On Rearrangements of Series in Locally Convex Spaces

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1. Introduction

This paper deals with the structure of the set of sums of a conditionally convergent series in locally convex metrizable topological vector spaces. The problem goes back to the well-known theorem in analysis due to Riemann [19] asserting that the set of sums $\mathcal{S}_{(a_k)}$ of a conditionally convergent series $\sum_{k=1}^{\infty} a_k$, $a_k \in \mathbb{R}$, under all its convergent rearrangements fills the whole real line $\mathbb{R}$. This was generalized to the case of finite-dimensional spaces by Levy and Steinitz.

Theorem 1. *Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series in a finite-dimensional normed space X.*

(a) [14] *The set $\mathcal{S}_{(a_k)}$ has the form*

$$\mathcal{S}_{(a_k)} = E + a, \tag{1}$$

where E is a linear subspace and $a \in X$ is a fixed element; equivalently, $\mathcal{S}_{(a_k)}$ together with any points x, y in $\mathcal{S}_{(a_k)}$ contains the whole straight line $\alpha x + (1-\alpha)y$, $\alpha \in \mathbb{R}$.

(b) [21] *$x \in \mathcal{S}_{(a_k)}$ if and only if, for each linear functional x^* on X, there exists a permutation σ of $\mathbb{N}$ (which may depend on x^*) such that $x^*(x) = \sum_{k=1}^{\infty} x^*(a_{\sigma(k)})$; equivalently,*

$$\mathcal{S}_{(a_k)} = F^{\perp} + s, \tag{2}$$

where $F = \{x^* \in X^* : \sum |x^*(a_k)| < \infty\}$ and $F^{\perp} = \{x \in X : x^*(x) = 0$ for all $x^* \in F\}$ with $s \in \mathcal{S}_{(a_k)}$.

There is an extensive bibliography devoted to the generalization of the Levy and Steinitz theorems to the case of infinite-dimensional spaces (cf. [8]). We note here some of the results.

Let X be a locally convex topological vector space (Hausdorff, over the field of real or complex numbers). We say that a series $\sum_{k=1}^{\infty} a_k$, $a_k \in X$, converges *unconditionally* if it converges for all rearrangements (automatically to the same

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element $s \in X$). We call a series $\sum_{k=1}^{\infty} a_k$ *conditionally convergent* if it converges but does not converge unconditionally. Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series in X. We consider all possible rearrangements of the series such that it converges, and denote by $\mathcal{S}_{(a_k)}$ the set of sums of the series obtained in this way. We say that a conditionally convergent series $\sum_{k=1}^{\infty} a_k$, $a_k \in X$, *satisfies the Levy theorem* if $\mathcal{S}_{(a_k)}$ has the form (1), where $E \subseteq X$ is a closed linear subspace and $a \in X$ is a fixed element, and that it *satisfies the Steinitz theorem* if $\mathcal{S}_{(a_k)}$ has the form (2), where X^* is the space of all continuous linear functionals on X. It is easy to see that if $\sum_{k=1}^{\infty} a_k$ satisfies the Steinitz theorem then it also satisfies the Levy theorem.

In the early 1930s Banach (see [20]) questioned the validity of the Levy theorem in an infinite-dimensional normed space. Marcinkiewicz [20] constructed an example of a conditionally convergent series in infinite-dimensional Hilbert space with a nonconvex set of sums. V. Kadets [12], making use of the Marcinkiewicz example along with the Dvoretzky lemma, has shown that examples of this type can be constructed for any infinite-dimensional Banach space. Today we know more: For any fixed elements x, y of an infinite-dimensional Banach space there exists a conditionally convergent series $\sum_{k=1}^{\infty} a_k$ such that $\mathcal{S}_{(a_k)} = \{x, y\}$ (cf. [11]).

In the light of previous results it was expected that, in the class of locally convex metrizable spaces, the Steinitz (and the Levy) theorem is true for nuclear spaces and only for them. The problem turned out to be difficult, but was solved in the affirmative by Banaszczyk [1]. The result for the particular case of $\mathbb{R}^{\infty}$ was obtained by Wald as early as 1940 (cf. [22; 13]).

