EXCEPTIONAL SETS FOR HOLOMORPHIC SOBOLEV FUNCTIONS

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We let B^n denote the unit ball in $\mathbf{C}^n$ and let S denote its boundary. For the most part we will follow the notation and terminology of Rudin [11]. For f holomorphic in B^n the radial derivative of f is defined to be

$$Rf(z)=\sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)=\sum_{k=0}^{\infty} kf_k(z),$$

if $f=\sum f_k$ is the homogeneous polynomial development of f. For $\beta>0$ one is led to the definition $R^\beta f(z)=\sum_{k=0}^{\infty}(k+1)^\beta f_k(z)$ (see [6]). For $p,\beta>0$ we define

$$H_\beta^p(B^n)=\{f\in H(B^n): R^\beta f\in H^p(B^n)\},$$

where $H^p(B^n)$ is the usual Hardy space [11]. $H_\beta^p(B^n)$ may be considered as a holomorphic version of a Sobolev space [6]. For $\zeta\in S$ and $\delta>0$ there is the Koranyi ball $B(\zeta,\delta)=\{\eta\in S: |1-\langle\zeta,\eta\rangle|<\delta\}$. There are also the admissible approach regions for $\zeta\in S$, $\alpha>1$,

$$D_\alpha(\zeta)=\{z\in B^n: |1-\langle z,\zeta\rangle|<(\alpha/2)(1-|z|^2)\}.$$

For each function $f: B^n\to\mathbf{C}$ we have the admissible maximal function

$$M_\alpha f(\zeta)=\sup_{z\in D_\alpha(\zeta)}|f(z)|.$$

The main result of this paper is the following "trace" theorem for Sobolev functions, which will be proved in Section 1.

THEOREM 1.1. *Suppose that* $0<p\le 1$ *and* $f\in H_\beta^p(B^n)$, *where* $m=n-\beta p>0$, *and that* ν *is a positive Borel measure on* S *that satisfies*

$$(*)\qquad \nu(B(\zeta,\delta))\le C\delta^m \quad \textit{for some constant } C.$$

Then for each $\alpha>1$ *there is a constant* $C=C(\alpha)$ *such that*

$$\int M_\alpha f(\zeta)^p\,d\nu(\zeta)\le C\|R^\beta f\|_p^p.$$

(*Here* $\|g\|_p$ *denotes the norm of* g *in* $H^p(B^n)$.)

Of course, the strong inequality of the theorem gives rise to a corresponding weak estimate which in turn yields the following.

COROLLARY 1. *Every* $f\in H_\beta^p(B^n)$ *has an admissible limit almost everywhere,* $d\nu$.

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There is also a real variable version of this theorem which can be proved by the same method. This is discussed in Section 1. The real variable version has been discovered independently, using another method, by Adams [1].

We give some examples involving curves to show how condition (*) on ν involves "directional" considerations. We conclude Section 1 by showing that the condition on ν is necessary for the conclusion of Theorem 1.1 to hold, even if $p>1$.

In the second section we turn our attention to the case $p>1$. By analogy with the real variable case we expect that condition (*) on ν should be necessary but not sufficient in order for the conclusion of Theorem 1.1 to hold. More precisely, one expects that if $p>1$ and $m=n-\beta p>0$ then there should exist a measure $\nu \not\equiv 0$ satisfying (*) and a function $f \in H_\beta^p(B^n)$ such that f has a radial limit at no point of the support of ν. We can show that this is the case only under the restriction that $0<m<(n+1)/2$. Indeed, $(n+1)/2$ is the natural limit for the method we use. We will concentrate our attention on the case $m=1$, since this is the case that has relevance to the rest of Section 2. We conclude Section 2 by showing that if $1<p\le 2$ and ν is arclength measure on a transverse curve, then every $f\in H_\beta^p(B^n)$ with $n-\beta p=1$ has an admissible limit almost everywhere, $d\nu$. The relevant point is that such a ν satisfies (*) with $m=1$. This result fails even for slices when $p>2$. Our examples show that this cannot follow from any general result like Theorem 1.1, but necessarily depends on the "almost analytic" structure associated with a transverse curve.

