Comparison of the Pluricomplex and the Classical Green Functions

MAGNUS CARLEHED

1. Introduction

1.1. The Classical Green Function

We define the fundamental solution for the Laplacian in $\mathbf{R}^N$ as

$$p(x) = \begin{cases} \log|x| & \text{if } N = 2, \\ -|x|^{2-N} & \text{if } N \geq 3. \end{cases}$$

Let Ω be a bounded domain in $\mathbf{R}^N$ with Lipschitz boundary, and fix $y \in \Omega$. Then Ω is regular for the Dirichlet problem

$$\begin{cases} \triangle u(x) = 0 \text{ in } \Omega, \\ u(x) = -p(x-y) \text{ on } \partial\Omega; \end{cases}$$

that is, there is a function $h_y(x)$, continuous on $\bar{\Omega}$, that solves this problem. Define

$$G(x, y) = p(x-y) + h_y(x).$$

This is the *classical Green function* for the Laplacian, with pole at y. It is negative and subharmonic in Ω, harmonic in $\Omega \setminus \{y\}$, and tends to zero on $\partial\Omega$. Near y, it behaves like $p(x-y)$. Furthermore, it is symmetric, that is, $G(y, x) = G(x, y)$.

Let $U(\Omega, y)$ be the class of subharmonic functions u in Ω such that $u(\zeta) \leq p(\zeta - y) + O(1)$ when $\zeta \to y$. Then, using the classical Perron method, one can easily see that

$$G(x, y) = \sup\{u(x);\ u \in U(\Omega, y),\ u \leq 0\}.$$

REMARK. In most texts, the Green function is defined to be the *negative* of our Green function.

1.2. The Pluricomplex Green Function

Let Ω be a bounded domain in $\mathbf{C}^n$. Let $V(\Omega, y)$ be the class of plurisubharmonic functions u in Ω such that $u(\zeta) \leq \log|\zeta - y| + O(1)$ when $\zeta \to y$. We define the *pluricomplex Green function* for Ω with pole in $y \in \Omega$:

$$g(x, y) = \sup\{v(x);\ v \in V(\Omega, y),\ v \leq 0\}.$$

The definition is due to Klimek [K2].

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The function $x \mapsto g(x, y)$ is continuous in $\Omega \setminus \{y\}$ if Ω is *hyperconvex*, which means that there exists a continuous, negative plurisubharmonic function ρ such that $\{z \in \Omega;\ \rho(z) < c\}$ is relatively compact in Ω for all $c < 0$. For example, a strictly convex set—or, more generally, a strictly pseudoconvex set (with C^2 boundary)—is always hyperconvex.

The function $x \mapsto g(x, y)$ is a negative plurisubharmonic function that is maximal (see [K1]) in $\Omega \setminus \{y\}$. It tends to zero on the boundary if and only if Ω is hyperconvex. Near y, it behaves like $\log|x - y|$.

It is not true in general that $g(x, y) = g(y, x)$ [BD]. However, the pluricomplex Green function is symmetric if Ω is strictly convex [L].

1.3. The Quotient

In this paper, we study the quotient $h(x, y) = g(x, y)/G(x, y)$ of the two Green functions. We let $N = 2n$, and will always assume that $n \geq 2$ ($n = 1$ is trivial). The function h can be extended to a nonnegative continuous function in $\Omega \times \Omega$, since $h(x, y) \to 0$ when $x, y \to \zeta \in \Omega$. The question is: When is h bounded in $\Omega \times \Omega$?

The case of the unit ball was treated in [C2], where it was shown that h is bounded by the constant $2^{2n-3}/(n-1)$ and that this constant is the best possible. In this paper we prove boundedness in strictly pseudoconvex domains (Sections 2–4). In fact, we have the following theorem.

THEOREM 1. *Let Ω be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$. Then there is a constant $C = C(\Omega) > 0$ such that*

$$0 \leq h(x, y) = \frac{g(x, y)}{G(x, y)} \leq C|x - y|^{2n-4}$$

for all $x, y \in \Omega$. In particular, the quotient is bounded.

It is obvious that hyperconvexity is a necessary condition for boundedness. However, it is not sufficient. In Section 5 we give a counterexample—namely, the bidisc in $\mathbf{C}^2$, which has only Lipschitz boundary.

For the purpose of proving Theorem 1, we show an estimate for the pluricomplex Green function (Theorem 3). Some results of this paper can also be found in [C1].

