Coherentlike Conditions in Pullbacks

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1. Introduction and Preliminaries

Let M be a (nonzero) maximal ideal of a domain T, let $k = T/M$ be the residue field, let $\phi: T \to k$ be the natural projection, and let D be a proper subring of k. Let $R = \phi^{-1}(D)$ be the domain arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & k = T/M. \end{array} \qquad \Box$$

We use K and F to denote the quotient fields of R and D, respectively. The case $k = F$ is of particular interest; in this case, we say that the diagram $\Box$ is of type $\Box^*$.

The goal of this paper is to characterize certain coherentlike properties of integral domains in pullback constructions of type $\Box$. In one sense, this work is a sequel to that of Brewer and Rutter [BR], in which coherence and several other properties are studied in so-called generalized $D+M$ constructions—that is, pullbacks of type $\Box$ in which it is assumed that $T = k + M$. ([BR] was in turn at least partly inspired by the work of Dobbs and Papick [DP] on coherence in the classical $D + M$ construction, in which $T = k + M$ is assumed to be a valuation domain.) Our work in this more general context is partly motivated by the fact that results which hold for the $D + M$ construction do not always extend to pullbacks of type $\Box$. For example, [FG, Thm. 4.2(b)] shows that the characterization of the GCD-property given in [BR, Thm. 11] requires modification, and [FG, Example 4.3] exploits pullbacks to give a counterexample to a conjecture of Anderson and Ryckaert [AR, Question 3.10].

The notion of v-finiteness figures prominently in [FG]. An ideal I of a domain R is said to be *v-finite* if $I^{-1} = J^{-1}$ for some finitely generated ideal J of R. We denote Section 2 to a study of divisoriality and v-finiteness in pullbacks of type $\Box$. We show, for example, that if T is quasilocal with maximal ideal M, then each

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nonprincipal divisorial ideal of T is a divisorial ideal of R; we also prove a partial converse. We then give a complete characterization of when M is v-finite as an ideal of R.

A domain R is said to be *v-coherent* if I^{-1} is a v-finite divisorial ideal for each finitely generated ideal I of R. This concept was introduced in the thesis of Nour El Abidine [N1], where it was proved that Nagata's theorem for class groups of Krull domains can be extended to v-coherent domains. Section 3 begins with a brief review of known facts, most of which are contained in [N1]. We then proceed to characterize when R is v-coherent. On the one hand, we show that in a pullback diagram of type $\square^*$, R is v-coherent if and only if D and T are v-coherent and M is a t-ideal of T; on the other hand, in a pullback diagram of type $\square$ where it is assumed that k properly contains the quotient field of D, we show that R is v-coherent if and only if D and T are v-coherent and M is either not a t-ideal of T or is a v-finite divisorial ideal of T. We also use our characterization to give a counterexample to a conjecture that appears in [N1].

In Section 4, we use results and techniques from Sections 2 and 3 (and from [FG]) to characterize several other coherentlike conditions in pullbacks of type $\square$. For example, we extend to this more general context the characterization of coherence given in [BR], and we also give characterizations of quasicoherence and the finite-conductor condition. The characterizations are as follows: R is coherent (quasicoherent, a finite conductor domain) if and only if exactly one of the following conditions holds: (1) k is the quotient field of D, D and T are coherent (quasicoherent, finite conductor domains), and T_M is a valuation domain; or (2) D is a field, $[k : D] < \infty$, T is coherent (quasicoherent, a finite conductor domain), and M is finitely generated in T. We also extend to v-domains the characterization of Prüfer v-multiplication domains (PVMDs) in pullbacks given in [FG], and we use this to give an example of a non–completely integrally closed v-domain that is not a PVMD. Finally, we summarize what is known for Mori domains in pullbacks, and we study the class of domains in which each divisorial ideal is v-finite.

Our methods throughout are ideal-theoretic.

2. Divisorial Ideals in Pullbacks

We begin by reviewing some terminology. Let R be a domain with quotient field K. For a nonzero fractional ideal I of R, the *inverse* of I is the fractional ideal $I^{-1} = (R : I) = \{ x \in K \mid xI \subseteq R \}$, the v or *divisorial closure* of I is given by $I_v = (I^{-1})^{-1}$, and the t-closure is given by $I_t = \bigcup \{ J_v \mid J$ is a finitely generated subideal of $I \}$. Recall from Section 1 that I is *v-finite* if $I^{-1} = J^{-1}$ (equivalently, $I_v = J_v$) for some finitely generated ideal J of R. For properties of the v- and t-operations, the reader is referred to [Gi, Secs. 32 and 34].

When studying pullbacks of type $\square$, it is often necessary to take inverses and v's in both R and T. For a fractional ideal I of R, we shall usually write I^{-1} for the inverse $(R : I)$ of I with respect to R and $(T : IT)$ for the inverse of IT with respect to T. However, because it is so convenient, we often write $(IT)_v$ for $(T : (T : IT))$ (the danger of confusion is slight).

We shall freely use the following properties of flat extensions (see [FG, Prop. 0.6]): If T is a flat overring of the domain R and J is a finitely generated ideal of R, then $(JT)_v = (J_vT)_v$; moreover, if J^{-1} is also v-finite in R then $(JT)_v = J_vT = (J_vT)_v$.

Recall that in the pullback diagram $\square$, M is the conductor of T, that is, $M = (R : T)$. It follows that R and T have the same quotient field K, that each ideal of T which is contained in M is also an ideal of R, and that M is a prime divisorial ideal of R with $M^{-1} = (M : M) \supseteq T$. Moreover, for each prime ideal P of R for which $P \nsubseteq M$, there is a unique prime ideal Q of T such that $P = Q \cap R$, and for this Q we have $R_P = T_Q$. In particular, for each maximal ideal $N \neq M$ in T we have $R_{N \cap R} = T_N$; if $k = F$ then $R_M = T_M$.

If $T = (T, M)$ is quasilocal, then every ideal of R is comparable with M; in particular, if D is also quasilocal (or a field) then R is quasilocal.

A general reference for basic facts about pullbacks is [F]. The term "ideal" means "integral ideal," and we often tacitly assume that an ideal is nonzero. Finally, because we refer to it so often, the reader is advised to have a copy of [FG] in hand.

LEMMA 2.1. *Consider a pullback diagram of type $\square$. Then:*

(1) *M is invertible in T if and only if $M^{-1} \subsetneqq (T : M)$; and*

(2) *if M is invertible in T, then $M^{-1} = T$.*

Proof. We have $M \subseteq M(T : M) \subseteq T$. Hence M is not invertible in T if and only if $M = M(T : M)$ if and only if $(M : M) = M^{-1} = (T : M)$, and (1) follows. Now suppose that M is invertible in T, and write $1 = m_1u_1 + \cdots + m_ku_k$ with $m_i \in M$ and $u_i \in (T : M)$ for $i = 1, \ldots, k$. Then for $x \in M^{-1} = (M : M)$ we have $x = (xm_1)u_1 + \cdots + (xm_k)u_k \in T$, from which it follows that $M^{-1} = T$. $\square$

PROPOSITION 2.2. *Consider a pullback diagram of type $\square$. Then M is divisorial in T if and only if exactly one of the following conditions holds:*

(i) *M is invertible in T; or*

(ii) *$M^{-1} \supsetneqq T$.*

Proof. Note that by Lemma 2.1, the two conditions cannot hold simultaneously.

($\Rightarrow$) If M is divisorial but not invertible in T then, by Lemma 2.1(1), $M^{-1} = (T : M) \supsetneqq T$.

($\Leftarrow$) If M is invertible then it is divisorial. If $M^{-1} \supsetneqq T$ then $(T : M) \supsetneqq T$ and, since M is maximal in T, this implies that M is divisorial. $\square$

In the case where T is quasilocal and D is a field, Proposition 2.2 is essentially [B, Remark 2.7].

PROPOSITION 2.3. *Consider a pullback of type $\square^*$. If M is invertible in T, then M is not v-finite in R.*

Proof. Let J be a finitely generated ideal of R with $J \subseteq M$. We shall show that $M^{-1} \neq J^{-1}$. For $u \in (T : M)$, we have $Ju \subseteq T$, so that $\phi(Ju) \subseteq k$. Since J

is finitely generated and $k = F$ (the quotient field of D), there is an element $r \in R \setminus M$ with $\phi(rJu) = \phi(r)\phi(Ju) \subseteq D$, whence $rJu \subseteq R$. It follows that $r(T:M) \subseteq J^{-1}$. On the other hand, $r(T:M) \nsubseteq T$, since $r \notin M = (T:(T:M))$. Because (by Lemma 2.1) $T = M^{-1}$, we have $J^{-1} \neq M^{-1}$. □

PROPOSITION 2.4. *In a pullback diagram of type* □, *the following are equivalent*:

(1) $k \neq F$;
(2) *there is an element* $x \in T \setminus R$ *with* $xR \cap R \subseteq M$;
(3) *there is an element* $x \in T \setminus R$ *with* $xR \cap R = xM$;
(4) *there is an element* $x \in T \setminus R$ *with* $M = (R:(1, x))$.

Proof. (1) ⇒ (2) We use $^{-}$ to denote images modulo M. Choose $x \in T \setminus R$ with $\bar{x} \notin F$. If $y = xr$ for $y, r \in R$, then $\bar{y} = \bar{x}\bar{r} \in D$, so that $\bar{r} = 0$ and we have $y \in M$.

(2) ⇒ (1) If $k = F$, then $T_M = R_M$ is flat over R. Hence for $x \in T \setminus R$, we have ($x \notin M$ and) $(xR \cap R)T_M = xT_M \cap T_M = T_M$, whence $xR \cap R \nsubseteq M$.