1. Let $\alpha>1$. Then, for each $E\subseteq S$, we define the α-tent over E to be $T_\alpha(E)=(\bigcup_{\zeta\notin E} D_\alpha(\zeta))^c$, the complement being taken in B^n. For each $\alpha>1$ there is a constant $C=C(\alpha)$ such that if $z\in T_\alpha(B(\zeta,\delta))$ then $1-|z|\le C\delta$. A complex-valued function defined in B^n is called an α-atom if

(i) there is a $\zeta\in S$ and a $\delta>0$ such that a is supported in $T_\alpha(B(\zeta,\delta))$,

(ii) $|a(z)|\le\delta^{-n}$ for all $z\in B^n$.

A type of atomic decomposition on spaces of homogeneous type is proved in [4, Lemma 2.1]. The proof follows closely the proof given in [8] for the case of $\mathbf{R}_+^{n+1}$. For the case of the ball, Lemma 2.1 of [4] reads as follows.

LEMMA 1.1. *For each $\alpha>1$ there is a constant $C=C(\alpha)$ such that, for every $f\colon B^n\to\mathbf{C}$ such that $\int_S M_\alpha f(\zeta)\,d\sigma(\zeta)<\infty$, there are nonnegative α-atoms a_k and nonnegative numbers λ_k such that*

$$|f(z)|\le\Sigma\lambda_k a_k(z)\quad \textit{and}\quad \Sigma\lambda_k\le C\int M_\alpha f(\zeta)\,d\sigma(\zeta).$$

We can now give the following.

Proof of Theorem 1.1. Let $F=R^\beta f$. Then a straightforward calculation shows that

$$f(z)=\frac{1}{\Gamma(\beta)}\int_0^1\left(\log\frac{1}{t}\right)^{\beta-1}F(tz)\,dt.$$

From this we see that it is enough to show that $M_\alpha G\in L^p(d\nu)$, where $G(z)=\int_0^1(1-t)^{\beta-1}|F(tz)|\,dt$. By the basic result of Koranyi and Vogi (see [11, p. 86]) we know that $|F(z)|^p$ satisfies the hypothesis of the lemma, so there are nonnegative α-atoms a_k and $\lambda_k \geq 0$ such that $|F(z)|^p \leq \Sigma \lambda_k a_k(z)$ and

$$\Sigma \lambda_k \leq C \int M_\alpha F(\zeta)^p \, d\sigma(\zeta) \leq C\|F\|_p^p = C\|R^\beta f\|_p^p.$$

Thus,

$$G(z)^p \leq \left\{ \int_0^1 (1-t)^{\beta-1} (\Sigma \lambda_k a_k(tz))^{1/p} \, dt \right\}^p.$$

Now, since $0 < p \leq 1$, $1/p \geq 1$ and we can apply Minkowski's inequality to obtain

$$G(z)^p \leq \Sigma \lambda_k \left(\int_0^1 (1-t)^{\beta-1} a_k(tz)^{1/p} \, dt \right)^p.$$

Now fix $\zeta \in S$ and suppose that $z \in D_\alpha(\zeta)$. Consider the kth integral

$$\int_0^1 (1-t)^{\beta-1} a_k(tz)^{1/p} \, dt,$$

and suppose a_k is supported on $T_\alpha(B(\zeta_k, \delta_k))$. Then certainly $a_k(tz) \equiv 0$ unless $\zeta \in B(\zeta_k, \delta_k)$; in any case, $a_k(tz) \equiv 0$ unless $1-t \leq C\delta_k$, so we have that the above integral is at most

$$\int_{1-C\delta_k}^1 (1-t)^{\beta-1} \delta_k^{-n/p} \, dt \leq C\delta_k^{\beta-n/p}.$$