REMARK. It is also natural to consider $x \mapsto 1/h(x, y)$, for a *fixed* pole y. Let Ω be bounded, with C^2 boundary. Then a classical lemma, due to Keldysh, Lavrent'ev, and Hopf (see [P]), states that $-g(x, y) > C\delta(x)$ for some constant $C = C(y)$, where $\delta(x)$ is the distance from x to the boundary of Ω. Combining this with well-known estimates for $G(x, y)$ (e.g., (3) in Section 3), we conclude that $x \mapsto 1/h(x, y)$ is bounded near $\partial\Omega$.

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2. Some Estimates for the Classical Green Function

Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 3$, with C^2 boundary. Let $\delta(\xi) = \delta_\Omega(\xi)$ denote the distance from ξ to the boundary of Ω. We adopt the convention that C is a uniform positive constant that may change in value from line to line. To prove boundedness for h, we need an estimate of $G(x, y)$ away from zero. Such an estimate was proved (for $C^{1,1}$ domains) by Zhao in 1986 [Z].

THEOREM 2 [Z]. *The following inequalities hold for* $x, y \in \Omega$:

$$-G(x, y) \geq \frac{C}{|x-y|^{N-2}} \quad \text{if} \quad |x-y| \leq \max\left(\frac{\delta(x)}{2}, \frac{\delta(y)}{2}\right); \tag{1}$$

$$-G(x, y) \geq \frac{C\delta(x)\delta(y)}{|x-y|^{N}} \quad \text{if} \quad |x-y| > \max\left(\frac{\delta(x)}{2}, \frac{\delta(y)}{2}\right). \tag{2}$$

Inequality (2) is also proved in [C3], using a different method. In addition, that paper contains sharper estimates in two special cases.

3. An Estimate for the Pluricomplex Green Function in Strictly Pseudoconvex Domains

We define strictly convex and strictly pseudoconvex domains in the standard way (see [Kr]). In particular, these domains have C^2 boundary. The main result of this section is the following estimate.

THEOREM 3. *Let* $\Omega \subset \mathbf{C}^n$ *be a bounded, strictly pseudoconvex domain. Then there exists a constant* $C = C(\Omega) > 0$ *such that*

$$-g(x, y) \leq C\frac{\delta(x)\delta(y)}{|x-y|^4}$$

for all $x, y \in \Omega$.

The following similar estimate for the classical Green function,

$$-G(x, y) \leq C\frac{\delta(x)\delta(y)}{|x-y|^N}, \tag{3}$$

was proved in [Ke] and can also be found in [Kr, pp. 324–331]. We will use some ideas and notation from these sources in this section.

LEMMA 4. *Let* Ω *be a bounded domain in* $\mathbf{R}^N$, $N \geq 2$, *with* C^2 *boundary. Then there exists an* η, $0 < \eta < \operatorname{diam}(\Omega)/2$, *such that*:

(1) *for each* $y \in \partial\Omega$ *there exist balls* $B(z_y, \eta) \subset \Omega$ *and* $\tilde{B}(\tilde{z}_y, \eta) \subset \Omega^c$ *that satisfy* $\bar{B}(z_y, \eta) \cap \overline{\Omega^c} = \{y\}$ *and* $\bar{\tilde{B}}(\tilde{z}_y, \eta) \cap \bar{\Omega} = \{y\}$.

(2) *for each point* ξ *in* $U = \{\xi \in \Omega;\ \delta(\xi) < \eta\}$ *there is a unique nearest point* $\pi\xi$ *in* $\partial\Omega$, *and* $\xi - \pi\xi$ *is an inner normal to the boundary at* $\pi\xi$.

Proof. The idea is to apply the inverse function theorem to the mapping

$$\partial\Omega \times (-1, 1) \to \mathbf{R}^N,$$
$$(\zeta, t) \mapsto \zeta + t\nu_\zeta,$$

where ν_ζ is the outer unit normal at ζ. See [Kr, p. 325]. □

REMARK. It follows from the proof of Lemma 4 that the mapping $x \mapsto \pi x$ is C^1 in $\bar{U}$.

LEMMA 5. *For $\Omega = B(0, R)$ there is a constant $C > 0$ (depending on the dimension only), such that*

$$-g_{B(0,R)}(x, y) \le C\frac{R\delta(y)}{|x-y|^2}.$$

Proof. The pluricomplex Green function for the ball $B(0, R)$ is

$$g(x, y) = \log|T_{y/R}(x/R)|,$$

where T_a denotes the Möbius transformation mapping a onto the origin. Explicitly,

$$T_a(x) = \frac{a - P_a(x) - \sqrt{1-|a|^2}Q_a(x)}{1 - \langle x, a\rangle},$$

where

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$$

and

$$P_a(x) = \frac{\langle x, a\rangle}{\langle a, a\rangle}a,\ Q_a(x) = x - P_a(x).$$