(3) ⇒ (2) Clear.

(2) ⇒ (3) Let $x \in T \setminus R$ with $xR \cap R \subseteq M$. If $y = xr$ with $y, r \in R$, then the assumption $xR \cap R \subseteq M$ implies that $y \in M$, whence $r \in M$ and we have $y \in xM$. Thus $xR \cap R \subseteq xM$. The reverse inclusion is clear.

(3) ⇔ (4) Clear. □

REMARK. In a pullback of type □, let $S = \phi^{-1}(F)$. Then it is clear from the proof of Proposition 2.4 that statement (4) can be replaced by "for each element $x \in T \setminus S$ we have $M = (R:(1, x))$." Statements (2) and (3) have similar replacements. In particular, if $D = F$ then the assumption $k \neq F$ is automatic (since we have assumed throughout that D is a proper subring of k), and we have $M = (R:(1, x))$ for each element $x \in T \setminus R$.

COROLLARY 2.5. *Consider a pullback diagram of type* □. *If* M *is finitely generated in the ring* $(M:M)$ *and* $k \neq F$, *then* M *is* v*-finite in* R.

Proof. Write $M = J(M:M)$ for some finitely generated ideal J of R with $J \subseteq M$. By Proposition 2.4, $M^{-1} = (M:M) = (1, x)_v$ for some $x \in T \setminus R$. Hence $M = J(1, x)_v \subseteq (J, Jx)_v \subseteq M$, and we have $M = (J, Jx)_v$. □

From Proposition 2.3 and Corollary 2.5, we obtain the following.

COROLLARY 2.6. *Consider a pullback diagram of type* □, *and assume that* M *is invertible in* T. *Then* M *is* v*-finite in* R *if and only if* $k \neq F$.

The following technical result is the key to proving the main results in this paper.

PROPOSITION 2.7. *Consider a pullback diagram of type* □, *and let* I *be a fractional ideal of* R.

(1) *If* IT_M *is not principal, then*
(a) $(M:I) = I^{-1} = (R:IT) = (M:IT) = (T:IT) = I^{-1}T$ *and*
(b) $I_v = I_vT \subseteq (I_vT)_v = (IT)_v$.

(Note that the latter two v's are taken with respect to T.) Moreover, if $T = (T, M)$ is quasilocal and $(IT)_v$ is not principal in T, then

$$I_v = I_vT = (I_vT)_v = (IT)_v.$$

(2) *If $I^{-1} = (M : I)$ and $I(M : M) = J(M : M)$ for some invertible fractional ideal J of R with $J \subseteq I$, then $M = JI^{-1} = J(M : I)$.*

Proof. (1) If IT_M is not principal then $((IT)(T : IT))T_M \subseteq IT_M(T_M : IT_M) \subseteq MT_M$. Hence $II^{-1} \subseteq (IT)(T : IT) \subseteq M$ and we have $I^{-1} = (M : I)$. It follows that $(T : IT) = (M : IT) \subseteq (R : IT) \subseteq I^{-1} = (M : I) \subseteq (T : IT)$. Also, $I^{-1} \subseteq I^{-1}T \subseteq (T : IT)$. This proves (a). Now $IT \subseteq I_vT = (R : I^{-1})T \subseteq (T : I^{-1}T) = (T : (T : IT)) = (IT)_v$; hence $(IT)_v = (I_vT)_v$. On the other hand, $I \subseteq (M : (M : I)) = (M : (R : I)) \subseteq I_v$, so that $I_v = (M : (M : I))$. Thus $I_vT = (M : (M : I))T \subseteq (M : (M : I)T) \subseteq (M : (M : I)) = I_v$, and (b) follows.

Now assume that $T = (T, M)$ is quasilocal and that $(IT)_v$ is not principal. Then, since IT is also not principal, it suffices by (b) to show that $(IT)_v \subseteq I_v$. However, $(IT)_vI^{-1} = (IT)_v(T : IT) \subseteq M \subseteq R$, whence $(IT)_v \subseteq I_v$ as desired.

(2) Assume that $I^{-1} = (M : I)$ and that J is as stated. From $I(M : M) = J(M : M)$ we obtain $IM = JM \subseteq J$. Thus (since J is invertible) we have $MJ^{-1} \subseteq I^{-1}$, whence $M \subseteq JI^{-1} = J(M : I) \subseteq M$. It follows that $M = JI^{-1} = J(M : I)$. □

REMARK 2.8. (1) In Proposition 2.7, suppose that T is quasilocal, IR_M is not principal, and IT is principal. It then follows (since T is quasilocal) that $IT = yT$ for some $y \in I$ and, since $T \subseteq (M : M)$, we may apply Proposition 2.7(2) to conclude that $M = yI^{-1}$. We shall make use of this fact several times in the remainder of the paper.

(2) By Proposition 2.7(1)(a), if IT_M is not principal then $I^{-1} = I^{-1}T$, whence I^{-1} is not principal (as a fractional R-ideal). It follows in this case that I_v is also not principal.

(3) In Proposition 2.7(1), it is possible to have $I_v \subsetneqq (IT)_v$. Indeed, if $M = IT$ for a finitely generated ideal I of R, but M is not divisorial in T, then $I_v \subseteq M \subsetneqq T = (IT)_v$. (It can actually be shown that $I_v = M$.) For an explicit example, see Example 3.6(2) in the next section.

COROLLARY 2.9. *Consider a pullback diagram of type □, and assume that $T = (T, M)$ is quasilocal. Then each nonprincipal divisorial ideal of T is a divisorial ideal of R. If, in addition, $T = (M : M)$, then each divisorial ideal of T is divisorial in R.*

Proof. Let I be a nonprincipal divisorial ideal of T. Then (I is also an ideal of R and), by Proposition 2.7(1)(b), we have $I_v = (IT)_v = I$ (where the first v is taken with respect to R). If $T = (M : M)$ $(= M^{-1})$ and $I = yT$ is principal, then $I = yT = yM^{-1}$ is a principal multiple of the divisorial ideal M^{-1} and is therefore divisorial. □

COROLLARY 2.10. *Consider a pullback diagram of type* $\square$, *and assume that* $T = (T, M)$ *is quasilocal. Let* I *be a nonprincipal divisorial ideal of* R. *Then* $I = IT$ *in each of the following cases:*

(1) IT *is not principal in* T;
(2) D *is a field.*

Moreover, JT *is divisorial in* T *for each divisorial ideal* J *of* R *if and only if* M *is divisorial in* T.

Proof. If IT is not principal in T then, by Proposition 2.7(1)(b), we have $I = I_v = I_vT = IT$. For (2), we may assume that $IT = yT$ is principal. Then, since T and R are quasilocal, we may apply Remark 2.8(1) to obtain $M = yI^{-1}$. Inverting both sides and using the fact that I is divisorial in R, we obtain $M^{-1} = y^{-1}I$, or $I = yM^{-1} = y(M : M)$, which is an ideal of T.

Now assume that M is divisorial in T, and let J be a divisorial ideal of R. We may assume that JT is not principal in T. Then $(JT)_v(T : JT) \subseteq (JT(T : JT))_v \subseteq (MT)_v = M$. Hence $(JT)_v$ is also not principal, and by Proposition 2.7(1)(b) we conclude that $J = (JT)_v$ is divisorial in T. The converse is clear. $\square$

REMARK 2.11. In the case where $D = F$, Corollary 2.9 yields [B, Prop. 2.4] and Corollary 2.10 is similar to [B, Prop. 2.6]. If, in addition, T is a valuation domain, then much more can be said. Recall that a *pseudo-valuation domain* (PVD) is a quasilocal domain (R, M) for which M^{-1} is a valuation overring of R satisfying $\operatorname{Spec}(M^{-1}) = \operatorname{Spec}(R)$; according to [AD, Prop. 2.6], a domain R is a PVD if and only if R arises in a pullback diagram of type $\square$, where T is a valuation domain and $D = F$. In this case we have by [HH1, Thm. 2.13 and Prop. 2.14] that every ideal of T is a divisorial ideal of R, and if J is a nonprincipal ideal of R then $J_v = JT$. It follows that every nonzero ideal of R is divisorial if and only if every nonprincipal ideal of R is an ideal of T.

We end this section with a complete characterization of when M is v-finite in R in a pullback of type $\square$. It is convenient to separate the cases $k = F$ and $k \neq F$, since in the former case v-finiteness of M in R precludes invertibility of M in T (by Corollary 2.6). See Examples 3.6 for simple examples relating to Proposition 2.12 and 2.14.

PROPOSITION 2.12. *Consider a pullback diagram of type* $\square^*$. *The following statements are equivalent:*

(1) M *is* v*-finite in* R;
(2) $(T : M) = (T : JT)$ *for some finitely generated ideal* J *of* R *with* $J \subseteq M$, *and* M *is not invertible in* T;
(3) *either* M *is not a* t*-ideal of* T *or* M *is a* v*-finite divisorial ideal of* T *that is not invertible.*

Proof. (1) $\Rightarrow$ (3) Suppose that M is a t-ideal of T. Write $M = J_v$, with J a finitely generated ideal of R, $J \subseteq M$. Since T is flat over R [FG, Lemma 0.3], we have $M = J_v T \subseteq (J_v T)_v = (JT)_v \subseteq M$, whence $M = (JT)_v$. Thus M is a v-finite divisorial ideal of T and, by Corollary 2.6, M is not invertible in T.