Along with the above results, the problem on additional conditions on the series in an infinite-dimensional space under which it satisfies the Levy and Steinitz theorems has also been solved. The first such restrictions were found by M. Kadets [9], who showed that a conditionally convergent series $\sum_{k=1}^{\infty} a_k$ in $L_p(T, \Sigma, \nu)$-spaces, $1 \le p < \infty$, satisfies the Levy theorem provided that $\sum_k \|a_k\|^d < \infty$, where $d = \min(2, p)$. Nikishin [16] later found a refined condition

$$\int_T \left(\sum_{k=1}^{\infty} |a_k(t)|^2 \right)^{p/2} d\nu(t) < \infty \tag{3}$$

for the case $1 \le p < 2$ (although (3) turned out to be the proper condition for the whole scale $1 \le p < \infty$).

It was noticed by Chobanyan [2] that the condition (3) can be given the invariant form

$$\text{the series } \sum a_k \varepsilon_k \text{ is a.e. convergent in } X, \tag{4}$$

where (ε_k) is the sequence of Rademacher random variables, and that condition (4) is sufficient for validity of the Steinitz theorem in the general case of a normed space X.

One disadvantage of the listed conditions is that they are not automatically satisfied in the particular case of $X = \mathbb{R}^n$; in other words, the Steinitz theorem is not a particular case of the corresponding results. The search for a condition weaker than (4) has led to the following: We say that a series $\sum a_k$ in a topological vector space X *satisfies the* (σ, θ)*-condition* if, for any permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$, there exists a sequence of signs $\theta = (\theta_k)$ such that the series $\sum a_{\sigma(k)}\theta_k$ is convergent in X.

It is easy to see that condition (4) implies the (σ, θ)-condition. In [18] and [4] the (σ, θ)-condition has been shown to be sufficient for the validity of the Steinitz theorem in the general case of normed spaces. In the particular case of $X = \mathbb{R}^n$ this implies the Steinitz theorem, since the (σ, θ)-condition in this case is automatically satisfied: for any null sequence (x_k), $x_k \in \mathbb{R}^n$, there exists a sequence of signs $\theta = (\theta_k)$ such that the series $\sum x_k\theta_k$ is convergent (cf. [6; 10]).

In [3], Chobanyan announced without proof that the (σ, θ)-condition is also sufficient for validity of the Steinitz theorem for series in locally convex metrizable topological vector spaces. One of the aims of this paper is to bring a complete proof of this result (Theorem 3). The application of Lemma 1 and making use in it of a special permutation seems to be the shortest way (even for the case of $\mathbb{R}^n$) of proving the Steinitz theorem.

Theorem 3 can be regarded as an assertion uniting most known results on the Steinitz theorem. One of the exceptions so far is the aforementioned result of Banaszczyk [1]. Its derivation from Theorem 3 is related to the following conjecture, which seems to be true but to date remains open: For any null sequence (x_n) in a nuclear metrizable locally convex space X, there exists a sequence of signs $\theta = (\theta_k)$ such that $\sum x_k\theta_k$ converges in X.

2. The Series Subsequence Problem and the Main Inequality

As a first step, all methods of proving the Steinitz and Levy theorems deal with an assertion of the following type. Suppose a series $\sum_{k=1}^{\infty} a_k$ in a topological vector space X is such that a subsequence of the sequence of partial sums tends to a certain limit s; then, under some conditions there exists a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} a_{\pi(k)}$ converges to the same limit s. Assertions of this type are of independent interest and are also considered in connection with other analytical problems (cf. [15; 24]).

THEOREM 2. *Let X be a locally convex metrizable topological vector space and let $\sum_{k=1}^{\infty} a_k$, $a_k \in X$, be a series such that a subsequence $S_{n_k} = \sum_1^{n_k} a_i$ of the sequence of partial sums converges to a limit s. If $\sum_{k=1}^{\infty} a_k$ satisfies the (σ, θ)-condition, then there exists a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} a_{\pi(k)} = s$.*

The proof of Theorem 2 is based on the following (main) inequality, which is a modification of a result of Chobanyan [3].

LEMMA 1. *Let $a_1, \dots, a_n$ be a fixed collection of elements of a linear space X, and let $\|\cdot\|$ be a seminorm on X. Then, for each collection of signs $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i = \pm 1$,*

$$\max_{1\le k\le n} \|a_{\pi(1)} + \cdots + a_{\pi(k)}\| \le \|a_1 + \cdots + a_n\| + \max_{1\le k\le n} \|a_{\sigma(1)}\theta_1 + \cdots + a_{\sigma(k)}\theta_k\|,$$

where $\pi\colon \{1,\dots,n\} \to \{1,\dots,n\}$ *is a permutation minimizing the left-hand side and* $\sigma(k) = \pi(n-k+1)$, $k = 1,\dots,n$.