(See [K1, p. 148, p. 224].) Because both sides of the inequality are invariant under the dilation $x \mapsto x/R$, we can take $R = 1$. Furthermore, since they are invariant under rotations, we may assume that $y = (t, 0, \ldots, 0)$ where $t \in \mathbf{R}^+$. Then

$$\begin{aligned}
-g(x, y) &= \frac{1}{2}\log\frac{|1-tx_1|^2}{|t-x_1|^2 + q(1-t^2)} \\
&= \frac{1}{2}\log\left(1 + \frac{(1-|x_1|^2-q)(1-t^2)}{|t-x_1|^2+q(1-t^2)}\right) \\
&\le \frac{1}{2}\frac{(1-|x_1|^2-q)(1-t^2)}{|t-x_1|^2+q(1-t^2)},
\end{aligned}$$

where $q = |x_2|^2 + \cdots + |x_n|^2$. Hence

$$\frac{-g(x, y)|x-y|^2}{\delta(y)} \le \frac{(1-|x_1|^2)(|t-x_1|^2+q)}{|t-x_1|^2+q(1-t^2)}.$$

For fixed t and x_1 this is an increasing function of q, as can easily be seen by differentiation. Because $q < 1 - |x_1|^2$, we obtain

$$\begin{aligned}\frac{-g(x,y)|x-y|^2}{\delta(y)} &\le \frac{(1-|x_1|^2)(|t-x_1|^2+1-|x_1|^2)}{|t-x_1|^2+(1-|x_1|^2)(1-t^2)} \\ &= (1-|x_1|^2)\left(1+\frac{t^2(1-|x_1|^2)}{|1-tx_1|^2}\right) \le 1+4\left(\frac{1-|x_1|}{|1-tx_1|}\right)^2.\end{aligned}$$

Since $|1-tx_1| \ge 1 - t|x_1| \ge 1 - |x_1|$, the lemma is proved. □

LEMMA 6. *Let Ω be a bounded, strictly convex domain in $\mathbf{C}^n$. Then there exists a constant $C > 0$ such that*

$$-g_\Omega(x,y) \le C\frac{\delta(y)}{|x-y|^2}.$$

Proof. There is a positive number R with the following property: For each point $\xi \in \partial\Omega$ we can find a ball B_ξ, with radius R, that is tangent to Ω at ξ and such that $\Omega \subset B_\xi$. Use Lemma 4 to produce a neigborhood U. Then, for each $y \in U \cap \Omega$, we use the ball $B_{\pi y}$. From the definition of the pluricomplex Green function, note that if $\Omega_1 \subset \Omega_2$ then $g_{\Omega_1}(x,y) \ge g_2(x,y)$. According to Lemma 5, we then have

$$-g_\Omega(x,y) \le -g_{B_R}(x,y) \le C\frac{\delta_{B_R}(y)}{|x-y|^2}.$$

But, as $\delta_{B_R}(y) = \delta_\Omega(y)$, the lemma follows when $y \in U \cap \Omega$. For $y \in \Omega \setminus U$, the lemma is obvious. □

LEMMA 7. *Theorem 3 is true if Ω is strictly convex.*

Proof. Let $\alpha > 0$ and let T_α be the half-ball $B(0,\alpha) \cap \{\operatorname{Im} z_n < 0\}$. Let $H_\alpha(z) = \operatorname{Im} z_n/\alpha^2$. H_α is a pluriharmonic function, negative in T_α. If $\tilde{x} \in \partial\Omega$, rotate and translate T_α such that 0 is sent to $\tilde{x}$, the flat part of the boundary is tangent to Ω at $\tilde{y}$, and the rest of the boundary is partly inside Ω. We call the resulting half-ball T_α^* and the corresponding function H_α^*. Let $\tau = \{z \in \overline{T_\alpha^*};\ |z-\tilde{x}| = \alpha\} \cap \Omega$; see Figure 1.

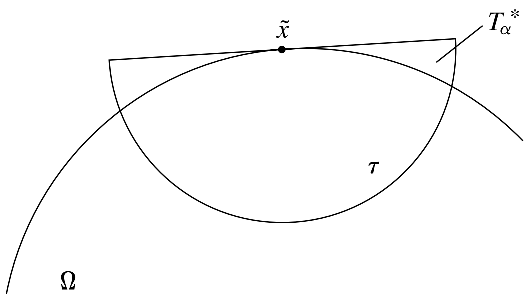

Figure 1

Let R be the greatest radius of curvature of $\partial\Omega$. Then $H_\alpha^*(z) \le -1/(2R)$ if $z \in \tau$, since in the worst case the portion of $\partial\Omega$ inside T_α^* is a portion of a sphere with radius R. The estimate follows by an elementary calculation. Note that the estimate is independent of $\tilde{x}$.

