Polynomial Hulls with Disk Fibers over the Ball in $\mathbf{C}^2$

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Let Y be a compact set in $\mathbf{C}^n$. We denote by $\hat{Y}$ the *polynomial* (*convex*) *hull* of Y; that is,

$$\hat{Y} = \{ z \in \mathbf{C}^n \mid |P(z)| \le \sup_{w \in Y} |P(w)| \text{ for all polynomials } P \text{ on } \mathbf{C}^n \}.$$

Let B_2 denote the open unit ball in $\mathbf{C}^2$. We shall be interested in the polynomial hulls of compact sets Y in $\mathbf{C}^3$, where Y projects onto $S = \partial B_2$.

The analogous problem of describing the polynomial hull of a compact set fibered over the unit circle Γ in $\mathbf{C}$ has been studied in [1; 3; 8; 9; 11]. A major issue in all of these works is to describe the extent of *analytic structure*—that is, when there exist analytic varieties contained in the polynomial hull with boundary contained in Y. One reason for this is that discovering such a variety in $\hat{Y}$ explains why the points on that variety lie in $\hat{Y}$ by virtue of the well-known local maximum modulus principle on analytic varieties. In this work, we shall examine when one can expect a higher degree of analytic structure, that is, when there exist analytic manifolds of dimension 2 in $\hat{Y}$ with boundary in Y. In particular, we shall examine when such an analytic manifold is in fact the graph of an analytic function over B_2.

Let Δ denote the closed unit disk in $\mathbf{C}$. In [11] Wermer showed that, for hulls of sets fibered over the circle, there need never exist analytic structure. However, it was shown by Alexander and Wermer in [1] and by Słodkowski in [8] that if Y has convex fibers Y_λ ($\lambda \in \Gamma$) over the circle then $\hat{Y} \cap \{|\lambda| < 1\}$ is the union of analytic graphs over int Δ of functions f in $H^\infty(\Delta)$ such that $f(\lambda) \in Y_\lambda$ for a.e. $\lambda \in \Gamma$. Furthermore, Alexander and Wermer proved the following.

THEOREM [1, Thm. 2]. *Suppose that α is a continuous complex-valued function on $\Gamma = \{|\lambda| = 1\}$, $\|\alpha\|_\infty \le 1$. Put $Y = \{ (\lambda, w) \mid |w - \alpha(\lambda)| \le 1,\ \lambda \in \Gamma \}$. Assume there is a b with $|b| < 1$ such that $\hat{Y}_b$ contains more than one point. Then there exist functions A, B, C, analytic on $\{|\lambda| < 1\}$ and in H^p $(0 < p < 1)$, as well as a $\phi_0 \in H^\infty$ such that*

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{A(\lambda)(w - \phi_0(\lambda)) + C(\lambda)}{B(\lambda)(w - \phi_0(\lambda)) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}.$$

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We shall prove a similar result for Y fibered over ∂B_2 (Theorem 2); however, we will need to assume more about the set Y because in general there is no reason to believe that $\hat{Y}$ contains any analytic graphs over B_2. We illustrate this with the following example.

EXAMPLE. Consider the function $w = f(z_1, z_2) = \bar{z}_2$ on ∂B_2 and its graph Y as a set fibered over S. Then $\hat{Y}$ is nontrivial (since it contains the graph of f over B_2); however, no point on this graph over B_2 can lie on an analytic graph with boundary in Y, because no analytic function on B_2 has the same boundary values as f.

Suppose that Λ is a 1-dimensional complex affine subspace of $\mathbf{C}^2$ that meets B_2, and let p be the point on Λ nearest to $(0,0)$. Then the points on Λ that meet S form a circle in Λ with center p, and $\Lambda \cap \bar{B}_2$ is a disk embedded complex affinely in $\mathbf{C}^2$. We write $L = \Lambda \cap \bar{B}_2$ and refer to L as a complex affine *slice* of $\bar{B}_2$.

A key requirement for being able to find an analytic graph X over B_2 passing through a point p in $\hat{Y}$ with boundary in Y is that, given any complex affine slice L of $\bar{B}_2$, p lies in the polynomial convex hull of the part of Y that lies over L. This condition is necessary because the part of X lying over L is a Riemann surface and so p must lie in the polynomial hull of its boundary. In a sense, p must be very strongly in the polynomial hull of Y. We shall be able to satisfy this requirement with the next definition.

Let $\Pi\colon \bar{B}_2 \times \mathbf{C} \to \bar{B}_2$ be a projection and let $z = (z_1, z_2)$ be a typical element of $\bar{B}_2$. If X is a compact set in $\mathbf{C}^3$ such that $\Pi(X) = \bar{B}_2$, then we shall say that X is *pseudoconcave* if $(B_2 \times \mathbf{C}) \setminus X$ is pseudoconvex. In order to obtain analytic graphs in $\hat{Y}$, we shall assume that $\hat{Y}$ is pseudoconcave. In particular, we shall show later that the points of X satisfy the strong condition of the previous paragraph with respect to $X \cap \{|z| = 1\}$.

We shall prove the following.

THEOREM 1. *Let Y be a compact subset of $(\partial B_2) \times \mathbf{C}$ of the form*

$$Y = \{\, (z, w) \mid |w - \alpha(z)| \le 1,\ z \in \partial B_2 \,\},$$

where α is a continuous complex-valued function on ∂B_2, $\|\alpha\|_\infty \le 1$, and

$$\hat{Y} \cap \{z = b\} \text{ has more than one point for some } b \in B_2. \tag{1}$$

Suppose also that $(B_2 \times \mathbf{C}) \setminus \hat{Y}$ is pseudoconvex. Let $(0, 0, w_0) \in \partial\hat{Y}$. Then there exists a unique $\phi \in H^\infty(B_2)$ such that $\phi(0,0) = w_0$ and, for all $z \in B_2$, $(z, \phi(z)) \in \partial\hat{Y}$.

THEOREM 2. *Let Y be as in Theorem 1. Then there exist analytic functions A, B, C, analytic on B_2, such that*

$$\hat{Y} \cap \Pi^{-1}(B_2) = \left\{ (z, w) \;\middle|\; \left| \frac{A(z)(w - \phi(z)) + C(z)}{B(z)(w - \phi(z)) + C(z)} \right| \le 1,\ |z| < 1 \right\},$$

where ϕ is the function found in Theorem 1.

One consequence of Theorems 1 and 2 is that $\hat{Y} \cap \Pi^{-1}(B_2)$ must be the union of analytic graphs over B_2. We note then that the condition that $(B_2 \times \mathbf{C}) \setminus \hat{Y}$ be pseudoconvex is necessary for the following reason. Given any point p in $\partial\hat{Y} \cap \Pi^{-1}(B_2)$, suppose there exists an analytic F whose graph is in $\hat{Y}$ and contains p. The function $1/(w - F(z))$ is then analytic in $(B_2 \times \mathbf{C}) \setminus \hat{Y}$ and singular at p; since p was arbitrary, this means $(B_2 \times \mathbf{C}) \setminus \hat{Y}$ must be pseudoconvex.

Our main approach is to apply what is known about polynomial hulls over the disk in $\mathbf{C}$ to the part of $\hat{Y}$ that lies over L, where again L is any complex affine slice of B_2. In Lemmas 1–3 we rewrite Alexander and Wermer's expression for hulls over the disk in two ways that will be convenient later. In Lemmas 4–10 we move to the case at hand and show that, if Y sits over ∂B_2 and has the properties stated in Theorem 1, then in fact $\hat{Y} \cap \{|z| < 1\}$ has disk fibers and has smooth boundary. This is the main technical difficulty in proving Theorems 1 and 2. We shall prove in Theorem 3 that essentially all sets having the form given in Theorem 2 are polynomially convex. Finally, in Theorem 4 we shall generalize the results of Theorems 1 and 2 to include compact sets Y with disk fibers Y_λ whose radii may vary.

LEMMA 1. *Let Y be as in Theorem 2 of* [1]. *Choose* $(\lambda_0, w_0) \in \partial\hat{Y}$ *with* $|\lambda_0| < 1$. *Then there exists a unique ϕ analytic in* int Δ *such that*

$$(\lambda, \phi(\lambda)) \in \hat{Y} \quad \textit{for all} \ \lambda \in \operatorname{int}\Delta, \qquad \phi(\lambda_0) = w_0. \tag{2}$$

Note. This lemma is a direct consequence of work of Forstnerič [3] and Słodkowski [9]. We provide here a short proof for the special case at hand.