Proof. Let $s = \sum_1^n a_i$ and consider the seminorm defined by

$$|(b_1,\dots,b_n)| = \max_{1\le k\le n} \left\| \sum_1^k b_i \right\|.$$

By the triangle inequality, for any permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ and any collection of signs $\theta = (\theta_1,\dots,\theta_n)$ we have

$$\begin{aligned} &|(-s, a_{\sigma(1)},\dots,a_{\sigma(n)})| + |(-s, a_{\sigma(1)}\theta_1,\dots,a_{\sigma(n)}\theta_n)| \\ &\qquad \ge |(-2s, a_{\sigma(1)} + a_{\sigma(1)}\theta_1,\dots,a_{\sigma(n)} + a_{\sigma(n)}\theta_n)| = 2|(-s, a_\sigma^+)| \end{aligned}$$

and

$$\begin{aligned} &|(-s, a_{\sigma(1)},\dots,a_{\sigma(n)})| + |(-s, a_{\sigma(1)}\theta_1,\dots,a_{\sigma(n)}\theta_n)| \\ &\qquad \ge |(0, a_{\sigma(1)} - a_{\sigma(1)}\theta_1,\dots,a_{\sigma(n)} - a_{\sigma(n)}\theta_n)| = 2|(a_\sigma^-)|, \end{aligned}$$

where (a_σ^+) is the subcollection of (a_σ) corresponding to indices with $\theta_i = 1$ while (a_σ^-) is that corresponding to $\theta_i = -1$.

It follows that

$$\begin{aligned} &|(-s, a_{\sigma(1)},\dots,a_{\sigma(n)})| + |(-s, a_{\sigma(1)}\theta_1,\dots,a_{\sigma(n)}\theta_n)| \\ &\qquad \ge 2\max(|(-s, a_\sigma^+)|, |(a_\sigma^-)|) = 2|(-s, a_{\sigma^*(\theta)})|, \qquad (1) \end{aligned}$$

where the permutation $\sigma^*(\theta)$ is defined as follows: First are indices $\sigma(i)$ of the subcollection (a_σ^+) and then, in reverse order, those of the subcollection (a_σ^-). The validity of the key inequality (1) follows from the following observation. If $b_1,\dots,b_n$ is a collection of elements from X with $\sum_1^n b_i = 0$ then, for any k $(0 < k < n)$ we have

$$\begin{aligned} &\max\{|(b_1,\dots,b_k)|, |(b_n, b_{n-1},\dots,b_{k+1})|\} \\ &\quad = \max\{|(b_1,\dots,b_k)|, \|b_n\| \|b_n + b_{n-1}\|,\dots,\|b_n + \cdots + b_{k+1}\|\} \\ &\quad = \max\{|(b_1,\dots,b_k)|, \|b_n + \cdots + b_{k+1}\|,\dots,\|b_1 + \cdots + b_n\|\} \\ &\quad = |(b_1,\dots,b_n)|. \end{aligned}$$

Hence (1) is satisfied for any permutation σ and any collection of signs θ. Take now as σ the permutation minimizing $|(-s, a_\lambda)|$. From (1) we then obtain

$$2|(-s, a_\sigma)| \le |(-s, a_\sigma)| + |(-s, a_\sigma\theta)|$$

and hence

$$|(-s, a_\sigma)| \le |(-s, a_\sigma\theta)|,$$

or

$$|(a_{\pi(1)},\dots,a_{\pi(n)})| \le \|s\| + |(a_{\sigma(1)}\theta_1,\dots,a_{\sigma(n)}\theta_n)|,$$

where π is the permutation that minimizes $|(a_\lambda)|$ and is related to σ by $\sigma(k) = \pi(n-k+1)$, $k = 1, \dots, n$.