Let η be the constant given by Lemma 4, and set $D = \operatorname{diam}(\Omega)$. Recall that $0 < \eta < D/2$. If $\delta(x) \ge \eta|x-y|/(2D)$ then, according to Lemma 6,

$$\frac{-g(x,y)|x-y|^4}{\delta(x)\delta(y)} \le C\frac{|x-y|^2}{\eta|x-y|/(2D)} \le C$$

and we are done. If $\delta(x) < \eta|x-y|/(2D)$ then there is a unique nearest point πx at $\partial\Omega$, according to Lemma 4. Let $\alpha = \eta|x-y|/D < |x-y|/2$, and construct T_α^* and the corresponding function H_α^* at $\tilde{x} = \pi x$. Note that:

(1) $x \in T_\alpha^* \cap \Omega$, since $\delta(x) < \alpha/2$; and
(2) $y \notin T_\alpha^* \cap \Omega$, since

$$\begin{aligned}|\pi x - y| &\ge |y-x| - |x - \pi x| = |y-x| - \delta(x)\\ &> |y-x|(1-\eta/(2D)) > \tfrac{3}{4}|y-x| > \alpha.\end{aligned}$$

Assume $t \in \tau$. Then

$$\begin{aligned}|y-t| &\ge |x-y| - |x-t| \ge |x-y| - (|x-\pi x| + |\pi x - t|)\\ &= |x-y| - (\delta(x)+\alpha) > |x-y| - 3\alpha/2\\ &= |x-y|(1-3\eta/(2D)) > \tfrac{1}{4}|x-y|.\end{aligned}$$

Hence, using Lemma 6,

$$-g(t,y) \le C|t-y|^{-2}\delta(y) \le C|x-y|^{-2}\delta(y).$$

Since $-2RH_\alpha^*(t) \ge 1$, we have

$$g(t,y) \ge 2CR|x-y|^{-2}\delta(y)H_\alpha^*(t) \ge C|x-y|^{-2}\delta(y)H_\alpha^*(t) \tag{4}$$

for all $t \in \tau$. The same inequality holds trivially for $t \in \partial\Omega \cap T_\alpha^*$, since $g(t,y) = 0$ there. Hence it holds for all $t \in \partial(\Omega \cap T_\alpha^*)$. Because $g(t,y)$ is a maximal plurisubharmonic function of t in $\Omega \cap T_\alpha^*$ and $C|x-y|^{-2}\delta(y)H_\alpha^*(t)$ is plurisubharmonic as a function of t (in fact, it is pluriharmonic), the inequality (4) holds true in $\Omega \cap T_\alpha^*$. In particular, it holds for $t = x$:

$$\begin{aligned}g(x,y) &\ge C|x-y|^{-2}\delta(y)H_\alpha^*(x)\\ &\ge C|x-y|^{-2}\delta(y)\frac{-\delta(x)}{\alpha^2} \ge -C|x-y|^{-4}\delta(x)\delta(y)\end{aligned}$$

and the lemma is proved. □

REMARK. The estimate in the lemma is sharp in the sense that no similar inequality can hold with a smaller exponent than 4 in the denominator. This can be seen from the example $\Omega = B(0,1) \subset \mathbf{C}^2$, $y = (t,0)$, $x = (t, \sqrt{1-t})$, $t \in \mathbf{R}^+$.

In proving the general case, we will use the following embedding theorem. It is a special case of a theorem due to Fornaess.

THEOREM 8 [F]. *Let Ω be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$. Then there exist a holomorphic map $\psi : \mathbf{C}^n \to \mathbf{C}^m$ for some $m \in \mathbf{Z}^+$ and a strictly convex, bounded domain $\hat{\Omega}$ in $\mathbf{C}^m$ such that*

(1) *ψ is biholomorphic onto a closed subvariety of $\mathbf{C}^m$,*
(2) *$\psi(\Omega) \subset \hat{\Omega}$ and $\psi(\partial\Omega) \subset \partial\hat{\Omega}$, and*
(3) *$\psi(\mathbf{C}^n)$ intersects $\partial\hat{\Omega}$ transversally.*

We remark that, in general, m is much larger than n.

