here,

$$\hat{X} := \{ (z_1, \dots, z_N) \in \mathbb{C}^N : |p(z_1, \dots, z_N)| \le \|p\|_X \text{ for all polynomials } p \}.$$

If X contains the boundary of an analytic disk—that is, if there exists a nonconstant holomorphic map $g = (g_1, \dots, g_N)$ from the unit disk $U \subset \mathbb{C}$ into $\mathbb{C}^N$ with $g^*(e^{it}) \in X$ for a.e. t (where $g^*(e^{it})$ denotes the radial limit value of g at e^{it})—then, by the maximum modulus principle, $\hat{X}$ contains the analytic disk $g(U)$. In [P1], the following result is proved.

THEOREM 1.1. *Let X be a compact set and let D be a Runge neighborhood of X. Fix $z_0 \in D$. Then $z_0 \in \hat{X}$ if and only if, for any open set $V \subset D$ containing X and for any $\varepsilon > 0$, there exists an analytic disk $g : \bar{U} \to D$ in D with $g(0) = z_0$ and*

$$m(\{ t \in [0, 2\pi] : g(e^{it}) \in V \}) > 2\pi - \varepsilon.$$

Here we write $g : \bar{U} \to D$ to mean g is holomorphic on U and continuous on $\bar{U}$. In Corollary 2.2 of the next section, we give an analogous characterization for a point z_0 to lie in the pluripolar hull E_D^- of a pluripolar set $E \subset D$.

This research was undertaken while the authors were visiting the mathematics departments at Indiana University (Bloomington) and the University of Toronto.

We wish to express our gratitude to these departments for their hospitality. We also are grateful to Eric Bedford for his suggestion to apply our methods to study the pluripolar hull of the set $\{w = z^\alpha, z \neq 0\}$.

2. Construction of Pluripolar Hulls

We write PSH(D) for the class of psh functions on D. Given a function ϕ on a domain D in $\mathbb{C}^N$, we define the *psh envelope of* ϕ to be

$$P_\phi(z) := \sup\{\, u(z) : u \in \mathrm{PSH}(D),\ u \le \phi \text{ in } D\,\}.$$

If ϕ is upper semicontinuous on D then $P_\phi(z)$ is psh in D and, by [P1],

$$P_\phi(z) = \inf\left\{ \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{it}))\,dt : f : \bar{U} \to D \text{ holomorphic},\ f(0) = z\,\right\}.$$

For a subset E of a domain $D \subset \mathbb{C}^N$, we define $\omega(z, E, D) := -P_\phi(z)$, where $\phi = -\chi_E$, and call this quantity the *harmonic measure of* E (relative to D) at z. If E is open then, by the preceding equation,

$$\omega(z, E, D) = \frac{1}{2\pi}\sup\{m\{\, t \in [0, 2\pi] : f(e^{it}) \in E\,\}\}, \tag{1}$$

where the supremum is taken over all $f : \bar{U} \to D$ with $f(0) = z$. In particular, if there exists an $f : \bar{U} \to D$ with $f(0) = z$ and $m\{\, e^{it} \in \partial U : f(e^{it}) \in E\,\} > 2\pi a$, then $\omega(z, E, D) > a$; and if $\omega(z, E, D) < a$ then, for any $f : \bar{U} \to D$ with $f(0) = z$, we have $m\{\, e^{it} \in \partial U : f(e^{it}) \in E\,\} < 2\pi a$.

It follows that, for a subset E of D,

$$\omega(z, E, D) = \inf\{\, \omega(z, V, D) : V \subset D \text{ is open and } E \subset V\,\}. \tag{2}$$

Indeed, clearly the right-hand side of (2) is greater than or equal to $\omega(z, E, D)$. On the other hand, for any $\varepsilon > 0$ and any point $z_0 \in D$, by definition of $\omega(z_0, E, D)$ we can find a psh function u on D with $u \le -\chi_E$ on D such that $\omega(z_0, E, D) + \varepsilon > -u(z_0)$. Let $V = \{\, z \in D : u(z) < -1 + \varepsilon\,\}$. Then V is open and contains E. Moreover,

$$\omega(z, V, D) \le -\frac{u(z)}{1-\varepsilon}$$

for all $z \in D$; thus,

$$\omega(z_0, V, D) \le -\frac{u(z_0)}{1-\varepsilon} < \frac{\omega(z_0, E, D) + \varepsilon}{1-\varepsilon}.$$

Since $\varepsilon > 0$ and $z_0 \in D$ are arbitrary, we obtain (2).

In the next three results (Theorems 2.1 and Corollaries 2.2 and 2.3), to avoid trivialities, we assume that D admits negative, nonconstant psh functions.