This proves the lemma. $\square$

Proof of Theorem 2. Let $\|\cdot\|_p$ be an increasing sequence of seminorms inducing the topology of X. For each p we can introduce the quantities

$$Q_p(N) = \sup_{f_1,\dots,f_q > N} \inf_\theta \max_{1\le i\le q} \left\| \sum_1^i a_{f_{\sigma(i)}}\theta_i \right\|_p,$$

where the supremum is taken over all finite collections of distinct indices exceeding N and the infimum over all collections of signs $\theta = (\theta_1, \dots, \theta_q)$; σ is the permutation determined by the minimizing permutation for $a_{f_1}, \dots, a_{f_q}$ as described in Lemma 1. Fulfillment of the (σ, θ)-condition implies that $Q_p(N) \to 0$ as $N \to \infty$ for any fixed p. Thus we can choose a subsequence (n_k) of (m_k) such that $Q_p(n_k) \le 1/p$ and $\|S_{n_{k+l}} - S_{n_k}\|_p \le 1/p$ for any $l \in \mathbb{N}$. For each collection $a_{n_p+1}, \dots, a_{n_{p+1}}$, choose the minimizing permutation $\pi_p\colon \{n_p+1, \dots, n_{p+1}\} \to \{n_p+1, \dots, n_{p+1}\}$. Then, according to Lemma 1,

$$\begin{aligned}
&\max_{1\le k\le n_{p+1}-n_p} \left\| \sum_{i=1}^k a_{\pi_p(n_p+i)} \right\|_p \\
&\quad\le \|S_{n_{p+1}} - S_{n_p}\|_p + \min_\theta \max_{1\le k\le n_{p+1}-n_p} \left\| \sum_{i=1}^k a_{\sigma_p(n_p+i)}\theta_i \right\|_p \\
&\quad\le \|S_{n_{p+1}} - S_{n_p}\|_p + Q_p(n_p),
\end{aligned}$$

where σ_p is uniquely determined by π_p. The collection of permutations (π_p) defines in a natural way a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$. Now we prove that the series, rearranged according to π, converges to s. It suffices to prove that the series is $\|\cdot\|_p$-Cauchy for every $p \in \mathbb{N}$. Take any integers $u < v$ with $n_{p_1} < u \le n_{p_1+1}$ and $n_{p_2} < v \le n_{p_2+1}$. Then, if $p < p_1 \le p_2$,

$$\begin{aligned}
\|S_v - S_u\|_p &\le \|S_v - S_{n_{p_2}}\|_p + \|S_{n_{p_2}} - S_{n_{p_1}}\|_p + \|S_u - S_{n_{p_1}}\|_p \\
&\le \|S_v - S_{n_{p_2}}\|_{p_2} + \|S_{n_{p_2}} - S_{n_{p_1}}\|_p + \|S_u - S_{n_{p_1}}\|_{p_1} \\
&\le \|S_{n_{p_2+1}} - S_{n_{p_2}}\|_{p_2} + Q_{p_2}(n_{p_2}) \\
&\quad + \|S_{n_{p_2}} - S_{n_{p_1}}\|_{p_1} + \|S_{n_{p_1+1}} - S_{n_{p_1}}\|_{p_1} + Q_{p_1}(n_{p_1}) \\
&\le \frac{2}{p_2} + \frac{3}{p_1} \to 0
\end{aligned}$$

as $u, v \to \infty$. The theorem is proved. $\square$

The following statements are obvious consequences of Theorem 2.

COROLLARY 1. *Theorem 2 remains valid if the (σ, θ)-condition is replaced by the condition of convergence a.e. in X of the series $\sum a_k\varepsilon_k$, where (ε_k) is a sequence of Rademacher random variables.*

COROLLARY 2 [15; 24]. *If X is a Banach space of type p, $1 \le p \le 2$, then Theorem 2 remains true provided the following condition is satisfied instead of the (σ, θ)-condition:* $\sum \|a_k\|^p < \infty$.

As pointed out in Section 1, the (σ, θ)-condition does not imply the convergence of the series $\sum a_k \varepsilon_k$, even in the case of scalar series $\sum a_k$ when the general term goes to zero sufficiently slowly. A source of more useful examples is provided by the following construction.

EXAMPLE. Let X be a locally convex topological vector space. Let $\sum a_k$ be an unconditionally convergent series such that for some $x^* \in X^* x^*(a_k) \neq 0$, $k \in \mathbb{N}$. Then consider the integers n_k such that $n_k |x^*(a_k)| > 1$. Construct now the series $\sum b_i$ consisting of a_1 n_1 times, $-a_1$ n_1 times, ..., a_k n_k times, $-a_k$ n_k times, ... Then the series $\sum b_k \varepsilon_k$ is obviously not convergent. In order to make sure that $\sum b_k$ satisfies the (σ, θ)-condition, consider any rearrangement $\sum b_{\pi(k)}$ of the series and arrange the signs $\theta = (\theta_k)$ as follows. If $b_{\pi(k)} = a_i$ then it is positive or negative according as the previous term a_i in $\sum b_{\pi(k)}$ was negative or positive, respectively; if a previous a_i does not exist then $b_{\pi(k)}$ is given an arbitrary sign. Clearly, $\sum_{k=1}^{\infty} b_{\pi(k)} \theta_k$ will be convergent—that is, $\sum b_k$ satisfies the (σ, θ)-condition.