In summary, we may conclude that

$$\int_0^1 (1-t)^{\beta-1} a_k(tz)^{1/p} \, dt \leq C\delta_k^{\beta-n/p} \chi_k(\zeta),$$

where χ_k is the characteristic function of $B(\zeta_k, \delta_k)$. It now follows that $G(z)^p \leq C\Sigma \lambda_k \delta_k^{\beta p - n} \chi_k(\zeta)$, all $z \in D_\alpha(\zeta)$. Hence $M_\alpha G(\zeta)^p \leq C\Sigma \lambda_k \delta_k^{\beta p-n} \chi_k(\zeta)$. Now, just integrate both sides with respect to ν and use the hypothesis on ν to obtain

$$\int M_\alpha G(\zeta)^p \, d\nu(\zeta) \leq C\Sigma \lambda_k \leq C\|R^\beta f\|_p^p.$$

This completes the proof. □

Here is the real-variable version of this theorem: Suppose $0 < p \leq 1$, $\beta > 0$, $m = n - \beta p > 0$, and ν is a positive Borel measure on $\mathbf{R}^n$ such that $\nu(B(x,r)) \leq Cr^m$ for some constant C. (Here $B(x,r)$ is the usual Euclidean ball.) Then we have $\int NF(x)^p \, d\nu(x) \leq C\|f\|_p^p$. Here $F(x,y) = (G_\beta * P_y * f)(x)$, where G_β is the Bessel kernel, P_y is the Poisson kernel, and NF denotes the nontangential maximal function of F. See [12, Chap. V] for a discussion of these notions.

The idea is to use an identity from the theory of Bessel functions, as in [4], to show that F is (up to a harmless term) majorized by $\int_0^2 t^{\beta-1} |u(x, y+t)| \, dt$, where u is the Poisson integral of f. Now in place of Lemma 1 above we use the atomic decomposition of [8] (which was the model for Lemma 1) and argue exactly as in the proof of Theorem 1.2. We omit the details.

Note what happens if $f\in H^p_\beta(B^n)$ when $n-\beta p=0$ and $p\le 1$. Then the argument used in the proof of Theorem 1.1 shows that

$$|f(z)|^p\le C\Sigma\lambda_k\le C\|R^\beta f\|^p_{H^p}\quad\text{for all } z\in B^n.$$

Hence,

$$\|f\|_\infty=\sup_{z\in B^n}|f(z)|\le C\|R^\beta f\|_{H^p}.$$

Let $f_r(z)=f(rz)$, $0<r<1$. Note that $R^\beta f_r=(R^\beta f)_r$ and apply the above inequality to $f-f_r$ to obtain

$$\|f-f_r\|_\infty\le C\|R^\beta f-(R^\beta f)_r\|_{H^p}\to 0\quad\text{as } r\to 1.$$

This shows that f is continuous on $\overline{B^n}$. This was proved earlier in [5], by a different method. See [3] and [5] for refinements.

To see how condition (*) of Theorem 1.1 involves directions, we consider the case of a curve φ in S. Let ν be arclength measure on φ. Recall that φ is called transverse if $|\langle\varphi',\varphi\rangle|\ge\epsilon>0$. It is not hard to see that if φ is transverse then $\nu(B(\zeta,\delta))\le c\delta$ and hence if $f\in H^p_\beta(B^n)$ with $n-\beta p=1$ and $p\le 1$, f has an admissible limit a.e. $d\nu$. For the general curve the best that can be said is that $\nu(B(\zeta,\delta))\le C\delta^{1/2}$ (and there are curves for which this is exactly right), and hence for the general curve we can say that if $f\in H^p_\beta(B^n)$ with $n-\beta p=1/2$ (a stronger requirement than $n-\beta p=1$) then f has admissible limit a.e. $d\nu$. The sharpness of these statements about curves can be shown by simple examples. It also follows from Theorem 1.2.