THEOREM 2.1. *Let D be a domain in $\mathbb{C}^N$, and let $E \subset D$ be pluripolar. Then $E_D^- = \{\, z \in D : \omega(z, E, D) > 0\,\}$.*

Proof. First of all, if $z_0 \in D$ and $\omega(z_0, E, D) > 0$ then, for any $v \in \mathrm{PSH}(D)$ with $v < 0$ in D and $v = -\infty$ on E, we have $u_n(z) := v(z)/n \leq -\omega(z_0, E, D)$ for each positive integer $n = 1, 2, \ldots$. Thus, in particular,

$$v(z_0) \leq -n\omega(z_0, E, D), \quad n = 1, 2, \ldots;$$

letting $n \to \infty$, we obtain $v(z_0) = -\infty$ and hence $z_0 \in E_D^-$. Conversely, if $z_0 \in D$ and $\omega(z_0, E, D) = 0$ then, by definition of $-\omega$, we can find a sequence of negative psh functions $\{u_j\}$ in D with $u_j \leq -1$ on E and $u_j(z_0) > -1/2^j$. Then

$$u(z) := \sum_{j=1}^{\infty} u_j(z)$$

is a negative psh function in D (the partial sums form a decreasing sequence of psh functions, since each u_j is nonpositive) that is not identically $-\infty$—indeed, $u(z_0) > -1$—but since $u_j \leq -1$ on E for each j, we have $u = -\infty$ on E. Since $u(z_0) > -1$, we have $z_0 \notin E_D^-$. □

REMARK. If $F \subset E \subset D$ with E pluripolar and if $E \subset F_D^-$, then of course $E_D^- = F_D^-$; thus, in this situation,

$$E_D^- = \{ z \in D : \omega(z, F, D) > 0 \}.$$

This observation will be used in the proof of Theorem 3.5.

Theorem 2.1, together with equation (2), immediately implies the following.

COROLLARY 2.2. *Let D be a domain in $\mathbb{C}^N$, and let $E \subset D$ be pluripolar. Fix $z_0 \in D$. Then $z_0 \in E_D^-$ if and only if there exists an $a > 0$ such that, for any open neighborhood $V \subset D$ of E, there exists a holomorphic map $f : \bar{U} \to D$ with $f(0) = z_0$ and*

$$m(\{ t \in [0, 2\pi] : f(e^{it}) \in V \}) > 2\pi a.$$

Proof. Suppose first that there does exist an $a > 0$. Then

$$\omega(z_0, V, D) > a$$

for every open neighborhood $V \subset D$ of E; from (2) we obtain

$$\omega(z_0, E, D) \geq a,$$

so that $z_0 \in E_D^-$ by Theorem 2.1. Conversely, suppose $z_0 \in E_D^-$ but that for all $a > 0$ there exists a neighborhood $V \subset D$ of E such that, for any holomorphic map $f : \bar{U} \to D$ with $f(0) = z_0$,

$$m(\{ t \in [0, 2\pi] : f(e^{it}) \in V \}) < 2\pi a.$$

Then $\omega(z_0, V, D) < a$. From (2), $\omega(z_0, E, D) < a$; this being valid for all $a > 0$, we have $\omega(z_0, E, D) = 0$, which contradicts Theorem 2.1. □

If E is compact, we can find a sequence of holomorphic maps through z_0 which (eventually) works for *any* neighborhood of E.

COROLLARY 2.3. *Let D be a domain in $\mathbb{C}^N$, and let $E \subset D$ be compact and pluripolar. Fix $z_0 \in D$. Then $z_0 \in E_D^-$ if and only if there exists an $a > 0$ and a sequence $\{f_j\}$ of holomorphic maps $f_j : \bar{U} \to D$ with $f_j(0) = z_0$ such that, for any open neighborhood $V \subset D$ of E, there exists j_0 such that, for all $j \geq j_0$,*

$$m(\{ t \in [0, 2\pi] : f_j(e^{it}) \in V \}) > 2\pi a.$$

Proof. The "if" follows from Corollary 2.2. For the "only if", suppose $z_0 \in E_D^-$. For each $j = 1, 2, \ldots$, set

$$V_j := \{ z \in D : \operatorname{dist}(z, E) < 1/j \}.$$

From Corollary 2.2, for each j we get a holomorphic map $f_j : \bar{U} \to D$ with $f_j(0) = z_0$ and

$$m(\{ t \in [0, 2\pi] : f_j(e^{it}) \in V_j \}) > 2\pi a.$$

The open sets $\{V_j\}$ are nested and, for any open neighborhood $V \subset D$ of E, there is an integer $j_0(V)$ such that $V_j \subset V$ for $j > j_0$; this completes the proof. □

To pass from local pluripolar hulls to global pluripolar hulls, we prove the following theorem.