3. Steinitz Theorem in Locally Convex Spaces

The aim of this section is to prove the following assertion.

THEOREM 3. *If $\sum_{k=1}^{\infty} a_k$ is a conditionally convergent series in a locally convex metrizable topological vector space X satisfying the (σ, θ)-condition, then it satisfies the Steinitz theorem.*

Proof. We follow the scheme suggested by Pecherski [17]; the main tools are Theorem 2 and the inequality of Lemma 1. Without loss of generality, we may assume that the topology of X is generated by an increasing sequence of seminorms $\|\cdot\|_p$, $p \in \mathbb{N}$. By Σ_m we denote the class of finite subsets of the set $\{m, m+1, m+2, \dots\}$ along with the empty set, and by $S(T)$, $T \in \Sigma_m$, we denote the sum $\sum_{k \in T} a_k$ if T is nonempty and 0 if T is empty. $S(\Sigma_m)$ stands for the set $S(T)$, $T \in \Sigma_m$. We denote by $\overline{\operatorname{conv}}\, M$ the closure of the convex hull of M in X.

LEMMA 2. *Let $\sum_{k=1}^{\infty} a_k$ be a series in X satisfying the (σ, θ)-condition, and let $m, p \in \mathbb{N}$. Then, for any $A, B \in \Sigma_m$, there exists $T \in \Sigma_m$ such that*

$$\|S(T) - \tfrac{1}{2}(S(A) + S(B))\|_p \le \tfrac{1}{2} Q_p(m).$$

Proof. Let us first prove that there exists a $C \subset A \triangle B = (A \cap B^c) \cup (A^c \cap B)$ such that

$$\|S(C) - \tfrac{1}{2} S(A \triangle B)\|_p \le \tfrac{1}{2} Q_p(m). \tag{1}$$

If $A \triangle B = \phi$ then one can put $C = \phi$. Suppose $A \triangle B \neq \phi$, and denote by $b_1, \dots, b_n$ the elements a_k, $k \in A \triangle B$. Then there exist signs $\theta_1, \dots, \theta_n = \pm 1$ such that

$$\left\| \sum_{k=1}^{n} \theta_k b_k \right\| \le Q_p(m).$$

Setting $\varepsilon_i = \frac{1}{2}(1+\theta_i)$, we obtain

$$\left\| \sum_{i=1}^{n} \varepsilon_i b_i - \frac{1}{2} \sum_{i=1}^{n} b_i \right\|_p \le \frac{1}{2} Q_p(m).$$

Now, as C one can take the subset of $A \triangle B$ corresponding to the indices for which $\varepsilon_i = 1$. For $T = (A \cap B) \cup C$ we will then have

$$\begin{aligned} \|S(T) - \tfrac{1}{2}(S(A)+S(B))\|_p &= \|S(A \cap B) + S(C) - \tfrac{1}{2} S(A \triangle B) - S(A \cap B)\|_p \\ &= \|S(C) - \tfrac{1}{2} S(A \triangle B)\|_p \le \tfrac{1}{2} Q_p(m), \end{aligned}$$

and the lemma is proved. □

COROLLARY. *For any $x \in \overline{\operatorname{conv}}\, S(\Sigma_m)$ and any $\varepsilon_m > 0$, there exists $x' \in S(\Sigma_m)$ such that*

$$\|x - x'\|_p \le Q_p(m) + \varepsilon_m, \quad p \in \mathbb{N}.$$

This corollary is a consequence of Lemma 2 and the following statement (due to Pecherski [17]).

PROPOSITION. *Let Σ be a ring of sets, v an additive set function on Σ with values in a normed space X, M the set of all values of v on Σ, and δ an arbitrary nonnegative number. Suppose M has the property that, for any two vectors $a, b \in M$, there exists a vector $c \in M$ such that*

$$\|c - \tfrac{1}{2}(a+b)\| \le \delta.$$

Then M is a $(2\delta + \varepsilon)$-net for $\overline{\operatorname{conv}}\, M$ for any $\varepsilon > 0$.