Proof. If we solve

$$\frac{A(\lambda)w + C(\lambda)}{B(\lambda)w + D(\lambda)} = \frac{A(\lambda_0)w_0 + C(\lambda_0)}{B(\lambda_0)w_0 + D(\lambda_0)} \equiv k \tag{3}$$

for w as a function of λ, then

$$w = \frac{kD(\lambda) - C(\lambda)}{A(\lambda) - kB(\lambda)}; \tag{4}$$

we call this last function $\phi(\lambda)$. Since $(\lambda_0, w_0) \in \partial\hat{Y} \cap \{|\lambda| < 1\}$, it follows that $|k| = 1$. Because $\hat{Y} \setminus Y$ has bounded fibers, the denominator of (4) is never zero. Hence (4) defines an analytic function with graph $\subset \partial\hat{Y} \cap \{|\lambda| < 1\}$, since

$$\left|\frac{A(\lambda)\phi(\lambda) + C(\lambda)}{B(\lambda)\phi(\lambda) + D(\lambda)}\right| = |k| = 1$$

for all $\lambda \in \operatorname{int}\Delta$.

Furthermore, $\phi(\lambda_0)$ is the value we get for w if we let $\lambda = \lambda_0$ in (3) and solve for w; clearly $w = w_0$ is a solution, and is the only one because in (4) this solution is unique. Thus $\phi(\lambda_0) = w_0$ and ϕ has the desired properties (2).

To show that ϕ is unique, suppose that $\tilde{\phi}$ also has the properties in (2). Let

$$M(\lambda, w) = \frac{A(\lambda)w + C(\lambda)}{B(\lambda)w + D(\lambda)}.$$

Then

$$|M(\lambda_0, \tilde{\phi}(\lambda_0))| = |M(\lambda_0, w_0)| = 1. \tag{5}$$

The mapping $\lambda \mapsto M(\lambda, \tilde{\phi}(\lambda))$ is analytic for $|\lambda| < 1$. By (5) and the local maximum modulus principle, $|M(\lambda, \tilde{\phi}(\lambda))| > 1$ for some λ, $|\lambda| < 1$, if $\lambda \mapsto M(\lambda, \tilde{\phi}(\lambda))$ is not constant. This is impossible, because if the graph of $\tilde{\phi}$ is in $\hat{Y} \cap \{|\lambda| < 1\}$ then $|M(\lambda, \tilde{\phi}(\lambda))| \le 1$ for all λ, $|\lambda| < 1$. Thus $M(\lambda, \tilde{\phi}(\lambda))$ is constant and, in fact, equal to $M(\lambda_0, \tilde{\phi}(\lambda_0)) = M(\lambda_0, w_0) = k$. Referring to (3) and (4), we see that this means $\phi = \tilde{\phi}$, as desired. ☐

Note that the graph of ϕ over int Δ lies in $\partial\hat{Y} \cap \{|\lambda| < 1\}$.

LEMMA 2. *Let α and Y be as in Theorem 2 of* [1]. *Let ϕ be any one of the analytic functions whose graph is in $\partial\hat{Y} \cap \{|\lambda| < 1\}$. Then*

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{\tilde{A}(\lambda)(w - \phi(\lambda)) + \tilde{C}(\lambda)}{\tilde{B}(\lambda)(w - \phi(\lambda)) + \tilde{C}(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}, \tag{6}$$

where $\tilde{C} = 2\tilde{F}$,

$$\tilde{F}/|\tilde{F}| = \alpha - \phi \ \ on\ \Gamma, \quad \|\tilde{F}\|_1 = 1, \tag{7}$$

$\tilde{F}$ is an outer function (and so is nonzero on int Δ), *$\tilde{F}$ is the only element in H^1 with properties* (7), *and*

$$\tilde{A} = -\tilde{\beta} - 1, \qquad \tilde{B} = -\tilde{\beta} + 1,$$

where

$$\tilde{\beta}(\lambda) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} |\tilde{F}(e^{i\theta})|\, d\theta, \quad |\lambda| < 1.$$

Proof. If ϕ has the smallest real part at zero of all the analytic functions whose graphs are $\subset \partial\hat{Y} \cap \{|\lambda| < 1\}$ then this lemma is just Theorem 2 of [1]. If not, then pick $\omega \in \Gamma$ such that, in the hull

$$\widehat{\omega Y} \cap \{|\lambda| < 1\} = \omega\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, \omega w) \;\middle|\; (\lambda, w) \in \hat{Y} \cap \{|\lambda| < 1\} \right\},$$

$\omega\phi$ has the smallest real part at zero of all the analytic functions with graphs in $\partial(\omega\hat{Y}) \cap \{|\lambda| < 1\}$. Then we proceed as in [1]: If F is the unique element of H^1 such that

$$\|F\|_1 = 1 \quad \text{and} \quad \frac{F}{|F|} = \omega\alpha - \omega\phi \tag{8}$$

and we define A, B, C as before, then

$$\omega\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{A(\lambda)(w - \omega\phi(\lambda)) + C(\lambda)}{B(\lambda)(w - \omega\phi(\lambda)) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}$$

from Theorem 2 of [1], so

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, \omega^{-1} w) \;\middle|\; \left| \frac{A(\lambda)(w - \omega\phi(\lambda)) + C(\lambda)}{B(\lambda)(w - \omega\phi(\lambda)) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}$$
$$= \left\{ (\lambda, w) \;\middle|\; \left| \frac{A(\lambda)(\omega w - \omega\phi(\lambda)) + C(\lambda)}{B(\lambda)(\omega w - \omega\phi(\lambda)) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}$$
$$= \left\{ (\lambda, w) \;\middle|\; \left| \frac{A(\lambda)(w - \phi(\lambda)) + \omega^{-1} C(\lambda)}{B(\lambda)(w - \phi(\lambda)) + \omega^{-1} C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}. \quad (9)$$

Now note that, from (8), $\omega^{-1}F$ is the unique element of H^1 with

$$\|\omega^{-1}F\|_1 = 1 \quad \text{and} \quad \frac{\omega^{-1}F}{|\omega^{-1}F|} = \alpha - \phi.$$

Also, $\omega^{-1}F$ is an outer function. Let $\tilde{F} = \omega^{-1}F$, $\tilde{C} = 2\tilde{F}$ $(= \omega^{-1}C)$,

$$\tilde{\beta}(\lambda) = \frac{1}{2\pi} \int_{\Gamma} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} |\tilde{F}(e^{i\theta})|\, d\theta, \quad |\lambda| < 1,$$

and

$$\tilde{A} = -\tilde{\beta} - 1, \qquad \tilde{B} = -\tilde{\beta} + 1.$$

Now $\beta(\lambda) = \tilde{\beta}(\lambda)$ since $|F| = |\tilde{F}|$, so $\tilde{A} = A$ and $\tilde{B} = B$. Combining this with the third equality of (9), we get

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{\tilde{A}(\lambda)(w - \phi(\lambda)) + \tilde{C}(\lambda)}{\tilde{B}(\lambda)(w - \phi(\lambda)) + \tilde{C}(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}$$

as desired. □

LEMMA 3. *Let Y be as in Lemma 1. Let $R(\lambda)$ and $S(\lambda)$ be unique functions such that*

$$\hat{Y} \cap \{|\lambda| < 1\} = \{ (\lambda, w) \mid |w - S(\lambda)| \le R(\lambda),\ |\lambda| < 1 \}.$$

Then

$$S(\lambda) = \frac{C(\lambda)}{2\operatorname{Re}\beta(\lambda)} + \phi(\lambda) \quad \textit{and} \quad R(\lambda) = \frac{|C(\lambda)|}{2\operatorname{Re}\beta(\lambda)},$$

where ϕ, β, C are the functions that appear as ϕ, $\tilde{\beta}$, $\tilde{C}$ in Lemma 2.

Proof. Since $|B(\lambda)| < |A(\lambda)|$ for all $|\lambda| < 1$ ($|A|^2 - |B|^2 = 4\operatorname{Re}\beta > 0$ for $|\lambda| < 1$), it follows that $A(\lambda) \neq 0$ for $|\lambda| < 1$. Thus B/A is a well-defined function on $|\lambda| < 1$ with modulus < 1. We have

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{A(\lambda)(w - \phi(\lambda)) + C(\lambda)}{B(\lambda)(w - \phi(\lambda)) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}.$$

Now

$$\left| \frac{A(\lambda)(w - \phi(\lambda)) + C(\lambda)}{B(\lambda)(w - \phi(\lambda)) + C(\lambda)} \right| \le 1$$

if and only if

$$\left| \frac{\dfrac{A(\lambda)(w-\phi(\lambda))+C(\lambda)}{B(\lambda)(w-\phi(\lambda))+C(\lambda)} - \dfrac{\overline{B(\lambda)}}{\overline{A(\lambda)}}}{1-\left[\dfrac{B(\lambda)}{A(\lambda)}\dfrac{A(\lambda)(w-\phi(\lambda))+C(\lambda)}{B(\lambda)(w-\phi(\lambda))+C(\lambda)}\right]} \right| \leq 1$$

if and only if

$$\left| \frac{\left(A(\lambda)-\dfrac{|B(\lambda)|^2}{\overline{A(\lambda)}}\right)(w-\phi(\lambda))+C(\lambda)\left(1-\dfrac{\overline{B(\lambda)}}{\overline{A(\lambda)}}\right)}{C(\lambda)\left(1-\dfrac{B(\lambda)}{A(\lambda)}\right)} \right| \leq 1.$$