THEOREM 2.4. *Let D be a pseudoconvex domain in $\mathbb{C}^N$. Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\bigcup_j D_j = D$. Let $E \subset D$ be pluripolar. Then*

$$E_D^* = \bigcup_j (E \cap D_j)_{D_j}^-.$$

Proof. Without loss of generality, we let ρ be a psh exhaustion function for D and assume that $D_j := \{ z \in D : \rho(z) < r_j \}$, $r_j \uparrow +\infty$, with $r_j - r_{j-1} \geq 1$. For if we have any increasing sequence of relatively compact subdomains $\{G_j\}$ with $\bigcup_j G_j = D$, then each G_j is contained in D_k for k sufficiently large. Take $z_0 \in \bigcup_j (E \cap D_j)_{D_j}^-$. Then $z_0 \in (E \cap D_j)_{D_j}^-$ for some j. For any $v \in \operatorname{PSH}(D)$ with $v = -\infty$ on E, we can find a constant $c = c(v)$ such that $v - c < 0$ on D_j. Since $z_0 \in (E \cap D_j)_{D_j}^-$, it follows that $v(z_0) - c = -\infty$ so $v(z_0) = -\infty$; that is, $z_0 \in E_D^*$. For the reverse inclusion, take $z_0 \in E_D^*$ and suppose $z_0 \notin \bigcup_j (E \cap D_j)_{D_j}^-$; for simplicity in notation, we assume $z_0 \in D_1$. Then, for each $j = 1, 2, \ldots$, we can find $u_j \in \operatorname{PSH}(D_j)$ with $u_j < 0$ in D_j and $u_j = -\infty$ on $E \cap D_j$ but $u_j(z_0) > -1/2^j$. We define the following (psh) functions in D:

$$p_j(z) := \begin{cases} \max[u_j(z), \rho(z) - r_j], & z \in D_j, \\ \rho(z) - r_j, & z \in D \setminus D_j. \end{cases}$$

Set $p(z) := \sum_{j=1}^{\infty} p_j(z)$. Note first of all that $p \not\equiv -\infty$ since $p_j(z_0) \geq u_j(z_0) > -1/2^j$ implies that $p(z_0) \geq -1$. Next, we claim that $p \in \operatorname{PSH}(D)$. For if $\omega \subset\subset D$ then we have $\omega \subset D_j$ for $j > j_0 = j_0(\omega)$. Since $p_j < 0$ on D_j, we have $p_j < 0$ on ω for $j > j_0$ and so the partial sums in the series defining p form a decreasing sequence of psh functions on ω; hence p is psh on ω. Finally, to show that $p = -\infty$ on E, from the assumption that $r_j - r_{j-1} \geq 1$ it follows that $p_j \leq -1$

on $E \cap D_{j-1}$. Thus, for any point $z \in E$, since $z \in D_{j-1}$ for $j > j_0(z)$ we have $p(z) = -\infty$. Thus $z_0 \notin E_D^*$, a contradiction. □

Suppose D is *hyperconvex*—that is, D admits a continuous *negative* psh exhaustion function ρ; thus $\{z \in D : \rho(z) < c\} \subset\subset D$ for all $c < 0$. Then we get a similar conclusion for the hull E_D^-.

THEOREM 2.5. *Let D be a hyperconvex domain in $\mathbb{C}^N$. Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\bigcup_j D_j = D$. Let $E \subset D$ be pluripolar. Then*

$$E_D^- = \bigcup_j (E \cap D_j)_{D_j}^-.$$

Proof. We may take $D_j := \{z \in D : \rho(z) < -1/2^j\}$, where ρ is a negative psh exhaustion function for D. The inclusion $\bigcup_j (E \cap D_j)_{D_j}^- \subset E_D^-$ is obvious from the definitions. For the reverse inclusion, take $z_0 \in E_D^-$ and suppose $z_0 \notin \bigcup_j (E \cap D_j)_{D_j}^-$; again we assume $z_0 \in D_1$. Then, for each $j = 1, 2, \dots$, we can find $u_j \in \mathrm{PSH}(D_j)$ with $u_j < 0$ in D_j and $u_j = -\infty$ on $E \cap D_j$ but $u_j(z_0) > -1/2^j$. As in the proof of Theorem 2.4, we define (psh) functions in D via

$$p_j(z) := \begin{cases} \max[u_j(z), \rho(z) + 1/2^j], & z \in D_j, \\ \rho(z) + 1/2^j, & z \in D \setminus D_j. \end{cases}$$