(3) $\Rightarrow$ (2) Suppose that M is not a t-ideal of T. Then $1 \in I_v$ for some finitely generated ideal I of T with $I \subseteq M$, and we have $(T : M) = (T : I) = T$. Of course, we may write $I = JT$ for some finitely generated ideal J of R. The other case is straightforward.

(2) $\Rightarrow$ (1) $(T : M) = M^{-1}$ by Lemma 2.1. Thus $J^{-1} \subseteq (T : JT) = (T : M) = M^{-1} \subseteq J^{-1}$, whence $M^{-1} = J^{-1}$ and $M = J_v$. □

REMARK. Proposition 2.12 shows, in particular, that M is v-finite in R when M is finitely generated but not invertible in T.

LEMMA 2.13. *Consider a pullback diagram of type □, and assume that T is quasilocal and that $k \neq F$. If M is v-finite in R, then $(T : M) = (T : JT)$ for some finitely generated ideal J of R with $J \subseteq M$.*

Proof. Write $M = I_v$ with I finitely generated. Then $I^{-1} = M^{-1} = (M : M)$. We may assume that M is not invertible in T, whence $M^{-1} = (T : M)$. If IT is not principal then, by Proposition 2.7(1)(a), $(T : IT) = I^{-1} = M^{-1} = (T : M)$ as desired. Suppose that $IT = yT$ is principal. By Remark 2.8(1), we have $M = yI^{-1} = yM^{-1}$. Thus, since M is not principal in T, we have $M^{-1} \neq T$, and we may choose $x \in M^{-1} \setminus T$. It is easy to see that $M = (T : (1, x))$. Thus $M^{-1} = (T : M) = (T : (T : (1, x)))$, whence $M = yM^{-1} = (T : (T : (y, yx)))$, from which it follows that $(T : M) = (T : (y, yx))$. Hence we may take $J = (y, yx)$. □

PROPOSITION 2.14. *Consider a pullback diagram of type □, and assume that $k \neq F$. Then the following statements are equivalent*:

(1) *M is v-finite in R*;

(2) *$(T : M) = (T : JT)$ for some finitely generated ideal J of R with $J \subseteq M$*;

(3) *either M is not a t-ideal of T or M is a v-finite divisorial ideal of T.*

Proof. (1) $\Rightarrow$ (2) Split the diagram □ into two parts:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ S & \longrightarrow & F \\ \downarrow & & \downarrow \\ T & \longrightarrow & k \end{array}$$

(here $S = \phi^{-1}(F)$). Since M is divisorial (and therefore a t-ideal) in S, the implication (1) $\Rightarrow$ (3) of Proposition 2.12 guarantees that M is v-finite in S. Thus (for this part of the proof) we may assume that $D = F$. Write $M = J_v$, where J is finitely generated in R. We may as well assume that M is not invertible in T. Localizing at M, we obtain the following pullback diagram:

$$\begin{array}{ccc} R_M & \longrightarrow & F \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & k. \end{array}$$

Hence MR_M is divisorial in R_M. Since R_M is flat over R, we have $MR_M = J_vR_M \subseteq (JR_M)_v \subseteq MR_M$, whence $MR_M = (JR_M)_v$. By Lemma 2.13, we have $(T_M : MT_M) = (T_M : IT_M)$ for some finitely generated ideal I of R with $I \subseteq M$. Replacing J by $I + J$ if necessary, we may assume that $(T_M : MT_M) = (T_M : JT_M)$ (and we continue to have $M = J_v$). We claim that JT_M is not principal. Suppose, on the contrary, that $JT_M = yT_M$ for some y. Then we have $MT_M = yT_M$ also. By Lemma 2.1, $(R_M : MR_M) = T_M$, whence $M^{-1} = \bigcap\{ R_Q \mid Q \in \operatorname{Max} R, Q \neq M \} \cap (R_M : MR_M) = \bigcap\{ T_N \mid N \in \operatorname{Max} T, N \neq M \} \cap T_M = T$. Since by assumption M is not invertible in T, Proposition 2.2 ensures that M is not divisorial in T. Hence $M^{-1} = T = (T : M)$. However, this leads to a contradiction. Indeed, since J is finitely generated and $JT_M = yT_M$, there is an element $s \in T \setminus M$ with $sJ \subseteq yT = yM^{-1}$. Thus $sM = sJ_v \subseteq yM^{-1} = yT$ (since M^{-1} is divisorial), and we have $s/y \in (T : M) = T$. That is, $s \in yT \subseteq M$, a contradiction. Hence JT_M is not principal. Thus, by Proposition 2.7(1)(a), we obtain $(T : M) \subseteq (T : JT) = J^{-1} = M^{-1} \subseteq (T : M)$, and we have $(T : M) = (T : JT)$ as desired.

(2) $\Rightarrow$ (1) If M is invertible in T, then M is v-finite in R by Corollary 2.6. If M is not invertible, then we may proceed exactly as in the proof of (2) $\Rightarrow$ (1) in Proposition 2.12.

(3) $\Rightarrow$ (2) As in the proof of Proposition 2.12.

(2) $\Rightarrow$ (3) Straightforward. □

3. v-Coherent Pullbacks

According to [FG], a domain R is *v-coherent* if the intersection of each pair of v-finite ideals is again v-finite. It is easy to see [FG, Prop. 3.6] that this is equivalent to the condition that I^{-1} be v-finite for each finitely generated ideal I of R. Thus the class of v-coherent domains coincides with the class of domains satisfying property P^* studied in [N1], where it was observed that Mori domains, PVMDs, and (quasi)coherent domains are all v-coherent. It was also shown: that a polynomial ring over an integrally closed v-coherent domain is again v-coherent; that each localization of a v-coherent domain is v-coherent; and that, if R is semiquasilocal and R_M is v-coherent for each maximal ideal M, then R is v-coherent. In Proposition 3.1 (which we state without proof), we extend this last result slightly, but in Example 3.3 we show that in fact v-coherence is not a local property.

PROPOSITION 3.1. *Let R be a domain, and assume that the intersection $R = \bigcap\{ R_M \mid M$ is a maximal ideal of $R \}$ has finite character (i.e., that each element of R lies in only finitely many of the maximal ideals M). Then R is v-coherent if and only if each R_M is v-coherent.*

Recall that a domain R is a *v-domain* if, for each finitely generated ideal I of R, we have $(II^{-1})_v = R$. (We characterize v-domains in pullbacks in Section 4b.)

A v-domain in which I^{-1} is v-finite for each finitely generated ideal I is called a *Prüfer v-multiplication domain* (PVMD). The following result is clear.

PROPOSITION 3.2. *A domain is a PVMD if and only if it is a v-coherent v-domain.*

EXAMPLE 3.3. We supply an example of a non-v-coherent domain R such that R_M is v-coherent for each maximal ideal M. Let R be the example of [HO] of an essential domain (i.e., a domain that is an intersection of valuation overrings each of which is a localization) which is not a PVMD. In [MZ], it is shown that R_M is a PVMD (and therefore v-coherent) for each maximal ideal M. However, it is well known that an essential domain R is a v-domain. Hence, by Proposition 3.2, R cannot be v-coherent.

With respect to pullbacks, [N2, Thm. 3.1] shows that if $T = D + M$, T_M is a valuation domain, and $k = F$, then $R = k + M$ is v-coherent if and only if D and T are v-coherent. Moreover, [N2, Thm. 4.2] shows that a similar result holds in a pullback of type $\square^*$ when T is assumed to be a valuation domain. In [N2, Example 1] this is used to give an example of a v-coherent domain that is not Mori, not coherent, and not a PVMD.

The goal of (the remainder of) this section is to give a complete characterization of v-coherence in pullbacks of type $\square$—that is, to prove the following two theorems.

THEOREM 3.4. *Consider a pullback diagram of type $\square^*$. Then R is v-coherent if and only if D and T are v-coherent and M is a t-ideal of T.*

THEOREM 3.5. *Consider a pullback diagram of type $\square$, and assume that $k \neq F$. Then R is v-coherent if and only if D and T are v-coherent and either M is not a t-ideal of T or M is a v-finite divisorial ideal of T.*

Before proving Theorems 3.4 and 3.5, we present some simple examples that illustrate the various conditions described in these results (as well as in Propositions 2.12 and 2.14). We observe that the first example settles in the negative a conjecture of Anderson that appeared in [N1, p. 33].

EXAMPLES 3.6. (1) Let $T = \mathbb{Q}[X, Y]$, $M = (X, Y)T$, and $D = \mathbb{Z}$. Here, $k = \mathbb{Q}$ ($= F$) and M is finitely generated in T. Of course, M is not a t-ideal of T, whence (by Theorem 3.4) $R = \mathbb{Z} + M$ is not v-coherent. Note also that, by Proposition 2.12, M is v-finite in R.

(2) Let $T = \mathbb{R}[X, Y]$, $M = (X, Y)T$, and $D = \mathbb{Q}$. In this case, $k = \mathbb{R} \supsetneq F = \mathbb{Q}$, and M is finitely generated in T. By Theorem 3.5, $R = \mathbb{Q} + M$ is v-coherent. Again, M is not a t-ideal of T, but M is v-finite in R.

(3) Let $T = (V, M)$ be a valuation domain. Then M is automatically a t-ideal of T. Note that M cannot satisfy condition (3) of Proposition 2.12 and that M satisfies condition (3) of Proposition 2.14 if and only if M is principal in T. Let $D \subseteq k = V/M$ be a v-coherent domain. Then:

(a) if $k = F$, then R is v-coherent; and
(b) if $k \neq F$, then R is v-coherent if and only if M is principal in T.

(See Corollaries 3.9 and 3.13.)

We need two more results before we prove Theorem 3.4.