The corollary to Theorem 1.1 suggests that if $0<p<1$ the exceptional sets for $H^p_\beta(B^n)$ are precisely the sets of non-isotropic m-dimensional Hausdorff measure 0. This is the case when $n=1$, as seen in what follows. Suppose $f\in H^p_\beta(B^1)$, $m=1-\beta p>0$, and E is the set where F fails to have a nontangential limit. By a basic result of Frostman [9], if E did not have measure 0 in dimension m then there would be a positive measure $\nu\not\equiv 0$ supported on E such that $\nu(B(\zeta,\delta))\le\delta^m$ for all $\zeta\in S$. This contradicts the corollary of Theorem 1.1. On the other hand, if E is a compact set of m-dimensional Hausdorff measure 0 then it follows from the examples given in [7] that there is a Blaschke product in $H^p_\beta(B^1)$ that fails to have a nontangential limit at any point of E.

We conclude this section by showing that (*) is the right condition on the measure ν.

THEOREM 1.2. *Suppose that* $0<p<\infty$, *that* α,β,m *are as in Theorem* 1.1, *and that* ν *is a positive measure on* S *such that*

$$\int M_\alpha f(\zeta)^p\,d\nu(\zeta)\le C\|R^\beta f\|^p_p,\quad \text{all } f\in H^p_\beta(B^n).$$

Then $\nu(B(\zeta,\delta))\le C\delta^m$.

Proof. Fix $0<r<1$, and let $\zeta\in S$. Define $F(z)=(1-r\langle z,\zeta\rangle)^{-((n+1)/p)}$. It follows from [11, p. 17] that $\|F\|^p_p\le C(1-r)^{-1}$. Next let $f(z)=\int_0^1(\log(1/t))^{\beta-1}F(tz)\,dt$. As we have seen, $R^\beta f=\Gamma(\beta)F$. We have

$$|f(z)| \geq \operatorname{Re} f(z) = \int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} \operatorname{Re} F(tz)\, dt$$

and

$$\begin{aligned}\operatorname{Re} F(tz) &= \operatorname{Re}(1 - tr\langle z, \zeta\rangle)^{-((n+1)/p)} \\ &= (1-tr)^{-((n+1)/p)} \operatorname{Re}\left(1 + \frac{tr(1-\langle z, \zeta\rangle)}{1-tr}\right)^{-((n+1)/p)}.\end{aligned}$$

Now there is a constant δ such that if $|\lambda| \leq \delta$ then $\operatorname{Re}(1+\lambda)^{-((n+1)/p)} \geq 1/2$. Now suppose $|1-\langle z, \zeta\rangle| < \delta(1-r)$; then $|(tr(1-\langle z, \zeta\rangle))/(1-tr)| \leq \delta$ and hence

$$\operatorname{Re}(1 - tr\langle z, \zeta\rangle)^{-((n+1)/p)} \geq \tfrac{1}{2}(1-tr)^{-((n+1)/p)},$$

so we see that

$$|f(z)| \geq C \int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} (1-tr)^{-((n+1)/p)}\, dt \geq C(1-r)^{\beta-(n+1)/p}.$$

This says that $|f(z)|^p \geq C(1-r)^{\beta p - n - 1}$ in $B(\zeta, \delta(1-r))$. Hence,

$$\begin{aligned}(1-r)^{-1} &\geq C\|f\|_p^p \geq C \int M_\alpha f(\eta)^p\, d\nu(\eta) \\ &\geq C \int |f(\eta)|^p\, d\nu(\eta) \geq C \int_{B(\zeta, \delta(1-r))} |f(\eta)|^p\, d\nu(\eta) \\ &\geq C(1-r)^{\beta p - n - 1} \nu(B(\zeta, \delta(1-r))).\end{aligned}$$