Set $p(z) := \left[\sum_{j=1}^{\infty} p_j(z)\right] - 1$. Note first of all that $p \not\equiv -\infty$ since $p_j(z_0) \geq u_j(z_0) > -1/2^j$ implies that $p(z_0) \geq -2$. Next, we claim that $p \in \mathrm{PSH}(D)$. For any $\omega \subset\subset D$ we have $\omega \subset D_j$ for $j > j_0 = j_0(\omega)$. Since $p_j < 0$ on D_j, we have $p_j < 0$ on ω for $j > j_0(\omega)$; hence the partial sums in the series defining p form a decreasing sequence of psh functions on ω and p is psh on ω. Clearly $p < 0$ on D, since each $p_j < 1/2^j$ on D. Finally, to show that $p = -\infty$ on E, fix $z \in E$. Since $z \in D_j$ for $j \geq j_0(z)$, it follows that $p_j(z) \leq \rho(z) + 1/2^j$ for $j \geq j_0(z)$. Thus, using the fact that $\rho(z) < 0$, we get

$$\begin{aligned} p(z) + 1 &= \sum_{j=1}^{j_0(z)} p_j(z) + \sum_{j > j_0(z)} p_j(z) \\ &\leq \sum_{j=1}^{j_0(z)} p_j(z) + \sum_{j > j_0(z)} \left(\rho(z) + \frac{1}{2^j}\right) = -\infty. \end{aligned}$$

We conclude that $z_0 \notin E_D^-$, a contradiction. □

REMARK. Note that the sets $(E \cap D_j)_{D_j}^-$ in Theorems 2.4 and 2.5 are monotone. That is,

$$(E \cap D_{j+1})_{D_{j+1}}^- \supset (E \cap D_j)_{D_j}^-, \quad j = 1, 2, \dots.$$

For $z \in D$, we denote by $\mathcal{J}_z(D)$ the set of all *Jensen measures* (with respect to psh functions on D) with barycenter at z; precisely, $\mu \in \mathcal{J}_z(D)$ if μ is a probability measure with compact support in D and, for each $u \in \mathrm{PSH}(D)$,

$$u(z) \le \int u\, d\mu.$$

It follows that if $\phi\colon D \to \mathbb{R}$ is Borel measurable then

$$P_\phi(z) \le \inf\left\{ \int \phi\, d\mu : \mu \in \mathcal{J}_z(D) \right\} := J_\phi(z). \tag{3}$$

Clearly, if $f : \bar{U} \to D$ is holomorphic with $f(0) = z$ then μ_f := push-forward of $dt/2\pi$ under f is an element in $\mathcal{J}_z(D)$.

COROLLARY 2.6. *Let D be a hyperconvex domain in $\mathbb{C}^N$, and let $E \subset D$ be compact and pluripolar. Fix $z_0 \in D$. Then $z_0 \in E_D^-$ if and only if there exists a $\mu \in \mathcal{J}_{z_0}(D)$ with $\mu(E) > 0$.*

Proof. Let $\phi = -\chi_E$. If $z_0 \in D$ and there exists a $\mu \in \mathcal{J}_{z_0}(D)$ with $\mu(E) > 0$, then

$$J_\phi(z_0) \le \int \phi\, d\mu = -\mu(E) < 0;$$

thus, by (3), $P_\phi(z_0) < 0$. Hence $z_0 \in E_D^-$ by Theorem 2.1. Conversely, if $z_0 \in E_D^-$ then, by Theorem 2.5 (and using the same notation), $z_0 \in (E \cap D_j)_{D_j}^-$ for j sufficiently large. Fix such a j. As in the proof of Corollary 2.3, we take $a > 0$ and $f_k : \bar{U} \to D_j$ holomorphic with $f_k(0) = z_0$ and

$$m(\{ t \in [0, 2\pi] : f_k(e^{it}) \in V_k \}) > a, \quad k = 1, 2, \dots, \tag{4}$$

where $V_k := \{ z \in D_j : \operatorname{dist}(z, E) < 1/k \}$. We take a subsequence of the mappings $\{f_k\}$ such that the corresponding measures $\{\mu_{f_k}\}$ converge weak-$*$ to a measure $\mu \in \mathcal{J}_{z_0}(D)$ supported in $\bar{D}_j$; by (4), $\mu(E) > a$. $\square$