Proof of Theorem 3. We shall carry out the proof in three steps as follows.

(1) First we show that $\mathcal{S}_{(a_k)}$ coincides with the set

$$P = \bigcap_{m=1}^{\infty} \overline{(S_m + S(\Sigma_{m+1}))}.$$

(2) We then show that $\mathcal{S}_{(a_k)}$ coincides with

$$L = \bigcap_{m=1}^{\infty} \overline{\operatorname{conv}}(S_m + S(\Sigma_{m+1})).$$

(3) Finally, we prove that

$$\mathcal{S}_{(a_k)} = \Gamma \equiv \{ s \in X : \forall x^* \in X^* \ \exists \pi \text{ s.t.} \sum x^*(A_{\pi(k)}) = x^*(s) \}.$$

Step 1. If $s \in \mathcal{S}_{(a_k)}$ then there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^{\infty} a_{\sigma(k)} = s$. For each fixed m there exists a partial sum of $\sum_{k=1}^{\infty} a_{\sigma(k)}$ that contains $a_1, \dots, a_m$. Consequently, all the partial sums of the rearranged series with sufficiently large indices are in $S_m + S(\Sigma_{m+1})$; that is, $s \in S_m + \overline{S(\Sigma_{m+1})}$.

We construct an increasing sequence of finite subsets of $\mathbb{N}$ as follows. Let $T = \{1\}$ and consider a finite subset $T_2 \supset \{1, 2\}$ such that $\|S(T_2) - s\|_2 \leq \frac{1}{2}$. Such a T_2 exists, because $s \in \overline{S_2 + S(\Sigma_3)}$. Suppose now that the number m' is such that $T_2 \cup \{3\} \subseteq \{1, \dots, m'\}$. Because $s \in \overline{S_{m'} + S(\Sigma_{m'+1})}$, there exists a finite subset $T_3 \supset T_2 \cup \{3\}$ such that $\|S(T_3) - s\|_3 < \frac{1}{3}$. Continuing this construction yields a nondecreasing sequence $T_1, T_2, \dots$ of finite sets of $\mathbb{N}$ such that $\|S(T_k) - s\|_k \leq 1/k$. Moreover, the union $\bigcup_1^\infty T_k$ exhausts all of $\mathbb{N}$, since $k \in T_k$ for each $k \in \mathbb{N}$. We now consider the sequence $T_1, T_2 - T_1, \dots, T_n - T_{n-1}, \dots$ of disjoint sets and form a permutation λ of $\mathbb{N}$ as follows. First go the elements of T_1, then the elements of $T_2 - T_1$, then those of $T_3 - T_2$, and so forth; the order of succession in each of these sets is immaterial. Clearly, $(S(T_k))$ is a subsequence of the sequence of partial sums of $\sum_{k=1}^\infty a_{\lambda(k)}$. Since $S(T_k) \to s$, according to Theorem 2 there exists a permutation σ of $\mathbb{N}$ such that $\sum_{k=1}^\infty a_{\sigma(k)} = s$.

Step 2. Obviously, if $s \in P$ then $s \in L$. Suppose $s \in L$. Then $s - S_m \in \overline{\operatorname{conv}} S(\Sigma_{m+1})$ for each m. Let us choose $(n_m)_{m \in \mathbb{N}}$ such that $n_m \geq m$ and $Q_m(n_m) \leq 1/m$. By the Corollary to Lemma 2, there exists $x_{n_m+1} \in S(\Sigma_{n_m+1})$ such that

$$\|s - S_{n_m} - x_{n_m+1}\| < Q_m(n_m) + \frac{1}{m} < \frac{2}{m}.$$

It follows that $S_{n_m} + x_{n_m+1} \to s$. Since $n_m \geq m$, we have that $S_{n_m} + x_{n_m+1} \in S_m + S(\Sigma_{m+1})$. Further, if $k \geq m$ then clearly $S_{n_k} + x_{n_k+1} \in S_{n_k} + S(\Sigma_{n_k+1}) \subseteq S_m + S(\Sigma_{m+1})$. Consequently $s \in \overline{S_m + S(\Sigma_{m+1})}$ for each m, so that $s \in \mathcal{S}_{(a_k)}$, as proved in the first step.