The result follows. □

2. We start this section by showing that if $1 < p$ and $n - \beta p = 1$ then there is a positive Borel measure on ν on S, $\nu \not\equiv 0$ and $\nu(B(\zeta, \delta)) \leq C\delta$, and an $f \in H_\beta^p(B^n)$ such that f fails to have a radial limit a.e. $d\nu$. This is especially easy for $p > 2$. In this case look for f of the form $f(z_1, \ldots, z_n) = F(z_1)$. Then $R^\beta f \in H^p(B^n)$ if

$$\int_{|z|<1} |R^\beta F(z)|^p (1-|z|)^{n-2}\, dA(z) < \infty.$$

Here dA is Lebesgue measure in the unit disc. By a result of Hardy and Littlewood,

$$\int |R^\beta F|^p (1-|z|)^{n-2}\, dA(z) < \infty$$

if and only if

$$\int |F'(z)|^p (1-|z|)^{p(1-\beta)+n-2}\, dA(z) = \int |F'(z)|^p (1-|z|)^{p-1}\, dA(z) < \infty.$$

We try $F(z) = \sum a_k z^{2^k}$, where $|a_k| = k^{-1/2}$. Then $|rF'(re^{i\theta})| \leq \sum 2^k k^{-1/2} r^{2^k}$. Now $2^k r^{2^k} \leq \sum_{2^{k-1} < l \leq 2^k} r^l$ and hence $|rF'(re^{i\theta})| \leq \sum (\log l)^{-1/2} r^l$. It is not hard to see (e.g., by comparing with an integral) that this last sum is bounded by a constant times $1/[(1-r)(\log[1/(1-r)])^{1/2}]$. It follows that $\int |F'(z)|^p (1-|z|)^{p-1}\, dA(z) < \infty$, since $p > 2$. However, since $\sum |a_k|^2 = \infty$, it follows from [13, Thm. 6.4, p. 203] that the set where F has a radial limit has measure 0. Hence the function $f(z) = F(z_1)$ fails to have radial limits a.e. with respect to arclength measure ν on the curve $(e^{i\theta}, 0, \cdot, 0)$. This measure satisfies $(*)$ with $m = 1$.

It doesn't seem quite so easy when $p \le 2$. Indeed, Theorem 2.2 will show that ν cannot be arclength measure on any slice or even any transverse curve. We proceed as follows: First assume that $n = 2$. We look for a function $f(z, w) = g(2zw)$, where g is holomorphic in the disc. We will actually find a function g such that f belongs to the Besov space

$$B_\beta^p(B^2) = \left\{ f : \int_{B^2} |R^{\beta+1} f(z)|^p (1 - |z|)^{p-1} \, dV(z) < \infty \right\};$$

here dV is Lebesgue measure on B^2. Since $p \le 2$, $B_\beta^p \subseteq H_\beta^p$. This is well known for $n = 1$ and follows by slice integration for $n > 1$. At this point it is convenient to state a lemma.

LEMMA 2.1. *There is a constant C such that if $F(z, w) = g(2zw)$ then*

$$\int_{B^2} |F(z, w)|^p (1 - |z|^2 - |w|^2)^{p-1} \, dV(z, w) \le C \int_{|z|<1} |g(z)|^p (1 - |z|)^{p-1/2} \, dA(z).$$

Proof. Using polar coordinates, the first integral above is equal to

$$\int_0^1 r^3 (1-r)^{p-1} \int_S |F(r\zeta)|^p \, d\sigma(\zeta) \, dr.$$

By Theorem 1 of [2], this last integral is at most a constant times

$$\int_0^1 r^3 (1-r)^{p-1} \int_0^1 \int_0^{2\pi} |g(r\rho e^{i\theta})|^p (1-\rho)^{-1/2} \, d\theta \, d\rho \, dr.$$

If we interchange the two inner integrals and make the substitution $t = r\rho$, then the above integral is at most a constant times

$$\begin{aligned} &\int_0^1 (1-r)^{p-1} \int_0^{2\pi} \int_0^r |g(te^{i\theta})|^p (1-t)^{-1/2} \, dt \, d\theta \, dr \\ &\qquad = \int_0^{2\pi} \int_0^1 |g(te^{i\theta})|^p (1-t)^{-1/2} \int_t^1 (1-r)^{p-1} \, dr \, dt \, d\theta \\ &\qquad \le C \int_{|z| \le 1} |g(z)|^p (1-|z|)^{p-1/2} \, dA(z). \end{aligned}$$