REMARK. We cannot replace $\mu \in \mathcal{J}_{z_0}(D)$ in Corollary 2.6 by μ_f for some holomorphic $f : \bar{U} \to D$ with $f(0) = z_0$. To see this, recall that Wermer [W] constructed a compact set X in $\partial U \times \mathbb{C} \subset \mathbb{C}^2$ with $\hat{X} \subset \bar{U} \times \mathbb{C}$ and such that $Y := \hat{X} \setminus X \subset U \times \mathbb{C}$ does not contain any analytic disk; that is, there is no nonconstant holomorphic $g : U \to \mathbb{C}^2$ with $g(U) \subset Y$. In [L], we showed that such a set can be constructed so that Y is pluripolar; then in [LS] we showed that any such pluripolar Wermer-type set Y is *complete pluripolar in $U \times \mathbb{C}$*; that is, there exists a u psh in $U \times \mathbb{C}$ with $E = \{ z \in U \times \mathbb{C} : u(z) = -\infty \}$. Let $M > 1$ be chosen sufficiently large so that $Y \subset D := \{ (z, w) : z \in U, |w| < M \}$. Then $Y = Y_D^*$. Fix $r < 1$ and let $Y_r := \{ (z, w) \in Y : |z| = r \}$. Using standard properties of polynomial hulls, $\hat{Y}_r = \{ (z, w) \in Y : |z| \le r \}$. Since D is Runge, it follows that $\hat{Y}_r \subset (Y_r)_D^*$ and so

$$Y_r \subset \hat{Y}_r \subset (Y_r)_D^* \subset (Y_r)_D^-.$$

Fix a point $(z_0, w_0) \in \hat{Y}_r \setminus Y_r \subset (Y_r)_D^- \setminus Y_r$. If $f : \bar{U} \to D$ is holomorphic with $f(0) = (z_0, w_0)$ and $\mu_f(Y_r) > 0$, then for any u psh in D that is $-\infty$ on Y_r we have $u = -\infty$ on $f(U)$; thus,

$$f(U) \subset (Y_r)_D^* \subset Y_D^* = Y,$$

which implies—since Y contains no nonconstant analytic disk—that f is constant. This contradicts the fact that $f(0) = (z_0, w_0) \in \hat{Y}_r \setminus Y_r$ while $\mu_f(Y_r) > 0$.

3. The Set $w = z^\alpha$

The goal of this section is to find the pluripolar hull of the set

$$\tilde{E}_\alpha := \{ (z, w) \in \mathbb{C}^2 : w = z^\alpha,\ z \neq 0 \}$$

when $\alpha > 0$ is irrational. A preliminary remark on the definition of this set is in order. Consider the real-analytic curve

$$E_\alpha := \{ (x, y) \in \mathbb{R}^2 : y = x^\alpha,\ x > 0 \}.$$

We consider all analytic continuations of $f(x) = x^\alpha$ on $x > 0$; then $\tilde{E}_\alpha$ is the Riemann surface generated by E_α. In particular, note that $\tilde{E}_\alpha$ contains no points of the form $(0, w)$.

We mention that if $\alpha = p/q$ is rational then, using the psh function $v(z, w) := \log|w^q - z^p|$, we see that the pluripolar hull of $\tilde{E}_\alpha$ with respect to $\mathbb{C}^2$ is contained in the union of $\tilde{E}_\alpha$ and the origin. But the origin also belongs to the pluripolar hull because, if a psh function $u(z, w)$ is equal to $-\infty$ on $\tilde{E}_\alpha$, then the function $U(\zeta) := u(\zeta^q, \zeta^p)$ equals $-\infty$ on $\mathbb{C} \setminus \{0\}$ and hence equals $-\infty$ everywhere. Thus the pluripolar hull of $\tilde{E}_\alpha$ equals the union of $\tilde{E}_\alpha$ and the origin.

We show (Theorem 3.5) that, when α is irrational, the pluripolar hull of $\tilde{E}_\alpha$ equals $\tilde{E}_\alpha$. We begin with the essential lemmas.

LEMMA 3.1. *Let $D \subset \mathbb{C}^N$ and $E \subset D$ and let $A \subset D$ be a closed, pluripolar set with $E \cap A = \emptyset$. Then $\omega(z, E, D) = \omega(z, E, D \setminus A)$ on $D \setminus A$.*

Proof. Clearly $\omega(z, E, D) \geq \omega(z, E, D \setminus A)$. On the other hand, if u is a negative psh function on $D \setminus A$ and $u \leq -1$ on E, then u extends to be psh and negative in D (cf. [K, Thm. 2.9.22]). Thus, since $E \cap A = \emptyset$, the extension is less than or equal to -1 on E and therefore $\omega(z, E, D) \leq \omega(z, E, D \setminus A)$. □

LEMMA 3.2. *Let $D \subset \mathbb{C}^N$ and $G \subset \mathbb{C}^M$ be domains, and let $h\colon D \to G$ be a holomorphic mapping. If $E \subset G$ then $\omega(z, h^{-1}(E), D) \leq \omega(h(z), E, G)$.*

Proof. If u is a negative psh function on G that is less than or equal to -1 on E, then $u \circ h$ is a negative psh function on D that is less than or equal to -1 on $h^{-1}(E)$. Thus, $\omega(z, h^{-1}(E), D) \leq \omega(h(z), E, G)$. □

We need equality to hold for holomorphic covering maps h in certain circumstances.