Thus

$$\begin{aligned} R(\lambda) &= \left| \frac{C(\lambda)\left(1-\dfrac{B(\lambda)}{A(\lambda)}\right)}{A(\lambda)-\dfrac{|B(\lambda)|^2}{\overline{A(\lambda)}}} \right| = \left| \frac{C(\lambda)\left(\dfrac{A(\lambda)-B(\lambda)}{A(\lambda)}\right)}{\dfrac{|A(\lambda)|^2-|B(\lambda)|^2}{\overline{A(\lambda)}}} \right| \\ &= \left| \frac{C(\lambda)\left(\dfrac{-2}{A(\lambda)}\right)}{\dfrac{4\operatorname{Re}\beta(\lambda)}{\overline{A(\lambda)}}} \right| = \frac{|C(\lambda)|}{2\operatorname{Re}\beta(\lambda)} \end{aligned}$$

and

$$\begin{aligned} S(\lambda) &= \phi(\lambda) - \frac{C(\lambda)\left(1-\dfrac{\overline{B(\lambda)}}{\overline{A(\lambda)}}\right)}{A(\lambda)-\dfrac{|B(\lambda)|^2}{\overline{A(\lambda)}}} \\ &= \phi(\lambda) - \frac{C(\lambda)(\overline{A(\lambda)}-\overline{B(\lambda)})}{|A(\lambda)|^2-|B(\lambda)|^2} = \phi(\lambda) + \frac{C(\lambda)}{2\operatorname{Re}\beta(\lambda)}. \end{aligned}$$ □

Note that, since $\operatorname{Re}\beta(\lambda) > 0$ for $|\lambda| < 1$, R and S are both C^∞ functions.

Now we move to the case of a Y fibered over ∂B_2 in $\mathbf{C}^3$.

LEMMA 4. *Let Y be as in Theorem 1. Then there exist C^∞ functions R and S on B_2 such that*

$$\hat{Y} \cap \{|z| < 1\} = \{ (z,w) \mid |w - S(z)| \leq R(z),\ |z| < 1 \}, \tag{10}$$

where R is positive.

Proof. Let Π be projection on $\bar{B}_2$, and let L be a complex affine slice of B_2. Then

$$\hat{Y} \cap \Pi^{-1}(\bar{L}) \text{ is polynomially convex,} \tag{11}$$

as the intersection of polynomially convex sets. Let $Z = Y \cap \Pi^{-1}(\bar{L})$. We shall show that

$$\hat{Y} \cap \Pi^{-1}(\bar{L}) = \hat{Z}. \tag{12}$$

Since $(B_2 \times \mathbf{C}) \setminus \hat{Y}$ is pseudoconvex, $\Pi^{-1}(L) \setminus (\hat{Y} \cap \Pi^{-1}(L))$ is pseudoconvex in $\Pi^{-1}(L)$. Hence we can say that $\hat{Y} \cap \Pi^{-1}(\bar{L})$ is pseudoconcave in $\bar{L} \times \mathbf{C}$. Now we use a theorem proved by Wermer in [10] and Słodkowski in [7]: On a compact pseudoconcave set X projecting onto Δ, polynomials satisfy a maximum modulus principle with respect to $X \cap \{|\lambda| = 1\}$. Thus, polynomials in $\Pi^{-1}(\bar{L})$ satisfy a maximum modulus principle on $\hat{Y} \cap \Pi^{-1}(\bar{L})$ with respect to Z, where we regard $\bar{L}$ as a copy of Δ. This means that

$$\hat{Y} \cap \Pi^{-1}(\bar{L}) \subset \hat{Z}. \tag{13}$$

We also have $Z \subset \hat{Y} \cap \Pi^{-1}(\bar{L})$, so

$$\hat{Z} \subset \hat{Y} \cap \Pi^{-1}(\bar{L}), \tag{14}$$

by (11). Combining (13) and (14), we have (12).

If $\hat{Y}$ has property (1), then by the 1-dimensional theory applied to any L passing through b, $\hat{Y} \cap \{\lambda = b\}$ has more than one point for all $b \in B_2$. From Theorem 2 of [1], this means that every fiber of $\hat{Y}$ is a disk with positive radius, so $\hat{Y}$ has the form (10) where R is indeed positive on B_2.

From the remark after the proof of Lemma 3, we know that $R|_L$ and $S|_L$ are in $C^\infty(L)$. That is, R and S are C^∞ on every complex affine slice of B_2.

LEMMA 5. *In fact, R and S are locally Lipschitz in B_2.*

Proof. Fix a concentric subball B_2' of B_2 such that $\bar{B}_2' \subset B_2$. We show R and S to be Lipschitz in B_2'. It will suffice to show R and S Lipschitz on every complex affine slice of B_2', where the same Lip constant works for every slice.

Let r denote the radius of B_2'. Fix a slice $L = \{ a + \lambda b \mid \lambda \in \Delta \}$, where $a, b \in B_2$, $a \perp b$, $|a|^2 + |b|^2 = 1$, and $|a| < r$. This is a slice of $\bar{B}_2$ that cuts B_2'. Let $I\colon \Delta \to L \cap \bar{B}_2$ and $I(\lambda) = a + \lambda b$, so that I parameterizes L. The part of L in B_2' is parameterized by those λ with

$$|\lambda| < \frac{\sqrt{r^2 - |a|^2}}{\sqrt{1 - |a|^2}} \equiv r', \quad \text{where } r' \leq r.$$

If we let

$$Y^L = \{ (\lambda, w) \mid (I(\lambda), w) \in Y \} \quad \text{and} \quad (\hat{Y})^L = \{ (\lambda, w) \mid (I(\lambda), w) \in \hat{Y} \},$$

then Y^L is a subset of $\Gamma \times \mathbf{C}$ and $(\hat{Y})^L$ is a subset of $\Delta \times \mathbf{C}$. From (12) and the fact that I is an affine transformation we get that $(\hat{Y})^L = \widehat{(Y^L)}$, and from now on we will write $\hat{Y}^L$ to mean either of these. From Lemma 3, we derive C^∞ functions $R_L, S_L, C_L, \ldots$ such that $\hat{Y}^L \cap \{|\lambda| < 1\} = \{ (\lambda, w) \mid |w - S_L(\lambda)| \leq R_L(\lambda), |\lambda| < 1 \}$. Furthermore, if $z \in L \cap B_2$ then $S(z) = S_L(I^{-1}(z))$ and $R(z) = R_L(I^{-1}(z))$. Thus,

$$\begin{aligned} \operatorname{Lip}(S|_{L \cap B_2'}) &\leq \operatorname{Lip}(S_L|_{\{|\lambda|<r'\}}) \operatorname{Lip}(I^{-1}|_{L \cap B_2'}) \\ &\leq \operatorname{Lip}(S_L|_{\{|\lambda|<r\}}) \frac{1}{\sqrt{1 - r^2}}, \end{aligned} \tag{15}$$

because $r' \leq r$ and

$$\operatorname{Lip}(I^{-1}|_L) = \frac{1}{\sqrt{1-|a|^2}} \leq \frac{1}{\sqrt{1-r^2}}.$$

Now we must bound $\operatorname{Lip}(S_L|_{\{|\lambda|<r\}})$. We have $S_L = \phi_L + C_L/(2\operatorname{Re}\beta_L)$ from Lemma 3. Since C_L has H^1 norm ≤ 2, C_L has Lipschitz constant K_1 on $\{|\lambda| < r\}$ where K_1 depends only on r. We will write $\operatorname{Lip}(C_L) \leq K_1(r)$. Because $|\phi_L| \leq |\phi_L - \alpha_L| + |\alpha_L| \leq 1+1 \leq 2$ a.e. on Γ, similarly $\operatorname{Lip}(\phi_L) \leq K_1(r)$. For $|\lambda| = 1$, $\operatorname{Re}\beta_L = |F_L|$ so $\operatorname{Lip}(\operatorname{Re}\beta_L) \leq K_2(r)$, since $\operatorname{Re}\beta_L$ is harmonic. We also claim that, on $\{|\lambda| < r\}$,

$$\operatorname{Re}\beta_L \geq K_3(r) > 0. \tag{16}$$

This is a consequence of Harnack's inequality; on $\{|\lambda| < r\}$,

$$\operatorname{Re}\beta_L \geq \frac{1-r}{1+r}\operatorname{Re}\beta_L(0) = K_3(r)\frac{1}{2\pi}\int_\Gamma |F|\,d\theta = K_3(r),$$

so (16) follows.