Now if we take g holomorphic in U and apply the lemma to $R^{\beta+1} g$, we see that

$$\int_{B^2} |R^{\beta+1} f|^p (1-|z|)^{p-1} \, dv(z) \le C \int_{|z|<1} |R^{\beta+1} g|^p (1-|z|)^{p-1/2} \, dA(z).$$

This last integral is finite if and only if $g \in B_{1/(2p)}^p(U)$. Now the exceptional sets for this class are sets of a certain capacity zero, not Hausdorff measure zero. Precisely, there is a compact set E and a positive measure ν on E with $\nu(E) > 0$ and ν satisfying $(*)$ with $m = 1/2$, and a function $g \in B_{1/(2p)}^p(U)$ that fails to have a radial limit at any point of E. Now $f(z,w)=g(2zw)$ fails to have a radial limit at any point $(\eta,\zeta)\in S$ such that $2\zeta\eta\in E$. Call this set K. Letting T denote the unit circle, define $\psi: E\times T\to K$ by

$$\psi(\lambda,\eta)=\left(\frac{\lambda\bar{\eta}}{\sqrt{2}},\frac{\eta}{\sqrt{2}}\right).$$

It is easy to see that ψ is a homeomorphism. Let μ be the measure on K obtained by transporting $d\nu\times d\theta$ on $E\times T$ over to K by means of ψ ($d\theta$ is arclength on T). We will show that $\mu(B(\zeta,\delta))\le C\delta$. Take $\zeta=(\lambda_0\bar{\eta}_0/\sqrt{2},\eta_0/\sqrt{2})\in K$. It is enough to show that $(\lambda\bar{\eta}/\sqrt{2},\eta/\sqrt{2})\in B(\zeta,\delta)$ implies $|\lambda-\lambda_0|<c\delta$ and $|\eta-\eta_0|<C\delta^{1/2}$. Now $(\lambda\bar{\eta}/\sqrt{2},\eta/\sqrt{2})\in B(\zeta,\delta)$ means that

$$\left|1-\frac{\lambda\bar{\lambda}_0\bar{\eta}\eta_0}{2}-\frac{\eta\bar{\eta}_0}{2}\right|<\delta.$$

Letting $\lambda\bar{\lambda}_0=e^{it}$ and $\eta\bar{\eta}_0=e^{is}$, we have

$$\left|1-\frac{e^{it}e^{-is}}{2}-\frac{e^{is}}{2}\right|<\delta \quad\text{or}\quad \left|e^{-it/2}-\cos\left(\frac{t}{2}-s\right)\right|<\delta,$$

which implies that

$$\left|\sin\frac{t}{2}\right|<\delta \quad\text{and}\quad \left|\cos\frac{t}{2}-\cos\left(\frac{t}{2}-s\right)\right|<\delta.$$

Since we may assume δ is small, the first of these inequalities says that $|t|\le C\delta$. The second says that $|(1-\cos(t/2))-(1-\cos(t/2-s))|<\delta$ and hence that $|1-\cos(t/2-s)|\le\delta+|1-\cos(t/2)|\le\delta+\delta^2\le C\delta$. Consequently, $|t/2-s|\le C\delta^{1/2}$ and hence

$$|s|\le\left|\frac{t}{2}\right|+C\delta^{1/2}\le C\delta+C\delta^{1/2}\le C\delta^{1/2}.$$

This finishes the proof that $\mu(B(\zeta,\delta))\le C\delta$.