LEMMA 3.3. *Let D and G be domains in $\mathbb{C}^N$, and let $h\colon D \to G$ be a holomorphic covering mapping. Suppose that a set $E \subset G$ has a simply connected open neighborhood V such that $h^{-1}(V)$ is the union of disjoint connected open sets V_j $(j = 1, 2, \dots)$ and that, for some point $z \in D$,*

$$\lim_{j\to\infty}\omega\big(z, \textstyle\bigcup_{k=j}^{\infty} V_k, D\big) = 0.$$

Then $\omega(z, h^{-1}(E), D) = \omega(h(z), E, G)$.

Proof. By Lemma 3.2, $\omega(z, h^{-1}(E), D) \le \omega(h(z), E, G)$. To verify the reverse inequality, we first fix $\varepsilon > 0$ and take j sufficiently large so that

$$\omega\big(z, \textstyle\bigcup_{k=j}^{\infty} V_k, D\big) < \varepsilon.$$

Take an open neighborhood W of $h^{-1}(E)$ such that

$$\omega(z, h^{-1}(E), D) \ge \omega(z, W, D) - \varepsilon. \tag{5}$$

Let $W_j = W \cap V_j$ and $W' = \bigcap_{k=1}^{j-1} h(W_k)$. Then W' is an open set containing E and $W' \subset V$. Thus, using (1) and (2), we can find a holomorphic mapping $f : \bar{U} \to G$ such that $f(0) = h(z)$ and

$$m(\{t \in [0, 2\pi] : f(e^{it}) \in W'\}) > 2\pi(\omega(h(z), E, G) - \varepsilon). \tag{6}$$

Let g be a lifting of f, that is, $h \circ g = f$ and $g(0) = z$. If $\tilde{W} = h^{-1}(W')$ and $A = \{t \in [0, 2\pi] : g(e^{it}) \in \tilde{W}\}$, then

$$m(A) = m(\{t \in [0, 2\pi] : f(e^{it}) \in W'\}). \tag{7}$$

Since $\tilde{W} = \bigcup_{k=1}^{\infty}(\tilde{W} \cap V_k)$ and this is a union of disjoint sets, we have

$$\omega\big(z, \textstyle\bigcup_{k=j}^{\infty}(\tilde{W} \cap V_k), D\big) < \varepsilon.$$

Therefore, the measure of those points t in A where $g(e^{it}) \in \bigcup_{k=j}^{\infty}(\tilde{W} \cap V_k)$ is less than $2\pi\varepsilon$. Thus,

$$\omega\big(z, \textstyle\bigcup_{k=1}^{j-1}(\tilde{W} \cap V_k), D\big) \ge \frac{1}{2\pi} m(A) - \varepsilon > \omega(h(z), E, G) - 2\varepsilon,$$

where the second inequality uses (6) and (7). But $\bigcup_{k=1}^{j-1}(\tilde{W} \cap V_k) \subset \bigcup_{k=1}^{j-1} W_k \subset W$, and from the preceeding inequality together with (5) we obtain

$$\omega(z, h^{-1}(E), D) \ge \omega(z, W, D) - \varepsilon > \omega(h(z), E, G) - 3\varepsilon.$$

Since ε is arbitrary, we deduce that $\omega(z, h^{-1}(E), D) = \omega(h(z), E, G)$. □

LEMMA 3.4. *Let $D \subset\subset G$ be domains in $\mathbb{C}^N$. Let $E \subset D$ be compact, and let V be a domain in G that contains a point $z \in D$ and does not intersect E. Let $K = \overline{\partial V \cap D}$. If $\omega(z, E, D) = a$ then there is a point $w \in K$ such that $\omega(w, E, G) \ge a$.*

Proof. Note that K separates z and E in D. To prove the lemma, we take a sequence of open $(1/j)$-neighborhoods $V_j \subset D$ of E so that $\omega(z, V_j, G) \to \omega(z, E, G)$ as $j \to \infty$. For each j, we take a holomorphic mapping $f_j : \bar{U} \to D$ such that $f_j(0) = z$ and such that the length of the set $A_j = \{t \in [0, 2\pi] : f_j(e^{it}) \in V_j\}$ is greater than or equal to $2\pi(a - 1/j)$. Let h_j be a harmonic function on U with boundary values equal to χ_{A_j}. Then $h_j(0) \ge a - 1/j$ and, by the maximum principle, there is a point $\zeta_j \in f_j^{-1}(K)$ with $h_j(\zeta_j) > a - 1/j$. Since K is compact, we may assume (by taking a subsequence if necessary) that the points $w_j = f(\zeta_j)$ converge to a point $w \in K$. We may also assume that $|w_j - w| < 1/j^2$ and that $f_j(\bar{U}) - w_j + w \subset G$ for all j. Let