Now if $|\lambda_1 - \lambda_2| \leq \delta$ for $|\lambda_1|, |\lambda_2| < r$, then

$$\begin{aligned}
|S_L(\lambda_1) - S_L(\lambda_2)| &\leq |\phi_L(\lambda_1) - \phi_L(\lambda_2)| + \left|\frac{C_L(\lambda_1)}{2\operatorname{Re}\beta_L(\lambda_1)} - \frac{C_L(\lambda_2)}{2\operatorname{Re}\beta_L(\lambda_2)}\right| \\
&\leq K_1(r)\delta + \frac{1}{2}\left(\frac{C_L(\lambda_1)\operatorname{Re}\beta_L(\lambda_2) - C_L(\lambda_2)\operatorname{Re}\beta_L(\lambda_1)}{\operatorname{Re}\beta_L(\lambda_1)\operatorname{Re}\beta_L(\lambda_2)}\right) \\
&\leq K_1(r)\delta + \frac{1}{2}\,\frac{\begin{array}{l}|C_L(\lambda_1)\operatorname{Re}\beta_L(\lambda_2) - C_L(\lambda_1)\operatorname{Re}\beta_L(\lambda_1)\\ \quad + C_L(\lambda_1)\operatorname{Re}\beta_L(\lambda_1) - C_L(\lambda_2)\operatorname{Re}\beta_L(\lambda_1)|\end{array}}{K_3(r)^2}.
\end{aligned}$$

Now C_L and $\operatorname{Re}\beta_L$ both have maximum moduli $\leq K_4(r)$ for $|\lambda| \leq r$, so the foregoing inequality is

$$\begin{aligned}
&\leq K_1(r)\delta + \frac{1}{2}\frac{K_4(r)}{K_3(r)^2}\big(|\operatorname{Re}\beta_L(\lambda_2) - \operatorname{Re}\beta_L(\lambda_1)| \\
&\qquad\qquad + |C_L(\lambda_2) - C_L(\lambda_1)|\big) \\
&\leq \left(K_1(r) + \frac{1}{2}\frac{K_4(r)}{K_3(r)^2}(K_1(r) + K_2(r))\right)\delta \\
&= K_5(r)\delta.
\end{aligned}$$

Thus $\operatorname{Lip}(S_L|_{\{|\lambda|<r\}}) \leq K_5(r)$, and from (15) we obtain

$$\operatorname{Lip}(S|_{L\cap B_2'}) \leq K_6(r);$$

since this is true for all L, we find that S_L is Lipschitz on B_2', as desired.

To do the same for R_L, we can show $\operatorname{Lip}(R|_{L\cap B_2'}) \leq \operatorname{Lip}(R_L|_{\{|\lambda|<r\}})(1-r^2)^{-1/2}$, following (15). Then $R_L = |S_L - \phi_L|$, the absolute value of a Lipschitz function on $\{|\lambda| < r\}$, where again the constant depends only on r. Proceeding as with S, we conclude that R is Lipschitz on B_2'. This proves Lemma 5. □

Our next goal will be to show that the partials

$$\frac{\partial^n R}{\partial x_1^n} \quad \text{and} \quad \frac{\partial^n S}{\partial x_1^n}$$

are continuous, where again $z = (z_1, z_2)$ and $x_1 = \operatorname{Re} z_1$. In a manner similar to that of Lemma 5, let

$$I_\zeta \colon \Delta \to B_2 \quad \text{and} \quad \lambda \mapsto \left(\lambda\sqrt{1-|\zeta|^2}, \zeta\right).$$

Also, let

$$Y^\zeta = \{\, (\lambda, w) \mid (I_\zeta(\lambda), w) \in Y \,\} \quad \text{and} \quad (\hat{Y})^\zeta = \{\, (\lambda, w) \mid (I_\zeta(\lambda), w) \in \hat{Y} \,\}.$$

Then Y^ζ is a subset of $\Gamma \times \mathbf{C}$ and $(\hat{Y})^\zeta$ is a subset of $\Delta \times \mathbf{C}$. From (12) we have that $(\hat{Y})^\zeta = \widehat{(Y^\zeta)}$, and from now on we will write $\hat{Y}^\zeta$ to mean either of these.

From [1] and Lemma 2 we get associated functions ϕ_0^ζ, F^ζ, C^ζ, and β^ζ, where ϕ_0^ζ satisfies the extremal property given in [1]—namely, that ϕ_0^ζ has the smallest real part at zero of all analytic functions whose graphs lie in $\hat{Y}^\zeta \cap \{|\lambda| < 1\}$. Define

$$\phi_0(\lambda, \zeta) = \phi_0^\zeta(I_\zeta^{-1}(\lambda, \zeta))$$

for all $(\lambda, \zeta) \in B_2$, and define

$$F(\lambda, \zeta), C(\lambda, \zeta), \beta(\lambda, \zeta) \tag{17}$$

similarly.

LEMMA 6.
$$\frac{\partial^n \phi_0}{\partial x_1^n} \text{ are continuous, } n \geq 0,$$
as functions of z_1 *and* z_2 *on* B_2, *where* $x_1 = \operatorname{Re} z_1$.

Proof. Suppose not. Then, for some n there exists a sequence $\{(\lambda^k, \zeta^k)\} \to (\lambda^0, \zeta^0)$ such that, if we let

$$a_k = \frac{\partial^n \phi_0}{\partial x_1^n}(\lambda^k, \zeta^k),$$

then either

$$\{a_k\} \text{ converges to something other than } \frac{\partial^n \phi_0}{\partial x_1^n}(\lambda^0, \zeta^0) \tag{18}$$

or

$$\text{no subsequence of } \{a_k\} \text{ converges.} \tag{19}$$

We may assume without loss of generality that $\{(\lambda^k, \zeta^k)\}$ and (λ^0, ζ^0) are contained in the interior of some polydisc $\{(z_1, z_2) \mid |z_1| < r_1, |z_2| < r_2\}$. Now

$$\frac{\partial^n \phi_0}{\partial x_1^n}(z_1, z_2) = \left(\frac{1}{\sqrt{1-|z_2|^2}}\right)^n \frac{\partial^n \phi_0^{z_2}}{\partial x_1^n}(z_1). \tag{20}$$

Since the $\{\phi_0^{\zeta^k}\}$ are uniformly bounded, there exists a subsequence converging uniformly on compact subsets of int Δ. Call the limit $\psi_0^{\zeta^0}$. Passing to this subsequence, we also have

$$\frac{\partial^n \phi_0^{\zeta^k}}{\partial x_1^n} \to \frac{\partial^n \psi_0^{\zeta^0}}{\partial x_1^n} \tag{21}$$

uniformly on compact subsets of int Δ. For all ζ^k we have

$$\{ (\lambda\sqrt{1-|\zeta^k|^2}, \zeta^k, \phi_0^{\zeta^k}(\lambda)) \mid \lambda \in \operatorname{int}\Delta \} \subset \partial\hat{Y} \cap \Pi^{-1}(B_2),$$

since the left side is the graph of $\phi_0^{\zeta^k} \circ I_{\zeta^k}^{-1}$ over $B_2 \cap \{z_2 = \zeta^k\}$. Because $\partial\hat{Y} \cap \Pi^{-1}(B_2)$ is relatively closed in $\Pi^{-1}(B_2)$, we may say the same for the graph of $\psi_0^{\zeta^0} \circ I_{\zeta^0}^{-1}$; that is,

$$\{ (\lambda\sqrt{1-|\zeta^0|^2}, \zeta^0, \psi_0^{\zeta^0}(\lambda)) \mid \lambda \in \operatorname{int}\Delta \} \subset \partial\hat{Y} \cap \Pi^{-1}(B_2).$$

Hence

$$\{ \lambda, \psi_0^{\zeta^0}(\lambda)) \mid \lambda \in \operatorname{int}\Delta \} \subset \partial\hat{Y}^{\zeta^0} \cap \{|\lambda| < 1\}. \tag{22}$$

Furthermore,

$$\phi_0^{\zeta^k}(0) = \phi_0^{\zeta^k} \circ I_{\zeta^k}^{-1}(0, \zeta^k) = S(0, \zeta^k) - R(0, \zeta^k),$$

by the extremal property of the $\phi_0^{z_2}$ functions. By continuity of R and S (from Lemma 5),

$$\psi_0^{\zeta^0}(0) = \psi_0^{\zeta^0} \circ I_{\zeta^0}^{-1}(0, \zeta^0) = S(0, \zeta^0) - R(0, \zeta^0). \tag{23}$$

However, from Lemma 2, $\phi_0^{\zeta^0}$ is the only analytic function on int Δ satisfying the same properties that $\psi_0^{\zeta^0}$ does in (22) and (23). Thus $\phi_0^{\zeta^0} = \psi_0^{\zeta^0}$, so (21) implies

$$\frac{\partial^n \phi_0^{\zeta^k}}{\partial x_1^n} \to \frac{\partial^n \phi_0^{\zeta^0}}{\partial x_1^n}$$

uniformly on compact subsets of int Δ. Therefore

$$\left(\frac{1}{\sqrt{1-|\zeta^k|^2}}\right)^n \frac{\partial^n \phi_0^{\zeta^k}}{\partial x_1^n} \to \left(\frac{1}{\sqrt{1-|\zeta^0|^2}}\right)^n \frac{\partial^n \phi_0^{\zeta^0}}{\partial x_1^n}$$

uniformly on compact subsets of int Δ and hence, from (20),

$$\frac{\partial^n \phi_0}{\partial x_1^n}(\lambda^k, \zeta^k) \to \frac{\partial^n \phi_0}{\partial x_1^n}(\lambda^0, \zeta^0),$$

which contradicts (18) and (19). This proves Lemma 6. □

LEMMA 7.
$$\frac{\partial^n S}{\partial x_1^n} \textit{ is continuous, } n \geq 0.$$

Proof. We have shown that $\partial^n \phi_0 / \partial x_1^n$ is continuous, so it suffices to show that the partials of

$$\frac{C}{\operatorname{Re}\beta} = 2(S - \phi_0)$$

are continuous. (See (17) for definitions of C and β.)