Proof of Theorem 3. It follows from the definition of the pluricomplex Green function that if $f : \Omega_1 \to \Omega_2$ is a holomorphic mapping then

$$g_{\Omega_1}(x, y) \geq g_{\Omega_2}(f(x), f(y)).$$

We apply this to ψ from the preceding theorem, and use Lemma 7. We obtain

$$-g_{\Omega}(x, y) \leq -g_{\hat{\Omega}}(\psi(x), \psi(y)) \leq C\frac{\delta_{\hat{\Omega}}(\psi(x))\delta_{\hat{\Omega}}(\psi(y))}{|\psi(x) - \psi(y)|^4}.$$

Since ψ is biholomorphic onto its image in a neigborhood of Ω, there is a constant $A > 1$ such that

$$\frac{1}{A} < \frac{|\psi(x) - \psi(y)|}{|x - y|} < A$$

for all $x, y \in \bar{\Omega}$. Furthermore, for all $x \in U \cap \Omega$ (for the notation, cf. Lemma 4), we have

$$\delta_{\hat{\Omega}}(\psi(x)) \leq |\psi(x) - \psi(\pi x)| < C|x - \pi x| = C\delta(x).$$

For $x \in \Omega \setminus U$ we have $\delta(x) \geq \eta$, and hence (by increasing C if necessary) the same estimate holds. Putting everything together, we obtain

$$-g_{\Omega}(x, y) \leq C\frac{\delta(x)\delta(y)}{|x - y|^4},$$

as desired. □

4. Proof of the Main Theorem

Proof of Theorem 1. There are two cases. (i) When

$$|x - y| \leq \max\left(\frac{\delta(x)}{2}, \frac{\delta(y)}{2}\right),$$

we use inequality (1) of Theorem 2 together with the trivial estimate

$$-g(x, y) \leq \frac{C}{|x - y|}.$$

(ii) When

$$|x-y| > \max\left(\frac{\delta(x)}{2}, \frac{\delta(y)}{2}\right),$$

we use Theorem 2, (2), and Theorem 3. □

5. The Quotient in the Bidisc

In the bidisc in $\mathbf{C}^2$, the pluricomplex and the classical Green functions cannot be compared, even when the pole is fixed. The quotient h will tend to infinity when z approaches a point at the distinguished boundary. To show this, we need the following "boundary Harnack inequality".

THEOREM 9 [W]. *Suppose D is a Lipschitz domain, P_0 is a point in D, E is a relatively open set on ∂D, and S is a subdomain of D satisfying $\partial S \cap \partial D \subseteq E$. Then there is a constant C such that, whenever u_1 and u_2 are two positive harmonic functions in D vanishing on E and $u_1(P_0) = u_2(P_0)$, it follows that $u_1(P) \leq Cu_2(P)$ for all $P \in S$.*

THEOREM 10. *In the bidisc in $\mathbf{C}^2$, $h(z, 0)$ is unbounded.*

Proof. Let

$$D = \{z = (z_1, z_2) \in \mathbf{C}^2;\ 1/3 < |z_1| < 1,\ 1/3 < |z_2| < 1\},$$
$$E = \{z = (z_1, z_2) \in \partial D;\ |z_1| = 1 \text{ or } |z_2| = 1\},$$
$$S = \{z = (z_1, z_2) \in \mathbf{C}^2;\ 2/3 < |z_1| < 1,\ 2/3 < |z_2| < 1\}.$$

Then D, E, and S fulfill the conditions of Theorem 9.

Let $G(z, 0)$ be the classical Green function for the bidisc with pole in the origin, and let $-u_1(z)$ be its restriction to D. Define

$$u_2(z) = k(\log|z_1|)(\log|z_2|),$$

where k is chosen so that $u_1(1/2, 1/2) = u_2(1/2, 1/2)$. Then u_1 and u_2 are as in Theorem 9, and hence $u_1(z)/u_2(z) \leq C$ for all $z \in S$.

The pluricomplex Green function for the bidisc with pole in the origin is

$$g(z, 0) = \max\{\log|z_1|, \log|z_2|\}.$$

Thus

$$\frac{g(z,0)}{G(z,0)} = \frac{-g(z,0)}{u_1(z)} = \frac{-g(z,0)}{u_2(z)}\frac{u_2(z)}{u_1(z)} \geq \frac{-\max\{\log|z_1|, \log|z_2|\}}{Ck(\log|z_1|)(\log|z_2|)}$$

for all $z \in S$. Let $z_1 = z_2 = t \in \mathbf{R}^+$. Then

$$\frac{g((t,t),0)}{G((t,t),0)} \geq \frac{-\log t}{Ck(\log t)^2} \to +\infty$$

when t tends to 1. □

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Mid Sweden University
831 25 Östersund
Sweden

magnus.carlehed@ter.mh.se