PROPOSITION 3.7. *Consider a diagram of type* $\square$ *and the following conditions*:

(1) *for each* t*-ideal* I *of* R, IT *is a* t*-ideal of* T;
(2) M *is a* t*-ideal of* T;
(3) *for each finitely generated ideal* J *of* R *such that* JT_M *is not principal, we have* $J_v = J_vT = (JT)_v$.

If T *is* v*-coherent, then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If, in addition,* T *is quasilocal, then* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) This follows from the fact that M is divisorial (and hence a t-ideal) in R.

(2) $\Rightarrow$ (3) If J is as given in (3) then, by Proposition 2.7(1)(b), $J_v = J_vT \subseteq (JT)_v$. Hence in this case it suffices to show that $(JT)_v \subseteq J_v$. Since T is v-coherent, $(T : JT)$ is v-finite in T. Hence $(JT(T : JT))_v$ is also v-finite. By Proposition 2.7(1)(a), we have $J^{-1} = (T : JT) = (M : JT)$. Since M is a t-ideal of T, this yields $(JT(T : JT))_v = (JT(M : JT))_v \subseteq M$, whence $(JT)_vJ^{-1} = (JT)_v(T : JT) \subseteq (JT(T : JT))_v \subseteq M \subseteq R$, from which it follows that $(JT)_v \subseteq J_v$, as desired.

Now assume (3) and that T is quasilocal, and let I be a t-ideal of R and H a nonprincipal finitely generated ideal of T with $H \subseteq IT$. We wish to show that $H_v \subseteq IT$. Write $H = JT$ with J a finitely generated ideal of R, $J \subseteq I$. Then $J_v \subseteq I$, and so $H_v = (JT)_v = J_vT \subseteq IT$. $\square$

PROPOSITION 3.8. *Let* R *be a* v*-coherent domain, and let* T *be a flat overring of* R. *Then*:

(1) *for each* t*-ideal* I *of* R, IT *is a* t*-ideal of* T; *and*
(2) T *is* v*-coherent.

Proof. (1) Let H be a finitely generated ideal of T with $H \subseteq IT$, and write $H = JT$ for some finitely generated ideal J of R, $J \subseteq I$. Since R is v-coherent, J^{-1} is v-finite and, since T is R-flat, we have $H_v = (JT)_v = J_vT \subseteq IT$.

(2) Again, let $H = JT$, where J is a finitely generated ideal of R. By flatness, we have $(T : H) = (T : JT) = J^{-1}T$. Since R is v-coherent, $J^{-1} = A_v$ for some finitely generated fractional ideal A of R, whence $(T : H) = J^{-1}T = A_vT = (AT)_v$ and H is v-finite. $\square$

Proof of Theorem 3.4. ($\Rightarrow$) Let I be a finitely generated ideal of D. Then $\phi^{-1}(I)$ is finitely generated in R [FG, Cor. 1.7(b)], and so $(\phi^{-1}(I))^{-1}$ is v-finite, say $(\phi^{-1}(I))^{-1} = J_v$ with J finitely generated. Note that $J_v = (\phi^{-1}(I))^{-1} = \phi^{-1}(I^{-1}) \subseteq T$ by [FG, Prop. 1.8(a1)]. Since $M \subsetneq \phi^{-1}(I) \subseteq R \subsetneq T$, it follows that $M \subsetneq R \subseteq J_v \subsetneq T$, and we have $I^{-1} = \phi((\phi^{-1}(I))^{-1}) = \phi(J_v) =$

$\phi(J)_v$ by [FG, Prop. 1.8(b2)]. Hence I^{-1} is v-finite. This shows that D is v-coherent. That T is v-coherent and M is a t-ideal of T follows from Proposition 3.8, in view of the fact that T is R-flat when $k = F$ [FG, Lemma 0.3].

($\Leftarrow$) Let I be a finitely generated ideal of R. If $I \not\subseteq M$, then $\phi(I)$ is a nonzero ideal of D. Since D and T are v-coherent, $(T : IT)$ and $(D : \phi(I))$ are v-finite. Hence I^{-1} is v-finite by [FG, Lemma 1.12]. Now assume that $I \subseteq M$. Since $k = F$, we have $R_M = T_M$. If $IT_M = IR_M$ is principal, then for some $x \in K$ we have $xI \subseteq R$ and $xI \not\subseteq M$. From what has already been proved, $(xI)^{-1} = x^{-1}I^{-1}$ is v-finite, from which it follows that I^{-1} is also v-finite. Thus we assume that IT_M is not principal. In this case, Proposition 2.7(1) yields $I^{-1} = I^{-1}T = (T : IT)$. Since T is v-coherent, we can write $(T : IT) = (JT)_v$ for some finitely generated ideal J of R with $J \subseteq I^{-1}$. We wish to show that $I^{-1} = J_v$. For this it suffices, by Proposition 3.7, to show that JT_M is not principal. Since T_M is T-flat (and T is v-coherent), we have $(JT_M)_v = ((JT)_v T_M)_v = ((T : IT)T_M)_v = (T_M : IT_M)$. Since IT_M is not invertible, this yields $IJT_M \subseteq IT_M(JT_M)_v = IT_M(T_M : IT_M) \subseteq MT_M$. By Proposition 3.8, MT_M is a t-ideal of T_M, whence $(IJT_M)_v \subseteq MT_M$. It follows that $(T_M : IT_M) = (JT_M)_v$ is not principal and therefore that JT_M is not principal. □

The semiquasilocal case of the following corollary appears in [N1, Cor. 2.11].

COROLLARY 3.9. *Consider a pullback diagram of type □*, and assume that T_M is a valuation domain. Then R is v-coherent if and only if D and T are v-coherent.*

Proof. Since T_M is a valuation domain, MT_M is a t-ideal, and it is well known that this implies that M is a t-ideal of T. The result now follows from Theorem 3.4. □

We prove Theorem 3.5 in stages. First, we state Proposition 3.10 as a convenience. Then, in Proposition 3.11, we prove Theorem 3.5 in the case where $D = F$ and T is quasilocal; in Proposition 3.12, we remove the quasilocal assumption. Recall that the condition given on M is equivalent to v-finiteness of M in R (Proposition 2.14).

PROPOSITION 3.10. *Consider a pullback diagram of type □. If R is v-coherent and $k \neq F$, then M is v-finite in R.*

Proof. By Proposition 2.4, $M = (1, x)^{-1}$ for some $x \in T$, and v-coherence of R implies that M is v-finite. □

PROPOSITION 3.11. *Consider a pullback diagram of type □, and assume that $D = F$ and that T is quasilocal. Then R is v-coherent if and only if T is v-coherent and M is v-finite in R.*

Proof. ($\Rightarrow$) Note that M is v-finite in R by Proposition 3.10. Let L be a nonprincipal finitely generated ideal of T, and write $L = JT$ with J a (nonprincipal) finitely generated ideal of R. Since R is v-coherent, $J^{-1} = H_v$ with H finitely generated in R. By Proposition 2.7(1), we have $H_v = J^{-1} = J^{-1}T = (T : L) = H_vT$. If

HT is not principal, we also have $H_vT \subseteq (H_vT)_v = (HT)_v$; since $(T:L)$ is divisorial, this yields $(T:L) = (HT)_v$, which is v-finite. Suppose that $HT = yT$ is principal. By Remark 2.8(2), $J^{-1} = H_v$ is not principal as a fractional ideal of R, whence H is also not principal. By Remark 2.8(1), $M = yH^{-1}$, whence $M^{-1} = y^{-1}H_v$. If $M^{-1} = T$, then $(T:L) = J^{-1} = H_v = yT$, and we are done. If $M^{-1} \supsetneqq T$ then, by Lemma 2.1, $M^{-1} = (T:M)$. Thus $(T:L) = J^{-1} = H_v = yM^{-1} = y(T:M)$, and it now suffices to show that $M^{-1} = (T:M)$ is v-finite in T. Pick $x \in (T:M) \setminus T$. Then it is easy to see that $M = (T:(1,x))$, and we have $(T:M) = (1,x)_v$ (the v-operation being taken with respect to T), as desired.

($\Leftarrow$) Let J be a nonprincipal finitely generated ideal of R. If $JT = yT$ is principal then, by Remark 2.8(1), $M = yJ^{-1}$; since M is v-finite in R, J^{-1} is also v-finite. If JT is not principal then, by Proposition 2.7(1)(a), $J^{-1} = J^{-1}T = (T:JT)$. Since T is v-coherent, $(T:JT) = (HT)_v$ for some finitely generated ideal H of R with $H \subseteq J^{-1}$. If $(HT)_v$ is not principal, then (HT is also not principal and) $(HT)_v = H_v$ by Proposition 2.7(1)(b), in which case $J^{-1} = H_v$ is v-finite. If $(HT)_v = J^{-1} = zT$ is principal, then $J_v = z^{-1}(R:T) = z^{-1}M$, whence $J^{-1} = zM^{-1}$. By Proposition 2.4, $M^{-1} = (1,x)_v$ for some $x \in T \setminus R$. Thus J^{-1} is v-finite also. □

PROPOSITION 3.12. *Consider a pullback diagram of type* □, *and assume that $D = F$. Then R is v-coherent if and only if T is v-coherent and M is v-finite in R.*

Proof. ($\Rightarrow$) M is v-finite in R by Proposition 3.10. To show that T is v-coherent, it suffices to show that $(T:IT)$ is v-finite, where I is a finitely generated ideal of R. Use v-coherence of R to write $I^{-1} = J_v$ with J finitely generated (and $J \subseteq I^{-1}$). Localize the diagram □ at M to obtain the following pullback diagram:

$$\begin{array}{ccc} R_M & \longrightarrow & F \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & k. \end{array}$$

Since R_M is v-coherent by Proposition 3.8, Proposition 3.11 guarantees v-coherence of T_M. Hence $(T_M:IT_M) = (T:I)T_M$ is v-finite, and we can write $(T_M:IT_M) = (HT_M)_v$ for some finitely generated ideal H of T, $H \subseteq (T:IT)$. Let $L = H + JT$; we shall show that $(T:IT) = L_v$. It is clear that $L_v \subseteq (T:IT)$. For the reverse inclusion, it suffices to show that $(T:IT)(T:L) \subseteq T_N$ for each maximal ideal N of T. Recall that, for $N \neq M$ and $Q = N \cap R$, we have $T_N = R_Q$. Hence in this case $(T:IT)(T:L) \subseteq (T:IT)(T:JT) \subseteq (T_N:IT_N)(T_N:JT_N) = (R_Q:IR_Q)(R_Q:JR_Q) = I^{-1}J^{-1}R_Q = J_vJ^{-1}R_Q \subseteq R_Q = T_N$. When $N = M$, we have $(T:IT)(T:L) \subseteq (T:IT)(T:H) \subseteq (T_M:IT_M)(T_M:HT_M) = (HT_M)_v(T_M:HT_M) \subseteq T_M$. Hence $(T:IT) \subseteq L_v$ as desired.