Recall that, for $|\zeta| < 1$, C^ζ has H^1-norm ≤ 2 and $\operatorname{Re}\beta^\zeta$ has L^1-norm 1 on $\partial\Delta$. Thus F^ζ and $\operatorname{Re}\beta^\zeta$ are locally uniformly bounded as a class of functions indexed by ζ. Now suppose that

$$\frac{\partial^n}{\partial x_1^n}\left(\frac{C}{\operatorname{Re}\beta}\right)$$

is not continuous. Then, as in Lemma 6, there exists a sequence $\{(\lambda^k, \zeta^k)\} \to (\lambda^0, \zeta^0)$ in B_2 such that, if we let

$$a_k = \frac{\partial^n}{\partial x_1^n}\frac{C}{\operatorname{Re}\beta}(\lambda^k, \zeta^k),$$

then either

$$\{a_k\} \text{ converges to something other than } \frac{\partial^n}{\partial x_1^n}\frac{C}{\operatorname{Re}\beta}(\lambda^0, \zeta^0) \tag{24}$$

or

$$\text{no subsequence of } \{a_k\} \text{ converges.} \tag{25}$$

Choose subsequences $\{C^{\zeta^j}\}$ of $\{C^{\zeta^k}\}$ and $\{\operatorname{Re}\beta^{\zeta^j}\}$ of $\{\operatorname{Re}\beta^{\zeta^k}\}$ that converge uniformly on compact subsets of int Δ, to $\hat{C}^{\zeta^0}$ and $\operatorname{Re}\hat{\beta}^{\zeta^0}$, respectively.

On compact sets, $\operatorname{Re}\hat{\beta}^{\zeta^0}$ is bounded away from zero since the $\operatorname{Re}\beta^{\zeta^j}$ are uniformly bounded away from zero. Thus all partials of C^{ζ^j} and $\operatorname{Re}\beta^{\zeta^j}$ tend to the respective partials of $\hat{C}^{\zeta^0}$ and $\operatorname{Re}\hat{\beta}^{\zeta^0}$, and since $\operatorname{Re}\hat{\beta}^{\zeta^0}$ is bounded away from zero we have

$$\frac{\partial^n}{\partial x_1^n}\frac{C^{\zeta^j}}{\operatorname{Re}\beta^{\zeta^j}} \to \frac{\partial^n}{\partial x_1^n}\frac{\hat{C}^{\zeta^0}}{\operatorname{Re}\hat{\beta}^{\zeta^0}} \tag{26}$$

uniformly on compact subsets of int Δ. However,

$$\frac{\hat{C}^{\zeta^0}}{\operatorname{Re}\hat{\beta}^{\zeta^0}} = \frac{C^{\zeta^0}}{\operatorname{Re}\beta^{\zeta^0}},$$

since

$$\frac{C}{\operatorname{Re}\beta} = 2(S - \phi_0)$$

is continuous from Lemmas 5 and 6. Thus (26) gives

$$\frac{\partial^n}{\partial x_1^n}\frac{C^{\zeta^j}}{\operatorname{Re}\beta^{\zeta^j}} \to \frac{\partial^n}{\partial x_1^n}\frac{C^{\zeta^0}}{\operatorname{Re}\beta^{\zeta^0}},$$

so

$$\left(\frac{1}{\sqrt{1-|\zeta^j|^2}}\right)^n \frac{\partial^n}{\partial x_1^n}\left(\frac{C^{\zeta^j}}{\operatorname{Re}\beta^{\zeta^j}}\right) \to \left(\frac{1}{\sqrt{1-|\zeta^0|^2}}\right)^n \frac{\partial^n}{\partial x_1^n}\left(\frac{C^{\zeta^0}}{\operatorname{Re}\beta^{\zeta^0}}\right)$$

and so

$$\frac{\partial^n}{\partial x_1^n}\frac{C}{\operatorname{Re}\beta}(\lambda^j, \zeta^j) \to \frac{\partial^n}{\partial x_1^n}\frac{C}{\operatorname{Re}\beta}(\lambda^0, \zeta^0),$$

contradicting (24) and (25). □

LEMMA 8.
$$\frac{\partial^n R}{\partial x_1^n} \text{ is continuous, } n \geq 0.$$

Proof. Since $R = |S - \phi_0|$ and $S = \phi_0$ is never zero, this follows from Lemmas 6 and 7. □

LEMMA 9. *Let $f: \mathbf{R}^n \to \mathbf{R}$. If $\partial^n f/\partial x_1^n$ $(n \geq 0)$ exists pointwise and is continuous, then the pointwise derivatives with respect to x_1 are the weak derivatives.*

Proof. Let $\phi \in C_0^\infty(\mathbf{R}^n)$. Then
$$\int f \frac{\partial^n \phi}{\partial x_1^n}\, dx = \iint \cdots \iint f \frac{\partial^n \phi}{\partial x_1^n}\, dx_1 \cdots dx_n$$
(by Fubini, since f is continuous)
$$= (-1)^n \iint \cdots \iint \left(\int \frac{\partial^n f}{\partial x_1^n} \phi\, dx_1 \right) dx_2 \cdots dx_n$$
(where $\partial^n f/\partial x_1^n$ denotes the pointwise derivative)
$$= (-1)^n \int \frac{\partial^n f}{\partial x_1^n} \phi\, dx$$
(by Fubini, since $\partial^n f/\partial x_1^n$ is continuous). Thus the pointwise $\partial^n f/\partial x_1^n$ serves as the weak partial of f, by definition. □

If ρ is a smooth function on $\mathbf{R}^n$ and $u \in \mathbf{R}^n$ then let
$$\frac{\partial \rho}{\partial u} = (\nabla \rho) \cdot u.$$
Lemma 9 now shows that R and S have continuous weak partials with respect to x_1 on B_2; however, these arguments hold not only for $\partial^n/\partial x_1^n$ but also for $\partial^n/\partial u^n$ in any direction u, simply because our arguments apply in any direction. Lemma 10 will allow us to conclude that all mixed weak partials of R and S are continuous functions.

LEMMA 10. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index of nonnegative integers, $|\alpha| = k$. Then there exist vectors $u_1, u_2, \ldots, u_p$ in $\mathbf{R}^n$ and real numbers $a_1, a_2, \ldots, a_p$ such that, for any distribution ϕ on an open set in $\mathbf{R}^n$,*
$$\frac{\partial^k \phi}{\partial x^\alpha} = \sum_{i=1}^{p} a_i \frac{\partial^k \phi}{\partial u_i^k}, \tag{27}$$
where the partials taken in (27) are weak partials.

Proof. It suffices to prove the lemma for classical partials of C^∞ functions ϕ. Let $D^k\phi$ be the kth derivative of ϕ (as defined in [6, Chap. XIII]), so that $D^k\phi(x) \in L_s^k(\mathbf{R}^n; \mathbf{R})$, the set of real symmetric k-linear maps on (the tangent space of) $\mathbf{R}^n$. In this notation, we have

$$D^k\phi(x)\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial x_n}\right)=\frac{\partial^k\phi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

(where each $\partial/\partial x_i$ appears α_i times) and extend multilinearly. In particular, if $\partial/\partial u$ is some element of the tangent space to $\mathbf{R}^n$ then

$$D^k\phi(x)\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u},\dots,\frac{\partial}{\partial u}\right)=\frac{\partial^k\phi}{\partial u^k}.$$

By Lemma 4.4.19 [4, p. 122] (the k-dimensional polarization formula), the value of a symmetric k-linear form (such as $D^k\phi(x)$) on an arbitrary k-tuple can be computed as a linear combination of the values of the form on constant k-tuples, where the linear combination is independent of the given k-linear form. Thus, there exist $u_i \in \mathbf{R}^n$ and $a_i \in \mathbf{R}$ independent of ϕ and x such that

$$\begin{aligned} D^k\phi(x)&(e_1,e_1,\dots,e_2,e_2,\dots,e_3,e_3,\dots,e_n,e_n) \\ &= \sum_{i=1}^{p} a_i D^k\phi(x)(u_i,u_i,\dots,u_i), \end{aligned} \tag{28}$$

where $e_i = \partial/\partial x_i$ and e_i appears α_i times. In standard terminology, (28) is exactly (27), which is what we need. □

Conclusion of Proof of Lemma 4. Given an arbitrary mixed weak partial derivative operator, we now have that it can be written as a linear combination of nonmixed operators. Hence, from Lemmas 7–9 and the Remark after the proof of Lemma 9, all weak partials of R and S will be continuous and so in $L^2_{\text{loc}}(B_2)$. From Sobolev's embedding theorem (see [5, Thm. 4.6.3]) and the fact (from Lemma 5) that R and S are continuous, we conclude that R and S must be C^∞ functions. □

We now prove Theorems 1 and 2, which we state again for the reader's convenience.