Step 3. We must show that $s \in \mathcal{S}_{(a_k)}$ if and only if, for each $x^* \in X^*$, there exists a permutation σ such that

$$x^*(s) = \sum x^*(a_{(\sigma(k))}). \tag{2}$$

If $s \in \mathcal{S}_{(a_k)}$ then (2) clearly holds. Conversely, suppose that (2) holds and assume the contrary—that is, that $s \notin \mathcal{S}_{(a_k)}$. As proved in Step 2, this means that

$$s \notin \bigcap_{m=1}^{\infty} \overline{\operatorname{conv}}(S_m + S(\Sigma_{m+1}));$$

in other words, $s \notin \overline{\operatorname{conv}}(S_m + S(\Sigma_{m+1}))$ for some m. It is well known (see [5, Cor. V.2.12]) that a point and a closed convex set can be separated by a hyperplane. This means that there exists an element $x_o^* \in X^*$ and a number $\varepsilon > 0$ such that

$$x_o^*(1) + \varepsilon < x_o^*(s)$$

for all $y \in \overline{\operatorname{conv}}(S_m + S(\Sigma_{m+1}))$. On the other hand, $x_o^*(s) = \sum x_o^*(a_{\sigma(k)})$ for some permutation σ. From this it follows that there exists a finite set $T \supseteq \{1, \dots, m\}$ such that $|x^*(s) - x^*(S(T))| < \varepsilon/2$. It is clear that $S(T) \in S_m + S(\Sigma_{m+1})$, which contradicts (13). The proof is complete. □

REMARK. Theorem 2 remains valid under the following (weaker) modified (σ, θ)-condition: For any seminorm $\|\cdot\|$ that is continuous in the topology of X and for any permutation σ of the series, there exists a collection of signs $\theta = (\theta_i)$ such that the series $\sum a_{\sigma(k)}\theta_k$ is $\|\cdot\|$-convergent. The proof can be carried out exactly as before.

The following example shows that the (σ, θ)-condition is not necessary for the Steinitz theorem to hold.

EXAMPLE. Let X be an infinite-dimensional Hilbert space, and let $e_{m_k}, e_{m_k+1}, \ldots, e_{m_{k+1}-1}$ ($k \in \mathbb{N}$) be nonoverlapping parts of an orthonormal system such that $m_{k+1} - m_k \to \infty$. Define the vectors

$$y_{ki}^+ = (m_{k+1} - m_k)^{-1/2} e_{m_k+i},$$
$$y_{ki}^- = -(m_{k+1} - m_k)^{-1/2} e_{m_k+i}, \quad k \in \mathbb{N}, \ i = 0, 1, \ldots, m_{k+1} - m_k - 1.$$

Our series $\sum a_k$ is made up of y_{ki}^+s and y_{ki}^-s. It is evident that this series will converge to zero if each y_{ki}^+ is followed by a y_{ki}^-. It is easy to see that $\mathcal{S}_{(a_k)} = \{0\}$ and that $\sum a_k$ satisfies the Steinitz theorem. However, the (σ, θ)-condition is not satisfied: if the arrangement of the series is such that, for each k, the positive terms occur in succession, then no arrangement of signs will enable the series to converge.

Although the (σ, θ)-condition is not necessary, the authors do not know any result dealing with the Steinitz theorem where a weaker condition is imposed. One possible exception, due to Banaszczyk [1], states that no condition is needed in the case of metrizable nuclear spaces. However, this result will also be a consequence of Theorem 3 if the following conjecture is true.

CONJECTURE. *Let X be a metrizable nuclear space, and let $(a_k) \subset X$ be a null sequence, that is, $a_k \to 0$ as $k \to \infty$. Then there exists a sequence of signs $\theta = (\theta_k)$ such that the series $\sum a_k\theta_k$ is convergent.*

THE LEVY THEOREM. For the case of non–locally convex spaces we have only the result of [7], which states that the Levy theorem holds under the (σ, θ)-condition for locally bounded metrizable topological vector spaces. It is not known, for example, whether the Levy theorem holds under the (σ, θ)-condition for L_0. The case of nonmetrizable spaces is not well studied, and the authors do not know of any method for dealing with it. The following particular result of a negative nature is known (V. Kadets, personal communication): There is an example of a series in a Hilbert space equipped with the weak topology possessing a nonconvex set of sums. However, this kind of counterexample was expected in the light of Banaszczyk's theorem. It is worth noting also that the (σ, θ)-condition makes sense for series in topological groups. One can investigate when it implies that $\mathcal{S}_{(a_k)}$ is a closed subgroup of the group.

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