($\Leftarrow$) Now assume that T is v-coherent and that M is v-finite in R. Since T_M is v-coherent (Proposition 3.8), we have that R_M is v-coherent by Proposition 3.11. Let I be a finitely generated ideal of R. Then $(IR_M)^{-1} = I^{-1}R_M$ is v-finite, and we can write $(IR_M)^{-1} = (JR_M)_v$ for some finitely generated ideal J of R with

$J \subseteq I^{-1}$. We also have $(T:IT) = (HT)_v$ with H finitely generated in R. Note that, since $H \subseteq (T:IT)$, we have $HIM = HIMT \subseteq MT = M \subseteq R$, so that $HM \subseteq I^{-1}$. Set $L = J + HM$; we claim that $I^{-1} = L_v$. Clearly, $I^{-1} \supseteq L_v$. For the reverse inclusion, we show that $I^{-1}L^{-1} \subseteq R_Q$ for each maximal ideal Q of R. For $Q \neq M$, there is a unique maximal ideal N of T with $N \cap R = Q$ and $R_Q = T_N$. For such Q, we have $I^{-1}L^{-1} \subseteq I^{-1}(HM)^{-1} \subseteq (IR_Q)^{-1}(HR_Q)^{-1} = (T_N:IT_N)(T_N:HT_N) = (T:IT)(T:HT)T_N = (HT)_v(T:H)T_N \subseteq T_N = R_Q$. We also have $I^{-1}L^{-1} \subseteq I^{-1}J^{-1} \subseteq (IR_M)^{-1}(JR_M)^{-1} = (JR_M)_v(JR_M)^{-1} \subseteq R_M$. Thus $I^{-1} = L_v$, as claimed. We must now show that L_v is v-finite. Since M is v-finite, we have $M = A_v$ for some finitely generated ideal A of R. Hence $(HM)_v = (HA_v)_v = (HA)_v$, and we have $L_v = (J + HM)_v = (J + (HM)_v)_v = (J + (HA)_v)_v = (J + HA)_v$, which is clearly v-finite. □

Proof of Theorem 3.5. By Proposition 2.14, it suffices to prove that R is v-coherent if and only if D and T are v-coherent and M is v-finite in R. Split the diagram into two parts (as in the proof of Proposition 2.14):

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ S & \longrightarrow & F \\ \downarrow & & \downarrow \\ T & \longrightarrow & k. \end{array}$$

Now suppose that R is v-coherent. Then M is v-finite in R by Proposition 3.10, whence M is v-finite in S by Proposition 2.12. Theorem 3.4 now ensures that D and S are v-coherent, whence T is also v-coherent by Proposition 3.12. For the converse, we again have M v-finite in S, and v-coherence of T implies that S is also v-coherent by Proposition 3.12. Finally, since M is divisorial in S, v-coherence of D implies v-coherence of R by Theorem 3.4. □

COROLLARY 3.13. *Consider a pullback diagram of type □. Assume that T is a valuation domain and that $k \neq F$. Then R is v-coherent if and only if D is v-coherent and M is principal in T.*

Proof. This follows immediately from Theorem 3.5 since, in a valuation domain T, the condition "M is not a t-ideal of T or M is a v-finite divisorial ideal of T" is clearly equivalent to principality of M. □

COROLLARY 3.14. *Let R be a PVD; that is, in Corollary 3.13, assume that $D = F$ (see Remark 2.11). Then the following conditions are equivalent:*

(i) *R is v-coherent;*
(ii) *M is principal in $T = M^{-1}$;*
(iii) *M is divisorial in T;*
(iv) *each nonzero ideal of T is divisorial in T;*

(v) $(J_vT)_v = J_v$ *for each nonprincipal ideal J of R (where the second v is taken with respect to T).*

Proof. The equivalence of (i) and (ii) follows from Corollary 3.13. The equivalence of (ii) and (iv) is [H, Lemma 5.2], and the equivalence of (ii) and (iii) is well known (and easy to show). Finally, the equivalence of (iv) and (v) follows from Remark 2.11. □

4. Other Coherentlike Conditions

In this section we apply the techniques and results already developed to the study of several other coherentlike properties. We begin with definitions of the terms not yet defined.

DEFINITION. A domain R is:

(1) *coherent* if the intersection of each pair of finitely generated ideals of R is again finitely generated or, equivalently, if each finitely generated ideal of R is finitely presented;
(2) *quasicoherent* if I^{-1} is finitely generated for each finitely generated ideal I of R;
(3) a *finite conductor domain* if each conductor to R of an element from K is finitely generated or, equivalently, if $Ra \cap Rb$ is finitely generated for each $a, b \in R$;
(4) a *Mori domain* if R satisfies the ascending chain condition on divisorial ideals or, equivalently, if for each ideal I of R we have $I^{-1} = J^{-1}$ for some finitely generated ideal J of R with $J \subseteq I$ [Q1, Thm. 1];
(5) a *DVF domain* if each divisorial ideal of R is v-finite.

We have the following implications:

$$
\begin{tikzcd}
\text{coherent} \arrow[r] & \text{quasicoherent} \arrow[r] \arrow[d] & \text{finite conductor} \\
\text{PVMD} \arrow[r] \arrow[d] & v\text{-coherent} & \\
v\text{-domain} & \text{DVF} \arrow[u] & \\
& \text{Mori}\,. \arrow[u] &
\end{tikzcd}
$$

4a. Coherent, Quasicoherent, and Finite Conductor Pullbacks

Of course, coherence has been studied extensively. For our purposes, we cite the paper by Greenberg [Gre]; the paper by Dobbs and Papick [DP], in which coherence was characterized in the classical $D + M$ construction; the paper by Brewer and Rutter [BR], in which coherence and several other properties were characterized in the general $D + M$ construction; and the book by Glaz [Gl].

We shall characterize the three properties in the title of this subsection in pullback constructions of type $\square$. (For the case of coherence, our result is an extension of [BR, Thm. 3]; our proof is entirely ideal-theoretic in spirit.) To our knowledge, there are no known examples which prove that these three properties are distinct, and one consequence of our results is that pullbacks of type $\square$ cannot produce such examples.

The nonreversibility of most of the other implications in the above diagram is discussed in what follows. We recall that [HH2, Thm. 1.6] shows that, in the notation of Corollary 3.14, a PVD R is coherent if and only if T is a finitely generated R-module and $M \neq M^2$ (i.e., M is principal) if and only if R is quasicoherent. Since finite generation of T over R is easy to avoid, we see that v-coherence of R is not equivalent to (quasi)coherence of R. For an explicit example, let $T = \mathbb{R}[[X]]$, $M = X\mathbb{R}[[X]]$, and $R = \mathbb{Q}+M$. Then R is v-coherent by Corollary 3.14 but R is not coherent, since T is not a finitely generated R-module (since $\mathbb{R}$ is not finitely generated over $\mathbb{Q}$).

Lemma 4.1 is a straightforward extension of [BR, Lemma 1]. Proposition 4.2 is stated for convenience; it is probably well known, but we do not know a reference. Proposition 4.3 is an extension of [BR, Prop. 2]; we include a proof of part of it.

LEMMA 4.1. *Consider a pullback diagram of type $\square$. If there exists a nonzero ideal of T that is finitely generated as an R-module, then $D = F$ and $[k : F] < \infty$.*

PROPOSITION 4.2. *Consider a pullback diagram of type $\square$, and assume that $F \neq k$. Then the following statements are equivalent:*

(1) *M is finitely generated in R;*
(2) *T is a finitely generated R-module, and M is finitely generated in T;*
(3) *$D = F$, $[k : F] < \infty$, and M is finitely generated in T.*

Proof. (1) $\Rightarrow$ (3) Lemma 4.1.

(3) $\Rightarrow$ (2) Since $[k : F] < \infty$, we can write $k = F + F\alpha_2 + \cdots + F\alpha_n$, with $\alpha_i \in k$ for $i = 2, \ldots, n$. For each i, pick $t_i \in T$ with $\alpha_i = \phi(t_i)$. Then for $t \in T$ we can write $t = r_1 + r_2t_2 + \cdots + r_nt_n + a$, where $r_i \in R$ for each i and $a \in M$. Hence $t = (r_1 + a) + r_2t_2 + \cdots + r_nt_n$. Thus $T = R + Rt_2 + \cdots + Rt_n$ is a finitely generated R-module.