THEOREM 1. *Let Y be a compact subset of $(\partial B_2)\times \mathbf{C}$ of the form*

$$Y=\{\,(z,w)\mid |w-\alpha(z)|\le 1,\ z\in\partial B_2\,\},$$

where α is a continuous complex-valued function on ∂B_2, $\|\alpha\|_\infty \le 1$, and

$$\hat{Y}\cap\{z=b\}\ \textit{has more than one point for some}\ b\in B_2.$$

Suppose also that $(B_2\times\mathbf{C})\setminus\hat{Y}$ is pseudoconvex. Let $(0,0,w_0)\in\partial\hat{Y}$. Then there exists a unique $\phi\in H^\infty(B_2)$ such that $\phi(0,0)=w_0$ and for all $z\in B_2$ and $(z,\phi(z))\in\partial\hat{Y}$.

Proof. Lemma 4 shows that $\partial\hat{Y}\cap\{|z|<1\}$ is a C^∞ hypersurface in $\mathbf{C}^3$. We show that it is Levi flat. Let $M=\partial\hat{Y}\cap\{|z|<1\}$. Let $\rho(z,w)=|w-S(z)|-R(z)$, where R and S are defined in Lemma 4. Then M is defined by the vanishing of ρ, which is C^∞ in a neighborhood of M in $B_2\times\mathbf{C}$. Also note that ρ has nonzero gradient on M for the following reason: if we fix z_1 and z_2 then the radial derivative in w is nonzero.

Now suppose that $u \in \mathbf{C}^3$ satisfies

$$\sum_{i=1}^{3} \frac{\partial \rho}{\partial z_i}(q) u_i = 0, \tag{29}$$

where $q \in M$ and $z_3 = w$. We wish to show

$$\sum_{j,k=1}^{3} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) u_j \bar{u}_k = 0. \tag{30}$$

Now if we write

$$\frac{\partial \rho}{\partial u}(z_1, z_2, z_3) = \frac{\partial \rho}{\partial \lambda}((z_1, z_2, z_3) + \lambda(u_1, u_2, u_3))$$

then (29) is the same as

$$\frac{\partial \rho}{\partial u}(q) = 0 \tag{31}$$

and (30) is the same as

$$\frac{\partial^2 \rho}{\partial u \partial \bar{u}}(q) = 0. \tag{32}$$

Consider a diametrical slice L of B_2 such that the direction of (u_1, u_2) is in L. Then $\partial \hat{Y} \cap \Pi^{-1}(L)$ is Levi flat, as a hypersurface of $\Pi^{-1}(L)$. This means (31) implies (32), which is what we want.

By Exercise 9 [5, p. 308], a Levi-flat hypersurface in $\mathbf{C}^3$ is foliated by analytic manifolds of complex dimension 2; thus there exists an analytically embedded ball in $\partial \hat{Y} \cap \Pi^{-1}(B_2)$ passing through $(0, 0, w_0)$:

$$I \colon B_2 \to \partial \hat{Y} \cap \Pi^{-1}(B_2), \qquad I(0,0) = (0, 0, w_0).$$

We want to make this analytic ball a graph over B_2. Let $T_{(0,0)}B_2$ be the tangent space to B_2 at $(0, 0)$. We note that

$$\begin{aligned} &dI|_{(0,0)}(T_{(0,0)}B_2) \\ &\qquad = \text{the entire space of complex tangents to } M \text{ at } (0,0,w_0). \end{aligned} \tag{33}$$

Claim. $d\Pi|_{(0,0,w_0)} \circ dI|_{(0,0)}(T_{(0,0)}B_2) = T_{(0,0)}B_2$.

Consider $L_1 = \{ (z_1, 0) \mid |z_1| < 1 \}$ and $L_2 = \{ (0, z_2) \mid |z_2| < 1 \}$, as well as the corresponding unique $\phi_1(z_1)$, $\phi_2(z_2)$ such that $\phi_1(0) = \phi_2(0) = w_0$ and the graphs of ϕ_1, ϕ_2 over L_1, L_2 are contained in $\partial \hat{Y} \cap \Pi^{-1}(B_2)$. Then $(1, 0, \partial \phi_1 / \partial z_1)$ and $(0, 1, \partial \phi_2 / \partial z_2)$ are linearly independent complex tangents in $T_{(0,0,w_0)}(M)$, so by (33) they generate $dI|_{(0,0)}(T_{(0,0)}B_2)$. Then

$$d\Pi|_{(0,0,w_0)}\left(c_1\left(1, 0, \frac{\partial \phi_1}{\partial z_1}\right) + c_2\left(0, 1, \frac{\partial \phi_2}{\partial z_2}\right)\right) = (c_1, c_2, 0),$$

which equals zero if and only if $c_1 = c_2 = 0$. Thus $d\Pi|_{(0,0,w_0)}$ has image equal to all of $T_{(0,0)}B_2$, as claimed.

We conclude, by the inverse function theorem, that $\Pi \circ I$ has a local inverse. That is, there exist open balls $B_2((0,0), r_1)$, $B_2((0,0), r_2)$ and analytic $f \colon B_2((0,0), r_1) \to B_2((0,0), r_2)$ such that $(\Pi \circ I) \circ f = \text{id}$ on $B_2((0,0), r_1)$.

Then $I \circ f \equiv ((I \circ f)_1, (I \circ f)_2, (I \circ f)_3)\colon B_2((0,0), r_1) \to \mathbf{C}^3$ maps $(z_1, z_2) \mapsto (z_1, z_2, (I \circ f)_3(z_1, z_2))$, and the graph of $(I \circ f)_3$ lies in $\partial\hat{Y} \cap \Pi^{-1}(B_2)$. Note also that $(I \circ f)_3(0,0) = w_0$. We claim that $g \equiv (I \circ f)_3$ extends analytically to all of B_2.

Before verifying this, we note that we need not have chosen the point $(0, 0, w_0) \in \partial\hat{Y}$; we could have chosen any other point in $(\partial\hat{Y}) \setminus Y$ and similarly found a (unique) local analytic graph in $(\partial\hat{Y}) \setminus Y$ passing through that point.

Suppose that r is the radius of the largest open ball centered at $(0, 0)$ to which g extends analytically. Then we claim $r = 1$. Suppose that $r < 1$. Take any point $p = (p_1, p_2)$ on the sphere of radius r. Let L be the complex 1-dimensional linear diametrical slice of B_2 passing through $(0, 0)$ and (p_1, p_2).

Consider the point $(0, 0, g(0, 0))$ on $\partial\hat{Y}$. Then there is a unique analytic graph $\{w = g_L(z)\}$ over L, lying in $\partial\hat{Y} \cap \Pi^{-1}(B_2)$ and passing through $(0, 0, g(0,0))$. Consider the point $(p_1, p_2, g_L(p_1, p_2))$ on $\partial\hat{Y} \cap \Pi^{-1}(B_2)$. Then we can find a neighborhood N of (p_1, p_2) and an analytic function h defined on N such that the graph of h passes through $(p_1, p_2, g_L(p_1, p_2))$ and is contained in $\partial\hat{Y}$. Then h and g_L coincide on L in some neighborhood of (p_1, p_2), and so coincide on $L \cap N$. However, g and g_L coincide on $L \cap B_2((0,0), r)$. Further, we assumed g to be analytic in $B_2((0,0), r)$, and the graph of g is contained in $\partial\hat{Y}$. It follows that g and h coincide on some open set contained in $N \cap B_2((0,0), r)$. Thus, h provides an analytic extension of g to $B_2((0,0), r) \cup N$.

This argument holds for any p in the boundary of $B_2((0,0), r)$. Finitely many of such N cover the boundary of $B_2((0,0), r)$. Consider two such neighborhoods N_p and N_q (where p and q are points in the boundary of $B_2((0,0), r)$) with corresponding functions h_p and h_q analytic on them. Then h_p and h_q coincide with each other on $N_p \cap N_q \cap B_2((0,0), r)$, since both equal g there. Hence h_p and h_q coincide with each other on all of $N_p \cap N_q$ by uniqueness of continuation. Considering all functions h_p together, we conclude that g extends to be analytic in a ball of radius greater than r about $(0, 0)$. This is a contradiction, so the assumption that $r < 1$ is false. Thus g extends to be analytic in all of B_2, providing the desired analytic ϕ. The graph of ϕ is in $\partial\hat{Y}$ because, given any diametrical affine slice L of B_2, the graph of $\phi|_L$ over $L \cap B_2$ is an analytic disk that passes through a point in the boundary of $\hat{Y} \setminus Y$, so it is entirely contained in $(\partial\hat{Y}) \setminus Y$.