$$e_j(\zeta) = \frac{\zeta + \zeta_j}{1 + \overline{\zeta_j}\zeta}$$

and set $g_j(\zeta) = f_j(e_j(\zeta)) - w_j + w$. Then $g_j : \bar{U} \to G$ and $g_j(0) = w$. If $A_j' = e_j^{-1}(A_j)$ then $m(A_j') \geq 2\pi(a - 1/j)$; furthermore, $g_j(A_j') \subset V_{j-1}$ since $|w_j - w| < 1/j^2$. Thus $\omega(w, E, G) \geq a$. □

Now we can commence with the proof of the following theorem.

THEOREM 3.5. *If* $E = \{(z, w) \in \mathbb{C}^2 : w = z^\alpha,\ z \neq 0\}$, *where* $\alpha > 0$ *is an irrational number, then* $E^*_{\mathbb{C}^2} = E$.

Proof. By Theorem 2.4 it suffices to prove that $(E \cap D)^-_D = E \cap D$ for each bidisk $D \subset \mathbb{C}^2$. For simplicity in exposition and notation, we take $D := U \times U$ and write E^- for $(E \cap D)^-_D$.

Note that if we take a nonpolar piece of a "branch" of E, for example, by setting $\Delta := \{z : |z - 1/2| \leq 1/4\}$ and taking

$$F := \{(z, w) : z \in \Delta,\ w = e^{\alpha \log|z| + i\alpha \operatorname{Arg} z}\},$$

then $E \subset F^-_D$. Thus $F^-_D = E^-$ and, by the remark after Theorem 2.1, our goal is to evaluate $\omega((z, w), F, D)$. *We show for points* $(z, w) \in D$ *that* $\omega((z, w), F, D) > 0$ *if and only if* $(z, w) \in E$. For future use, we set $T := \{z : |z - 1/2| < 3/8\}$ so that $0 \notin \bar{T}$.

We first consider a point $(z, w) \in D \setminus E$ with $z \neq 0$. Let $A = D \cap \{z = 0\}$. By Lemma 3.1,

$$\omega((z, w), F, D \setminus A) = \omega((z, w), F, D).$$

Let $H := \{\xi \in \mathbb{C} : \Re\xi < 0\}$ and $G =: H \times U$, and define $h : G \to D \setminus A$ via $h(\xi, w) = (e^\xi, w)$. Then h is a holomorphic covering mapping.

The open set $V = T \times U$ is simply connected and contains F. Clearly $h^{-1}(V) = \bigcup_{j=-\infty}^{\infty}(T_j' \times U)$, where the set T_j' lies in the semi-infinite strip $\{\Re\xi < 0,\ (2j-1)\pi < \Im\xi < (2j+1)\pi\}$. These sets are open and disjoint. Thus, for every $R > 0$ we can choose j sufficiently large such that $\bigcup_{|k| \geq j} T_k'$ lies outside the disk of radius R centered at 0. Hence

$$\lim_{j\to\infty} \omega\big(\xi, \textstyle\bigcup_{|k|\geq j} T_k', H\big) = 0$$

for every $\xi \in H$. From Lemma 3.2, using the projection map $(\xi, w) \to \xi$ we conclude that

$$\lim_{j\to\infty} \omega\big((\xi, w), \big(\textstyle\bigcup_{|k|\geq j} T_k'\big) \times U, G\big) = 0$$

for every point $(\xi, w) \in G$. Thus, by Lemma 3.3,

$$\omega((\xi, w), h^{-1}(F), G) = \omega((e^\xi, w), F, D). \tag{8}$$

The set $h^{-1}(E \cap (D \setminus A))$ is the disjoint union of the analytic sets

$$E_j = \{ (\xi, w) \in G : w = e^{2j\alpha\pi i} e^{\alpha\xi} \} \quad (j = 0, \pm 1, \pm 2, \dots)$$

in G. For each j, we consider the negative psh function

$$u_j(\xi, w) := \ln|w - e^{2j\alpha\pi i} e^{\alpha\xi}| - 2$$

on G. Since u_j is $-\infty$ precisely on E_j, we conclude that

$$\omega((\xi, w), E_j, G) = 0 \quad \text{for } (\xi, w) \notin E_j;$$

thus, we conclude that $\omega((\xi, w), h^{-1}(E), G) = \sum_{j=-\infty}^{\infty} \omega((\xi, w), E_j, G) = 0$ when $(\xi, w) \notin h^{-1}(E)$. Then, because

$$\omega((\xi, w), h^{-1}(E), G) \geq \omega((\xi, w), h^{-1}(F), G) = \omega((e^{\xi}, w), F, D)$$

on G (here we use (8)), we deduce that $\omega((z, w), F, D) = 0$ if $(z, w) \in D \setminus E$ and $z \neq 0$.