(2) $\Rightarrow$ (1) Clear. $\square$

PROPOSITION 4.3. *Consider a pullback diagram of type $\square$. If R is a finite conductor domain, then exactly one of the following conditions holds:*

(1) *$D = F$, $[k : F] < \infty$, and M is finitely generated in T; or*
(2) *$k = F$ and T_M is a valuation domain.*

Proof. If $k \neq F$, then by Proposition 2.4 we have $xM = xR \cap R$ for some $x \in T \setminus R$. If R is a finite conductor domain, it follows that xM—and hence M also—is finitely generated in R. By Lemma 4.1, $D = F$ and $[k : F] < \infty$. If $k = F$, we may proceed as in the proof of [BR, Prop. 2] to conclude that T_M is a valuation domain. □

We proceed to characterize coherent, quasicoherent, and finite conductor pullbacks when $k = F$.

PROPOSITION 4.4. *Consider a pullback diagram of type* $\square^*$. *If R is quasicoherent* (*coherent*), *then D and T are quasicoherent* (*coherent*) *and T_M is a valuation domain.*

Proof. Proposition 4.3 guarantees that T_M is a valuation domain. Suppose that R is quasicoherent, and let I be a finitely generated ideal of D. Then $\phi^{-1}(I)$ is finitely generated in R [FG, Cor. 1.7(b)], and so $J = (\phi^{-1}(I))^{-1}$ is also finitely generated. By [FG, Prop. 1.8(a1)], $J = \phi^{-1}(D : I)$, whence $(D : I) = \phi(J)$ is finitely generated in D. Thus D is quasicoherent. Now suppose that R is coherent, and let H, I be finitely generated ideals of D. Then $\phi^{-1}(H)$ and $\phi^{-1}(I)$ are finitely generated in R, whence $\phi^{-1}(H) \cap \phi^{-1}(I)$ is also finitely generated. Since $M \subseteq \phi^{-1}(I)$, we have by [FG, Prop. 1.6(b)] that $H \cap I = \phi(\phi^{-1}(H) \cap \phi^{-1}(I))$ is finitely generated in D. Thus D is coherent.

Now let A, B be finitely generated ideals in T, and write $A = IT$, $B = JT$ with I, J finitely generated ideals of R. Since T is flat over R, we have $I^{-1}T = (T : IT) = (T : A)$ and $(I \cap J)T = IT \cap JT = A \cap B$. It follows that quasicoherence or coherence of R implies the corresponding property for T. □

It is convenient to record the following (easily proved) result.

LEMMA 4.5 (cf. [BAD, Prop. 2.1]). *A domain R is quasicoherent if and only if R is v-coherent and each v-finite divisorial ideal of R is finitely generated.*

PROPOSITION 4.6. *Consider a pullback diagram of type* $\square^*$, *and assume that T is a valuation domain. Then R is quasicoherent* (*coherent*) *if and only if D is quasicoherent* (*coherent*).

Proof. The assumption that D has either property implies that R is v-coherent, by Corollary 3.9. Suppose that D is quasicoherent, and let J be a finitely generated ideal of R. We claim that J_v is finitely generated. Since $T = R_M$ and JT is principal, we may choose $x \in K$ with $xJ \subseteq R$ and $xJ \nsubseteq M$. Thus $(xJ)_v \nsubseteq M$, whence $(xJ)_v \supseteq M$, and by [FG, Prop. 1.8(b2)] we have $\phi((xJ)_v) = \phi(xJ)_v$. Since D is quasicoherent, $\phi((xJ)_v)$ is finitely generated, whence $xJ_v = (xJ)_v = \phi^{-1}(\phi((xJ)_v))$ is also finitely generated [FG, Cor. 1.7(b)]. It follows that J_v is finitely generated. By Lemma 4.5, R is quasicoherent.

Now assume that D is coherent, and let I, J be finitely generated ideals of R. By flatness, $(I \cap J)T = IT \cap JT$, whence $(I \cap J)T = zT$ for some $z \in T$. Thus $z^{-1}(I \cap J)T = T$, and by [FG, Prop. 1.1] we have $z^{-1}(I \cap J) \supsetneq M$. Hence by [FG, Prop. 1.6(b)], $\phi(z^{-1}(I \cap J)) = \phi(z^{-1}I \cap z^{-1}J) = \phi(z^{-1}I) \cap \phi(z^{-1}J)$. Coherence of D now implies that $\phi(z^{-1}(I \cap J))$ is finitely generated, whence

$z^{-1}(I \cap J) = \phi^{-1}(\phi(z^{-1}(I \cap J)))$ is finitely generated in R [FG, Cor. 1.7(b)]. This completes the proof in this direction. The converse follows from Proposition 4.4. □

To handle the nonquasilocal case, we recall the notion of the complete preimage extension of R with respect to M [BS]; this is the domain $R(M)$ defined by the following pullback diagram:

$$\begin{array}{ccc} R(M) & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_M = T_M & \xrightarrow{\phi} & k = F \approx T_M/MT_M. \end{array}$$

(We continue to use ϕ for the canonical projection.) For each ideal I of R, we have $I = IT \cap IR(M)$ [FG, Lemma 1.3]. Moreover, $R(M)$ is flat over R, since $R(M)_N = R_{N \cap R}$ for each $N \in \mathrm{Max}(R(M))$.

THEOREM 4.7. *Consider a pullback diagram of type* $\square^*$. *Then R is quasicoherent (coherent) if and only if D, T are quasicoherent (coherent) and T_M is a valuation domain.*

Proof. ($\Rightarrow$) This follows from Proposition 4.4.

($\Leftarrow$) $R(M)$ is quasicoherent (coherent) by Proposition 4.6, and R is v-coherent by Theorem 3.4. For the quasicoherent case, let J be finitely generated in R. By flatness (and since J^{-1} is v-finite), $J_vR(M) = (JR(M))_v$ is v-finite and therefore finitely generated in $R(M)$; likewise, $J_vT = (JT)_v$ is finitely generated in T. Set $J_vR(M) = J_1R(M)$ and $J_vT = J_2T$, where J_1, J_2 are finitely generated ideals of R (which are contained in J_v), and let $H = J_1 + J_2$. Then $H \subseteq J_v = J_vR(M) \cap J_vT = J_1R(M) \cap J_2T \subseteq HR(M) \cap HT = H$. Hence $H = J_v$ is finitely generated. By Lemma 4.5, R is quasicoherent. For the coherent case, note that $R(M)$ is coherent by Proposition 4.6. Let I, J be finitely generated in R, and set $H = I \cap J$. Again by flatness, we use coherent to see that $HR(M)$ and HT are finitely generated, from which we conclude as before that H is finitely generated. □

For the generalized $D + M$ construction, the coherent case of Theorem 4.7 appears in [BR, Thm. 3(ii)]; for the classical $D + M$ construction, it appears in [DP, Thm. 3(1)].

We wish to prove a result similar to Theorem 4.7 for the finite conductor condition. In one direction, we can proceed as in the (quasi)coherent case, but the other direction requires more care.

THEOREM 4.8. *Consider a pullback diagram of type* $\square^*$. *Then R is a finite conductor domain if and only if D, T are finite conductor domains and T_M is a valuation domain.*

Proof. ($\Rightarrow$) By flatness, $R(M)$ and T are finite conductor domains, and T_M is a valuation domain by Proposition 4.3. Now, by [FG, Thm. 2.3 and Prop. 2.9],

$\phi^{-1}(xD)$ is principal in $R(M)$ for each nonzero element $x \in k$. It is then easy to show that D is a finite conductor domain by proceeding as in the coherent case of Proposition 4.4.

($\Leftarrow$) A proof similar to that of the coherent case of Proposition 4.6 shows that $R(M)$ is a finite conductor domain. The proof can then be completed as in the coherent case of Theorem 4.7. $\square$

We now turn to the case of a pullback of type $\square$ with $k \neq F$.

THEOREM 4.9. *Consider a pullback diagram of type $\square$ with $k \neq F$. Then R is quasicoherent if and only if T is quasicoherent, $D = F$, $[k : F] < \infty$, and M is finitely generated in T.*

Proof. ($\Rightarrow$) By Proposition 4.3, we need only prove that T is quasicoherent. Accordingly, let $L = JT$ be a nonprincipal ideal of T, with J finitely generated in R. If JT_M is not principal then, by Proposition 2.7(1)(a), $(T : L) = J^{-1} = J^{-1}T$, and since J^{-1} is finitely generated, so is $(T : L)$. Suppose that JT_M is principal. Then we may pick $z \in (T : JT)$ so that z generates $(T_M : JT_M) = (T : JT)T_M$. We claim that (J^{-1}, z) generates $(T : JT)$. It suffices to check this locally. If N is a maximal ideal of T with $N \neq M$, then $T_N = R_{N\cap R}$, whence $(T_N : JT_N) = (R_{N\cap R} : JR_{N\cap R}) = J^{-1}R_{N\cap R} = J^{-1}T_N$. Because z generates at M, this proves the claim. Hence T is quasicoherent.