To show that ϕ is unique, suppose we have another ψ that satisfies the requirements of Theorem 1. Then, over every diametrical slice L, the graph of ψ_L is the unique analytic graph in $\hat{Y} \cap \Pi^{-1}(L)$ passing through $(0, 0, w_0)$. Thus $\phi|_L = \psi|_L$ for all L, so $\phi = \psi$ and we are done. $\square$

THEOREM 2. *Let Y be as in Theorem 1. Then there exist analytic functions A, B, C analytic on B_2, such that*

$$\hat{Y} \cap \Pi^{-1}(B_2) = \left\{ (z, w) \;\middle|\; \left|\frac{A(z)(w - \phi(z)) + C(z)}{B(z)(w - \phi(z)) + C(z)}\right| \le 1,\ |z| < 1 \right\}, \tag{34}$$

where ϕ is the function found in Theorem 1.

Proof. From Lemma 3 we note that, if L is any complex linear slice of $\bar{B}_2$ and L is identified with the unit disk Δ, then we can apply the 1-dimensional result to obtain C_L and β_L such that, on $L \cap B_2$,

$$S(z) = \frac{C_L(z)}{2\operatorname{Re}\beta_L(z)} + \phi(z) \quad \text{and} \quad R(z) = \frac{|C_L(z)|}{2\operatorname{Re}\beta_L(z)}.$$

Hence, regardless of which L we take through a particular $z \in B_2$,

$$\frac{C_L(z)}{2\operatorname{Re}\beta_L(z)} = S(z) - \phi(z)$$

is well-defined. Thus $\arg C_L(z)$ is independent of the choice of L through z, and $\arg C_L$ is a well-defined pluriharmonic function on B_2. Hence there exist analytic C on B_2 such that $\arg C(z) = \arg C_L(z)$ for all $z \in B_2$ and all L passing through z. (C is unique up to multiplication by a positive constant.)

On every L, C is a positive multiple of C_L, and $C(z) \neq 0$ for all $z \in B_2$. Define $P(z)$ (to be our $2\operatorname{Re}\beta(z)$) as

$$P(z) = \frac{C(z)}{S(z) - \phi(z)}$$

on B_2. Since $S(z) \neq \phi(z)$ and $C(z)$ is never zero, $P(z)$ is well defined and nonzero on B_2. Because C is a positive multiple of C_L on every L, it follows that P is a positive multiple of $\operatorname{Re}\beta_L$ and is thus positive harmonic on L. Thus P is positive pluriharmonic on B_2. Choose β analytic on B_2 with $\operatorname{Re}\beta = P$, and let

$$A = -\beta - 1, \qquad B = -\beta + 1$$

on B_2. We claim that (34) holds for the A, B, and C just defined. To verify this, we merely go through the procedure of Lemma 3 to obtain the center and radius of the region defined by the right side of (34). We find the radius at z to be $|C(z)|/(2\operatorname{Re}\beta(z))$, which is equal to $|C_L(z)|/(2\operatorname{Re}\beta_L(z))$ on every slice L, as required; we find the center at z to be $\phi(z) + C(z)/(2\operatorname{Re}\beta(z))$, which is equal to $\phi(z) + C_L(z)/(2\operatorname{Re}\beta_L(z))$ on every slice L, as required. Thus (34) holds, as desired. □

We are now in a positive to verify the remark made earlier that $\hat{Y} \cap \Pi^{-1}(B_2)$ must be the union of analytic graphs over B_2 whose boundaries lie in Y. For $(z, w) \in B_2 \times \mathbf{C}$, let

$$M(z, w) = \frac{A(z)(w - \phi(z)) + C(z)}{B(z)(w - \phi(z)) + C(z)}.$$

Choose any point $(z_0, w_0) \in \hat{Y} \cap \Pi^{-1}(B_2)$; then $M(z_0, w_0) = k$ where $|k| \leq 1$. Since $A(z)C(z) - B(z)C(z) \neq 0$ for all $z \in B_2$, we can solve $M(z, w) = k$ for w in terms of z:

$$w = \frac{C(z)(k-1)}{A(z) - kB(z)} + \phi(z) \equiv f(z).$$

This is a well-defined complex analytic function of z, because $|A(z)|^2 - |B(z)|^2 = 4\operatorname{Re}\beta(z) > 0$ for $|z| < 1$ guarantees that $|B(z)| < |A(z)|$ and so $A(z) - kB(z) \neq 0$.

Since $M(z, f(z)) = k$ for all $z \in B_2$, we have that the graph of f over B_2 is contained in $\hat{Y}$. Thus $f \in H^\infty(B_2)$. Also, $M(z_0, f(z_0)) = k$ and so, because $M: \{z_0\} \times \mathbf{C} \to \mathbf{C}$ is invertible for fixed $z_0 \in B_2$, we have $f(z_0) = w_0$. Finally, we claim that the boundary values of f on ∂B_2 are in Y_z for a.e. $z \in \partial B_2$. Select a sequence of points z_n in B_2 tending radially to $z \in \partial B_2$ such that $\{f(z_n)\}$ converges. Then $\{(z_n, f(z_n))\}$ converges as well; since $\hat{Y}$ is closed, $\{(z_n, f(z_n))\}$ converges to an element of $\hat{Y}$ that sits over z. This means that $\{f(z_n)\}$ converges to an element of Y_z, as desired.

THEOREM 3. *Suppose that Y is a compact subset of $\bar{B}_2 \times \mathbf{C}$ of the form*

$$Y = \left\{ (z, w) \;\middle|\; \left|\frac{A(z)w + B(z)}{C(z)w + D(z)}\right| \le 1,\ z \in \bar{B}_2 \right\},$$

where A, B, C, D are analytic in a neighborhood of $\bar{B}_2$, $A(z)D(z) - B(z)C(z) \neq 0$ on $\bar{B}_2$, and $|A(z)| > |C(z)|$ for $z \in \bar{B}_2$ (so that Y is compact with disk fibers.) Then Y is pseudoconcave and polynomially convex.

Proof. For fixed $e^{i\theta}$, $0 \le \theta \le 2\pi$, we may solve

$$\frac{A(z)w + B(z)}{C(z)w + D(z)} = e^{i\theta}$$

for w; this shows that the boundary of Y over B_2 is foliated by graphs of functions analytic in a neighborhood of $\bar{B}_2$. From this we easily obtain the fact that Y is pseudoconcave: Given an analytic ϕ whose graph is in the foliation, the function $1/(w - \phi(z))$ is analytic in $(B_2 \times \mathbf{C}) \subset Y$ and singular at every point of the graph of ϕ. Since the graphs of such ϕ foliate $\partial Y \cap \{|z| < 1\}$, this shows that $(B_2 \times \mathbf{C}) \subset Y$ is pseudoconvex and hence Y is pseudoconcave.

Let us consider any two different functions ϕ_1 and ϕ_2 whose graphs lie in the foliation of $\partial Y \cap \{|z| < 1\}$. Since the fibers of Y are strictly convex, $(\phi_1(z) + \phi_2(z))/2$ lies in the interior of Y_z for all $z \in \bar{B}_2$. Hence we may assume without loss of generality (by change of variable in w) that $0 \in \operatorname{int} Y_z$ for all $z \in \bar{B}_2$. We also note that, since Y is the union of graphs over $\bar{B}_2$ of functions analytic in a neighborhood of $\bar{B}_2$,

$$Y \text{ is contained in the polynomial hull of } Y \cap \{|z| = 1\};$$

hence

$$\hat{Y} \text{ is contained in the polynomial hull of } Y \cap \{|z| = 1\}. \tag{35}$$