Now suppose that $(z, w) \in E$. We take a point $(\xi, w) \in E_0$ such that $h(\xi, w) = (z, w)$. Since $\omega((\xi, w), E_j, G) = 0$ when $j \neq 0$, we see that

$$\omega((\xi, w), h^{-1}(F), G) = \omega((\xi, w), F_0, G),$$

where $F_0 = h^{-1}(F) \cap E_0$. Note that $F_0 = \{ (\xi, w) : \xi \in \Delta_0,\ w = e^{\alpha\xi} \}$, where Δ_0 is the connected component of the preimage of Δ under the mapping $z = e^{\xi}$ lying in the strip $\{\Re\xi < 0, -\pi < \Im\xi < \pi\}$. Here we are using the hypothesis that α is irrational to conclude that F_0 consists of a single component; clearly, then, $\omega(\xi, \Delta_0, H) \to 0$ as $\Re\xi \to -\infty$. By Lemma 3.2 applied to the projection map $(\xi, w) \to \xi$, it follows that $\omega((\xi, w), F_0, G) \to 0$ as $\Re\xi \to -\infty$. Finally, using Lemma 3.3 we conclude that $\omega((z, w), F, D) \to 0$ as $|z| \to 0$. This statement remains valid if we replace D by a larger (but fixed) polydisk.

To finish the proof, we consider points of the form $(0, w) \in D$. Suppose that there is a point $(0, w) \in D$ with $\omega((0, w), F, D) = a > 0$. Let $\tilde{G} = \{ (z, w) : |z| < 2,\ |w| < 2 \}$. By the previous paragraph, we can choose $r > 0$ sufficiently small so that $\omega((z, w), F, \tilde{G}) < a/2$ when $|z| \leq r$. Take

$$V =: \{ (z, w) : |z| < r,\ |w| < 2 \}.$$

By Lemma 3.4, there is a point $(r, w) \in \tilde{G}$ such that $\omega((r, w), F, \tilde{G}) \geq a > a/2$. Hence $a = 0$.

This concludes the proof that, for points $(z, w) \in D$, $\omega((z, w), F, D) > 0$ if and only if $(z, w) \in E$. By Theorem 2.1, $E_D^- = E \cap D$; finally, by Theorem 2.4, $E^*_{\mathbb{C}^2} = E$. □

This fact also answers an old question of Sadullaev. A set $E \in \mathbb{C}^N$ is called *plurithin* at a point $z_0 \in \bar{E}$ if there exists a psh function u on $\mathbb{C}^N$ such that

$$\limsup_{z \to z_0,\ z \in E} u(z) < u(z_0).$$

For example, every real-analytic curve is not plurithin at each of its points (see [S, Prop. 4.1]). Sadullaev asked whether the set E in Theorem 3.5 is plurithin at the origin (see [S, 5.3]).

COROLLARY 3.6. *The set $\tilde{E}_\alpha$ is plurithin at the origin when $\alpha > 0$ is irrational.*

Proof. Since $E^*_{\mathbb{C}^2} = E$, there is a psh function u on $\mathbb{C}^2$ such that $u(z) = -\infty$ when $z \in E$ and $u(0) > -\infty$. □

References

[B] E. Bedford, *Survey of pluripotential theory,* Several complex variables (Stockholm, 1987–88), Math. Notes, 38, Princeton Univ. Press, Princeton, NJ, 1993.

[K] M. Klimek, *Pluripotential theory,* Clarendon Press, Oxford, 1991.

[L] N. Levenberg, *On an example of Wermer,* Ark. Mat. 26 (1988), 155–163.

[LS] N. Levenberg and Z. Slodkowski, *Pseudoconcave pluripolar sets in* $\mathbb{C}^2$, Math. Ann. 312 (1998), 429–444.

[P1] E. Poletsky, *Holomorphic currents,* Indiana Univ. Math. J. 42 (1993), 85–144.

[P2] ———, *Analytic geometry on compacta in* $\mathbb{C}^n$, Math. Z. 222 (1996), 407–424.

[S] A. Sadullaev, *Plurisubharmonic measures and capacities on complex manifolds,* Uspekhi Mat. Nauk 36 (1981), 53–105.

[W] J. Wermer, *Polynomially convex hulls and analyticity,* Ark. Mat. 20 (1982), 129–135.

N. Levenberg
Department of Mathematics
University of Auckland
Auckland, New Zealand

E. A. Poletsky
Department of Mathematics
Syracuse University
Syracuse, NY 13244

eapolets@mailbox.syr.edu