($\Leftarrow$) We proceed as in the proofs of Propositions 3.11 and 3.12. First, suppose that T and (hence) R are quasilocal. Let I be finitely generated (and nonprincipal) in R. If IT is not principal then, by Proposition 2.7(1), $I^{-1} = (T : IT) = HT$ for some finitely generated ideal $H \subseteq I$ (since T is quasicoherent). By Proposition 4.2, T is a finitely generated R-module, whence I^{-1} is finitely generated in this case. If IT is principal then, by Remark 2.8(1), $M = yI^{-1}$ for some $y \in I$, whence (again) $I^{-1} = y^{-1}M$ is finitely generated. To handle the general (non-quasilocal) case, again let I be finitely generated in R. Since T_M is quasicoherent (by flatness), we can apply the local case to conclude that $I^{-1}R_M = (IR_M)^{-1}$ is finitely generated, say $I^{-1}R_M = JR_M$ with $J \subseteq I^{-1}$, J finitely generated. Since T is quasicoherent, $(T : IT) = HT$ for some finitely generated ideal H of R. It now follows by an argument similar to (but easier than) that used in the proof of Proposition 3.12 that $I^{-1} = J + HM$, whence I^{-1} is finitely generated. Hence R is quasicoherent. $\square$

THEOREM 4.10. *Consider a pullback diagram of type $\square$, and assume that $k \neq F$. Then R is a finite conductor domain if and only if T is a finite conductor domain, $D = F$, $[k : F] < \infty$, and M is finitely generated in T.*

Proof. ($\Rightarrow$) According to Proposition 4.3, it is enough to prove that T is a finite conductor domain, and for this it suffices to show that $(T : (1, x))$ is finitely generated for each $x \in K \setminus T$. We first assume that T is quasilocal, in which case we may as well assume that $(T : (1, x)) \subseteq M$. If $tx \notin M$ for some $t \in (T : (1, x))$, then $x^{-1} = t(tx)^{-1} \in M \subseteq T$ and we have that $(T : (1, x)) = Tx^{-1}$, a principal ideal. Hence we may assume that $x(T:(1,x)) \subseteq M$, from which it follows that $(T:(1,x)) = (R:(1,x))$, whence, again, $(T:(1,x))$ is finitely generated. For general T, we use the fact that conductors localize well, together with what has already been proved, to conclude that $(T:(1,x))T_M = (T_M:(1,x))$ is finitely generated, say $(T:(1,x))T_M = AT_M$, where A is a finitely generated ideal of T with $A \subseteq (T:(1,x))$. For N maximal in T with $N \neq M$ and $Q = N \cap R$, we have $(T:(1,x))T_N = (T_N:(1,x)) = (R_Q:(1,x)) = (R:(1,x))R_Q = (R:(1,x))T_N$. It follows that $(T:(1,x)) = A + (R:(1,x))T$, and since $(R:(1,x))$ is finitely generated in R, this completes the proof in this direction.

($\Leftarrow$) Let $x \in K \setminus R$. We shall show that $(R:(1,x))$ is finitely generated. If $x \in T$ then, by Proposition 2.4 and the remark that follows it, $(R:(1,x)) = M$, which is finitely generated by Proposition 4.2. Hence we may assume that $x \notin T$. Again, we begin by supposing that T is quasilocal. Since $x \notin T$, we have that $(T:(1,x)) \subseteq M$. If $x(T:(1,x)) \subseteq M$ then, as above, we have $(R:(1,x)) = (T:(1,x))$, and finite generation of $(R:(1,x))$ follows from the fact that T is a finitely generated R-module (Proposition 4.2). If $x(T:(1,x)) \nsubseteq M$ then, as before, $x^{-1} \in M \subseteq R$ and so $(R:(1,x)) = Rx^{-1}$ is principal. For general T, we have $(R:(1,x))R_M = (R_M:(1,x))$ finitely generated by the local case, say $(R:(1,x))R_M = JR_M$ for some finitely generated ideal J of R, $J \subseteq (R:(1,x))$. With the notation above, we have $(R:(1,x))R_Q = (R_Q:(1,x)) = (T_N:(1,x)) = (T:(1,x))T_N = (T:(1,x))R_Q$. Now $(T:(1,x))$ is finitely generated as a T-module and hence also as an R-module. Thus, since M is finitely generated in R, $M(T:(1,x))$ is also finitely generated, and it is clear that $M(T:(1,x)) \subseteq (R:(1,x))$. Finally, since $M(T:(1,x))$ and $(T:(1,x))$ agree at each (maximal ideal) Q (of R with $Q \neq M$), we can use the usual localization argument to show that $(R:(1,x)) = M(T:(1,x)) + J$, which is finitely generated. □

THEOREM 4.11. *Consider a pullback diagram □ with $k \neq F$. Then R is coherent if and only if T is coherent, $D = F$, $[k:F] < \infty$, and M is finitely generated in T.*

Proof. By Proposition 4.3, it suffices to show that T is coherent. Let IT and JT be (finitely generated) ideals in T, with I, J finitely generated in R. Since T is a finitely generated R-module, IT and JT are both finitely generated over R and therefore, since R is coherent, $IT \cap JT$ is also finitely generated over R. Of course, this implies that $IT \cap JT$ is finitely generated in T, as well.

For the converse, we first assume that T is quasilocal. Let I and J be finitely generated in R. Then $IT \cap JT$ is a finitely generated ideal of T, and since T is a finitely generated R-module (Proposition 4.2), $IT \cap JT$ is also a finitely generated ideal of R. Suppose that $IT \cap JT$ can be generated by n elements, and among all generating sets of size n, let p denote the largest number of generators that can lie inside $I \cap J$. We may assume that $p < n$; otherwise, $I \cap J = IT \cap JT$ and there is nothing to prove. Let $a_1, \ldots, a_n$ generate $IT \cap JT$ as an R-module, where $a_1, \ldots, a_p \in I \cap J$. Note that $IM \cap JM$ is finitely generated in (T and hence in) R. Thus it suffices to show that $I \cap J = Ra_1 + \cdots + Ra_p + (IM \cap JM)$. (If $p = 0$, this equation is to be interpreted as $I \cap J = (IM \cap JM)$.) One inclusion is obvious. Let $x \in I \cap J$, and write $x = r_1a_1 + \cdots + r_na_n$ with $r_i \in R$. If for some $i > p$ we have $r_i \notin M$, then a_i can be replaced by x in the generating set, contradicting the maximality of p. Thus, for $i > p$, we have $r_ia_i \in M(IT \cap JT) \subseteq IM \cap JM$. This completes the proof in the local case. For general T, use the local case to find a finitely generated subideal A of $I \cap J$ that generates at M. For Q a maximal ideal of R with $Q \neq M$, the usual argument shows that $IM \cap JM$ (which is a finitely generated ideal of T and hence of R) generates at R_Q. It follows that $I \cap J = A + (IM \cap JM)$. □

REMARK. Theorems 4.9, 4.10, and 4.11 generalize the characterization of coherent PVDs given in [HH2, Thm. 1.6].

We end this subsection by applying our techniques to characterize the Noetherian condition in pullbacks of type □ (cf. [BR, Thm. 4] and [F, Thm. 2.3]).

THEOREM 4.12. *Consider a pullback diagram of type □. Then R is Noetherian if and only if T is Noetherian, $D = F$, and $[k : F] < \infty$.*

Proof. Assume that T is Noetherian and local, that $D = F$, and that $[k : F] < \infty$. Let I be an ideal of R. Then IT is finitely generated as an ideal of R (Proposition 4.2). Let $a_1, \dots, a_n$ generate IT as an R-module, where $a_1, \dots, a_p \in I$ and p is the largest number of elements of any n-generating set of IT that can lie in I. With an argument similar to (but simpler than) the above, we can show that $I = Ra_1 + \cdots + Ra_p + IM$. The general case then follows in the usual way. The converse is easy [BR, Thm. 4]. □

4b. Pullbacks of PVMDs and v-Domains

PVMD pullbacks were characterized for $D + M$ constructions in [AR, Thm. 4.1] and more generally in [FG], where the following theorem is proved.

THEOREM 4.13 [FG, Thm. 4.1]. *Consider a pullback diagram of type □. Then R is a PVMD if and only if $k = F$, D and T are PVMDs, and T_M is a valuation domain.*

An example of a domain that is completely integrally closed (and hence a v-domain) but not a PVMD was given by Dieudonné [D]. The example from [HO], discussed in Example 3.3, is (easily seen to be) another completely integrally closed non-PVMD. In Theorem 4.15 we determine precisely which pullbacks are v-domains, and as applications we give a new proof of Theorem 4.13 and an example of a non–completely integrally closed v-domain that is not a PVMD.

LEMMA 4.14. *Consider a pullback diagram of type □. If I is a v-invertible ideal of R, then IT is a v-invertible ideal of T.*

Proof. Let I be v-invertible in R, so that $(II^{-1})_v = R$. By the proof of [FGH, Prop. 3.1], we have $(IT^{-1}T)_v = T$ (the v-operation being taken with respect to

T). Thus, since $(T : IT) \supseteq I^{-1}T$, we have $(IT(T : IT))_v = T$ also, and IT is v-invertible. □

THEOREM 4.15. *Consider a pullback diagram of type □. Then R is a v-domain if and only if $k = F$, D and T are v-domains, and T_M is a valuation domain.*

Proof. ($\Rightarrow$) Let R be a v-domain. Suppose $k \neq F$. Then, by Proposition 2.4, $M = (1, x)^{-1}$ for some $x \in T$. Hence M must be v-invertible, which contradicts the facts that M is divisorial and that $M^{-1} = (M : M)$. Hence $k = F$; that is, we are dealing with a diagram of type □*. Now let I be a finitely generated ideal of D. By [FG, Cor. 1.7(b)], $\phi^{-1}(I)$ is finitely generated in R, and since R is a v-domain, $\phi^{-1}(I)$ is v-invertible. By [FG, Prop. 1.8(b2)], we have $\phi(\phi^{-1}(I)_v) = (\phi(\phi^{-1}(I)))_v = I_v$. Hence I is v-invertible by [FG, Cor. 1.11]. Finally, let J be a finitely generated ideal of R. Then v-invertibility of J in R implies v-invertibility of JT in T by Lemma 4.14, and so T is a v-domain. Furthermore, $JJ^{-1} \nsubseteq M$ (since M is divisorial in R and J is v-invertible). It follows that $(JT(T : JT)) \nsubseteq M$, whence JT_M is principal. Therefore, T_M is a valuation domain.