Now suppose that $\hat{Y} \neq Y$. We proceed to a contradiction using an argument of Oka. Consider the class of sets tY given by

$$tY = \{ (z, tw) \mid (z, w) \in Y \}.$$

Since $0 \in \operatorname{int} Y_z$ for all $z \in \bar{B}_2$, there exist $t \ge 1$ such that $\hat{Y} \subset tY$. Let us choose the smallest such t. If we assume $t \neq 1$ then $\hat{Y}$ and tY have a boundary point $p = (z_0, w_0)$ in common that sits over B_2. Choose a ϕ whose graph lies in $\partial Y \cap \Pi^{-1}(B_2)$ such that p lies in the graph of $t\phi$. We shall be interested in the functions $\psi_s = 1/(w - s\phi)$ for $s > t$. We note that ψ_s is analytic in a neighborhood of $\hat{Y}$ and so is uniformly approximable by polynomials on $\hat{Y}$. Because $(z_0, w_0) \in \hat{Y}$ we have that, for all $s > t$,

$$\begin{aligned}
&|\psi_s(z_0, w_0)| \\
&\quad \le \sup_{(z,w)\in Y\cap\{|z|=1\}} |\psi_s(z,w)| \quad \text{from (35)} \\
&\quad = \sup_{(z,w)\in Y\cap\{|z|=1\}} \left| \frac{1}{w - s\phi(z)} \right| \\
&\quad \le \sup_{(z,w)\in Y\cap\{|z|=1\}} \left| \frac{1}{w - t\phi(z)} \right| \quad \text{since } Y_z \text{ is closer to } t\phi(z) \text{ than } s\phi(z) \\
&\quad = M,
\end{aligned}$$

where M is independent of s. However, if we choose s arbitrarily close to t then $|\psi_s(z_0, w_0)|$ can be made arbitrarily large, since $s\phi(z_0) \to w_0$ as $s \to t$. This is a contradiction, so the assumption that $t > 1$ was false and we have $\hat{Y} = Y$. □

We note that Theorem 3 is a partial converse to Theorem 2: the former shows that sets of the form (34) are pseudoconcave and polynomially convex for reasonably general A, B, and C. The latter shows that sets of the form (34) are essentially all of the pseudoconcave polynomially convex sets over $\bar{B}_2$ that have disk fibers.

We now generalize Theorems 1 and 2 to compact Y with disk fibers Y_λ of variable radius.

THEOREM 4. *Let Y be a compact subset of $(\partial B_2) \times \mathbf{C}$ of the form*

$$Y = \{\, (z,w) \mid |w - \alpha(z)| \le R(z),\ z \in \partial B_2 \,\},$$

where α is a continuous complex-valued function on ∂B_2, R is positive and in $C^2(\partial B_2)$, $|\alpha(z)| \le R(z)$, and

$$\hat{Y} \cap \{z = b\} \text{ has more than one point for some } b \in B_2.$$

Suppose also that $(B_2 \times \mathbf{C}) \setminus \hat{Y}$ is pseudoconvex. Let $(0, 0, w_0) \in \partial\hat{Y}$. Then there exists a unique $\phi \in H^\infty(B_2)$ such that $\phi(0,0) = w_0$ and for all $z \in B_2$, $(z, \phi(z)) \in \hat{Y}$. Furthermore, Theorem 2 holds for such Y.

Proof. One may proceed as in the case where $R = 1$. We sketch how the details are different; essentially, one must use Theorem 3 of [1] instead of Theorem 2 of [1]. The proof of Lemma 1 is the same. We replace Lemma 2 with the following.

LEMMA 11. *Let α, R, and Y be as in Theorem 3 of* [1]. *Let ϕ be any one of the analytic functions whose graph is in $\partial\hat{Y} \cap \{|\lambda| < 1\}$. Then*

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{\tilde{A}(\lambda)(w - \phi(\lambda)) + \tilde{C}(\lambda)}{\tilde{B}(\lambda)(w - \phi(\lambda)) + \tilde{C}(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}, \tag{36}$$

where $\tilde{C} \in H^1(\Delta)$, $\tilde{C}$ is never zero on int Δ, $\|\tilde{C}\|_1 \le 2\|R\|_\infty$, *and we have*

$$\tilde{A} = -\tilde{\beta} - 1, \qquad \tilde{B} = -\tilde{\beta} + 1,$$

where

$$\tilde{\beta}(\lambda) = \frac{1}{2\pi} \int_{\Gamma} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} |\tilde{F}(e^{i\theta})| \, d\theta, \quad |\lambda| < 1,$$

and $\tilde{F}$ *is an outer function with* H^1 *norm equal to* 1.

Proof. We follow Alexander and Wermer's arguments in [1], keeping track of the properties that $\tilde{A}$, $\tilde{B}$, ... will possess. We have

$$Y = \{ (\lambda, w) \mid |w - \alpha(\lambda)| \le R(\lambda),\ \lambda \in \Gamma \}.$$

As in [1], we write $R(\lambda) = e^{u(\lambda)} = |e^{G(\lambda)}|$, where $G = u + iv \in A(\Gamma)$. Then

$$Y = \left\{ (\lambda, w) \;\middle|\; \left| \frac{w}{e^{G(\lambda)}} - \frac{\alpha(\lambda)}{e^{G(\lambda)}} \right| \le 1,\ \lambda \in \Gamma \right\}.$$

If we let $w' = w/(e^{G(\lambda)})$ and $\alpha' = \alpha/(e^{G(\lambda)})$, then

$$Y = \{ (\lambda, w' e^{G(\lambda)}) \mid |w' - \alpha'(\lambda)| \le 1,\ \lambda \in \dot{\Gamma} \}.$$

We denote by $Y/e^{G(\lambda)}$ the set $\{ (\lambda, w') \mid (\lambda, w' e^{G(\lambda)}) \in Y \}$, so that

$$\frac{Y}{e^{G(\lambda)}} = \{ (\lambda, w') \mid |w' - \alpha'(\lambda)| \le 1,\ \lambda \in \Gamma \}.$$

Then $Y/e^{G(\lambda)}$ satisfies the properties of Theorem 2 of [1] and hence (according to our Lemma 2) there exist analytic $\tilde{A}$, $\tilde{B}$, C, $\tilde{F}$, and $\tilde{\beta}$ such that

$$\begin{aligned}
&\widehat{\left(\frac{Y}{e^{G(\lambda)}}\right)} \cap \{|\lambda| < 1\} \\
&= \frac{\hat{Y}}{e^{G(\lambda)}} \cap \{|\lambda| < 1\} \\
&= \left\{ (\lambda, w') \;\middle|\; \left| \frac{\tilde{A}(\lambda)\left(w' - \dfrac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)}{\tilde{B}(\lambda)\left(w' - \dfrac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\},
\end{aligned}$$

where we note that the graph of ϕ/e^G is $\subset \partial\widehat{\left(\frac{Y}{e^{G(\lambda)}}\right)} \cap \{|\lambda| < 1\}$. Further, we have that $\tilde{A}$, $\tilde{B}$, $\tilde{F}$, and $\tilde{\beta}$ satisfy the requirements of the lemma and $C = 2\tilde{F}$. Then

$$\begin{aligned}
&\frac{\hat{Y}}{e^{G(\lambda)}} \cap \{|\lambda| < 1\} \\
&= \left\{ (\lambda, w') \;\middle|\; \left| \frac{\tilde{A}(\lambda)\left(w' - \dfrac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)}{\tilde{B}(\lambda)\left(w' - \dfrac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\} \\
&= \left\{ \left(\lambda, \frac{w}{e^{G(\lambda)}}\right) \;\middle|\; \left| \frac{\tilde{A}(\lambda)\left(\dfrac{w}{e^{G(\lambda)}} - \dfrac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)}{\tilde{B}(\lambda)\left(\dfrac{w}{e^{G(\lambda)}} - \dfrac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)} \right| \le 1,\ |\lambda| < 1 \right\}
\end{aligned}$$

and

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \,\middle|\, \left| \frac{\tilde{A}(\lambda)\left(\frac{w}{e^{G(\lambda)}} - \frac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)}{\tilde{B}(\lambda)\left(\frac{w}{e^{G(\lambda)}} - \frac{\phi(\lambda)}{e^{G(\lambda)}}\right) + C(\lambda)} \right| \leq 1,\ |\lambda| < 1 \right\}$$
$$= \left\{ (\lambda, w) \,\middle|\, \left| \frac{\tilde{A}(\lambda)(w - \phi(\lambda)) + C(\lambda)e^{G(\lambda)}}{\tilde{B}(\lambda)(w - \phi(\lambda)) + C(\lambda)e^{G(\lambda)}} \right| \leq 1,\ |\lambda| < 1 \right\}.$$

Now, if we let $\tilde{C} = Ce^G$, we find that $\|\tilde{C}\|_1 \leq \|C\|_1 \|e^G\|_\infty = 2\|R\|_\infty$ and that $\tilde{C}$ is never 0 on int Δ; hence we have (36). $\square$

Conclusion of Proof of Theorem 4. Replacing Lemma 2 with Lemma 11, we find that Lemma 3 follows for our more general Y. The only remaining differences occur in Lemmas 5 and 7. We must replace the bound of 2 on the H^1-norm of C_L and C^ξ by the bound $2\|R\|_\infty$. Here, $\|R\|_\infty$ remains constant through our discussion and so functions as a uniform bound on the C_L and C^ξ, just as 2 did before. Also, in Lemma 5, in obtaining a bound on $\mathrm{Lip}(\phi_L)$ we find instead that $|\phi_L| \leq |\phi_L - \alpha_L| + |\alpha_L| \leq R + R \leq 2\|R\|_\infty$. Hence we still obtain $\mathrm{Lip}(\phi_L) \leq K_1(r)$. Otherwise we may proceed as before and prove Theorem 4. $\square$

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