($\Leftarrow$) Let J be a finitely generated integral ideal of R. Since $T_M = R_M$ is a valuation domain, there is an element $x \in K$ with $xJ \subseteq R$ and $xJ \nsubseteq M$. Thus there is no loss of generality in assuming that $J \nsubseteq M$. Then $\phi(J)$ and JT are both v-invertible, whence J is v-invertible by [FG, Prop. 1.13(a)]. □

REMARK. Using Proposition 3.2, we recover Theorem 4.13 by combining Theorems 3.4 and 4.15. (Recall that for a PVMD T, T_M is a valuation domain if and only if M is a t-ideal of T [Gri, Thm. 5].)

EXAMPLE 4.16. Let D be a completely integrally closed domain which is not a PVMD [D; HO], let k denote the quotient field of D, let $T = k[[X]]$, and let $R = D + Xk[[X]]$. Then, according to Theorem 4.15, R is a v-domain. It is clear that R is not completely integrally closed, and R is not a PVMD by Theorem 4.13.

4c. Mori and DVF Pullbacks

Recall that in a Mori domain every divisorial ideal is v-finite, so that Mori domains are automatically v-coherent. Moreover, in a Mori domain, the v- and t-operations are the same. Thus in a pullback diagram of type □ in which T is assumed to be a Mori domain, the conditions on M in Theorem 3.5 are automatically satisfied (when $k \neq F$), and they are satisfied in Theorem 3.4 precisely when M is divisorial in T (when $k = F$). These observations yield the following restatements of Theorems 3.4 and 3.5 when T is assumed to be a Mori domain.

PROPOSITION 4.17. *Consider a pullback diagram of type □, and assume that T is a Mori domain.*

(1) *If $k \neq F$, then R is v-coherent if and only if D is v-coherent.*
(2) *If $k = F$, then R is v-coherent if and only if D is v-coherent and M is divisorial in T.*

According to [B, Prop. 3.4], in a pullback of type $\square$ in which T is quasilocal, R is Mori if and only if T is Mori and D is a field. It is easy to extend this to the general case as follows. We can define $R(M)$ as in the discussion just before Theorem 4.7 (even in a pullback of type $\square$). If R is Mori, then, as a generalized quotient ring of R, $R(M)$ is also Mori [Q2, Sec. 2, Thm. 2]. Hence, by the local case [B, Prop. 3.4], $D = F$. The extension to the general case then follows from [R, Thm. 4.15] and the remarks following. That is, we have our next theorem.

THEOREM 4.18. *Consider a pullback diagram of type $\square$. Then R is a Mori domain if and only if T is a Mori domain and $D = F$.*

We now turn our attention to DVF domains; recall that R is a DVF domain if each divisorial ideal of R is v-finite. (Actually, according to our definition of v-finite, *every* nonzero ideal of a DVF domain is v-finite.) As already noted, Mori domains are DVF domains; [G, Example 2.8], which is a pullback of the type discussed in Theorem 4.20, is an example of a non-Mori DVF domain. It is clear that Prüfer DVF domains have the property that each divisorial ideal is invertible. Domains with this property were studied by Zafrullah [Z] (where they were called "generalized Dedekind domains"). Using ideas from [Z], we can easily characterize when a valuation domain is a DVF domain. First, note that if V is a DVF-valuation domain then, by [Z, Cor. 1.4], V is completely integrally closed and must therefore have rank 1. Suppose that V is a rank-1 valuation domain, and consider the value group G of V to be a subgroup of $\mathbb{R}$. It is clear that V is a DVF domain if V is discrete, and Zafrullah shows [Z, Thm. 2.6] that V is also a DVF domain if $G = \mathbb{R}$. On the other hand, if G is dense in $\mathbb{R}$ but $G \neq \mathbb{R}$ then, by choosing $\alpha \in \mathbb{R} \setminus G$ and reasoning as in [Z, Example 2.7], one can show that the ideal I consisting of those elements of V having value greater than α is divisorial but not principal. This proves the following.

PROPOSITION 4.19. *A valuation domain V is a DVF domain if and only if* (i) *V has rank* 1 *and* (ii) *either V is discrete or the value group G of V, when considered as a subgroup of $\mathbb{R}$, satisfies $G = \mathbb{R}$.*

In particular, a valuation domain with value group equal to $\mathbb{R}$ is a non-Mori DVF domain. For still another example of a non-Mori (Prüfer) DVF domain, we can take the ring of entire functions [Z, Example 2.1]; this is also an example of a DVF domain with non-DVF localizations (cf. [Z, Cor. 2.3]).

Our final result allows the construction of many more (non-Mori) DVF domains. Combined with Theorem 3.4, part (1) of Theorem 4.20 also allows the construction of v-coherent domains that are not DVF-domains (e.g., $\mathbb{Z} + XQ[[X]]$). We do not know the extent to which the quasilocal assumption on T is necessary.

THEOREM 4.20. *Consider a pullback diagram of type $\square$, and assume that T is quasilocal.*

(1) *If $k = F$, then R is a DVF-domain if and only if D and T are DVF domains and M is a nonprincipal v-finite divisorial ideal of T.*

(2) *If $k \neq F$, then R is a DVF-domain if and only if D and T are DVF domains and either M is a v-finite divisorial ideal of T or M is not a t-ideal of T.*

Proof. Split the diagram $\square$ as usual:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ S \arrow[r] \arrow[d] & F \arrow[d] \\ T \arrow[r] & k. \end{tikzcd}$$

(1) ($\Rightarrow$) Suppose that R is a DVF domain. Then, in particular, R is v-coherent, whence M is a t-ideal of T by Theorem 3.4. Hence, since M is a divisorial v-finite ideal of R, M is a nonprincipal v-finite divisorial ideal of T by Proposition 2.12. Let I be a divisorial ideal of D. By [FG, Prop. 1.8(a2)], $\phi^{-1}(I)$ is divisorial in R, and we have $\phi^{-1}(I) = H_v$ for some finitely generated ideal H of R. Since $H_v = \phi^{-1}(I) \supsetneqq M$, [FG, Prop. 1.8(b2)] yields $I = \phi(\phi^{-1}(I)) = \phi(H_v) = \phi(H)_v$, and I is v-finite in D. Thus D is a DVF domain. Now let L be a nonprincipal divisorial ideal of T. By Corollary 2.9, L is a divisorial ideal of R, and we have $L = J_v$ for some finitely generated ideal J of R. Since $L = J_vT = (J_vT)_v$, we have $L = (JT)_v$ by Proposition 2.7(1)(b), and L is v-finite. Thus T is a DVF domain.

($\Leftarrow$) Let I be a nonprincipal divisorial ideal of R. We first supose that $IT = yT$ is principal in T. In this case, if $II^{-1} \subseteq M$ (i.e., if $I^{-1} = (M : I)$), then the hypotheses of Proposition 2.7(2) hold and we obtain $M = yI^{-1}$. Then, since R is v-coherent by Theorem 3.4 and since M is v-finite in R by Proposition 2.12, M^{-1} is also v-finite, whence $I = I_v = yM^{-1}$ is v-finite. On the other hand, if $II^{-1} \nsubseteq M$, pick $x \in I^{-1}$ with $xI \nsubseteq M$. Then, since each ideal of R is comparable to M (since T is quasilocal), we have $xI \supsetneqq M$. By [FG, Prop. 1.8(b2)], $\phi(xI)$ is divisorial in D, and we can write $\phi(xI) = J_v$ for some finitely generated ideal J of D. By [FG, Cor. 1.7(b)], $\phi^{-1}(J)$ is finitely generated, whence by [FG, Prop. 1.8(a2)], $xI = \phi^{-1}(J_v) = \phi^{-1}(J)_v$ and I is v-finite. Now suppose that IT is not principal. Then, by Proposition 2.7(1) and the fact that T is a DVF domain, we have $I^{-1} = (T : IT) = (HT)_v$ for some finitely generated fractional ideal H of R. If $(HT)_v$ is principal in T, then $I^{-1} = (HT)_v = zT$ for some $z \in K$, whence $I = I_v = z^{-1}(R : T) = z^{-1}M$, which is v-finite. If $(HT)_v$ is not principal, then by Proposition 2.7(1) we have $I^{-1} = (HT)_v = H_v$, whence $I = I_v = H^{-1}$ is v-finite (since R is v-coherent).

(2) ($\Rightarrow$) By Theorem 3.5, it suffices to prove that D and T are DVF domains. By concentrating on the upper diagram, we obtain from (1) that D is a DVF domain, and the proof above that T is a DVF domain works in this case as well.

($\Leftarrow$) Of course, M is divisorial in S, and the assumption on M in T implies that M is also v-finite in S by Proposition 2.14. Moreover, if I is a nonprincipal divisorial ideal of S, then $II^{-1} \subseteq M$ (since S is quasilocal), and a simplified version of the argument used to prove the implication ($\Leftarrow$) of (1) above (replacing the appeal to Theorem 3.4 by an appeal to Theorem 3.5) shows that I is v-finite in S. Hence S is a DVF domain. That R is a DVF domain now follows from (1). $\square$

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