The case $n>2$ now follows easily. Look for $f(z)=F(z_1,z_2)$. By formula 1 of [11, p. 14], $R^\beta f\in H^p(B^n)$ if and only if

$$\int_{B^2}|R^\beta F|^p(1-|z|)^{n-3}\,dV(z)<\infty.$$

But this last condition is equivalent to $F\in B^p_{1/p}(B^2)$, so we have an example. □

Finally we show that, in spite of the above discussion, it is still true that if $f\in H^p_\beta(B^n)$ with $n-\beta p=1$ and $p\le 2$ then f has admissible limits almost everywhere with respect to arclength on any transverse curve. Recall that a curve φ in S is called transverse if $|\langle\varphi',\varphi\rangle|>\epsilon>0$. By proper choice of orientation we may assume that $-i\langle\varphi',\varphi\rangle\ge\epsilon>0$. We recall the construction of the associated almost analytic disc. For some large integer N, define

$$\Phi(z)=\Phi(x+iy)=\sum_{k=0}^{N}\frac{(iy)^k}{k!}\varphi^{(k)}(x).$$

Then, if (a,b) is the parameter interval for φ, there is a $\delta>0$ such that $Q=(a,b)\times(0,\delta)$ is mapped into B^n by Φ. Note that $\Phi(x)=\varphi(x)$. Moreover, for each $x\in(a,b)$ the curve $y\mapsto\Phi(x+iy)$ lies in a nontangential region with vertex at $\varphi(x)$. Also,

$$\frac{\partial\Phi}{\partial\bar z}(x+iy)=\frac{(iy)^N}{N!}\varphi^{(N+1)}(x).$$

Moreover, $(1/C)y\le 1-|\Phi(x+iy)|\le Cy$. Finally, if $F\in H^p(B^n)$ for some $p>0$ and if a positive integer k is given, then there is an N such that if Φ is defined as above then $(\partial/\partial\bar z)(F\circ\Phi)\in C^k(\bar Q)$. See [12] for more details.

We will also need the gradient estimates of [4]. We will be explicit since one of the definitions given in [4, p. 370] is incorrect. We define, for F holomorphic in B^n, $\nabla_N F(r\zeta)=(1/r)RF(r\zeta)\bar\zeta$ and $\nabla_T F(r\zeta)=\nabla F(r\zeta)-\nabla_N F(r\zeta)$. Then Lemma 3.2 [4, p. 370] should read as follows.

LEMMA 2.2. *There is a constant C such that if $z\in D_\alpha(\zeta)$ and if $\langle w,\zeta\rangle=0$, $|w|=1$, then*

$$|\langle w,\nabla F(z)\rangle|\le C(|\nabla_T F(z)|+(1-|z|)^{1/2}|\nabla F(z)|).$$

Now suppose that ν denotes arclength on a transverse curve. Then from Theorem 3.2 of [4] we have the following.

THEOREM 2.1. *There is a constant C such that*

$$\int M_\alpha H(\zeta)^p\,d\nu(\zeta)\le C\|F\|_{H^p}^p,$$

where $H(z)=(1-|z|)^{(n-1)/p+1}|\nabla F(z)|+(1-|z|)^{(n-1)/p+1/2}|\nabla_T F(z)|$.

LEMMA 2.3. *Suppose h is holomorphic on a rectangle $Q=(a,b)\times(0,\delta)$ and $p\le 2$, and for some integer k we have*

$$\int_Q |h^{(l)}(z)|^p y^{kp-1}\,dx\,dy<\infty$$

for all $l\le k$. Then for any subinterval $[c,d]\le(a,b)$ we have

$$\int_c^d Nh(x)^p\,dx\le C\sum_{l\le k}\int_Q |h^{(l)}(z)|^p y^{kp-1}\,dx\,dy,$$

where $Nh(x_0)=\sup\{|h(x+iy)|: y\ge\alpha|x-x_0|\}$.

Proof. Let $I=[c,d]$. Then there is a simple closed smooth curve contained in $\bar Q$ whose boundary contains I. Let τ be a conformal mapping of U onto the interior of this curve. Then τ is conformal up to the boundary. Let $f(z)=h(\tau(z))$. Then $|f^{(k)}(z)|\le C\sum_{l\le k}|h^{(l)}(\tau(z))|$, so

$$\int_{|z|\le 1}|f^{(k)}(z)|^p(1-|z|)^{kp-1}\,dA(z)\le C\sum_{l\le k}\int_{|z|\le 1}|h^{(l)}(\tau(z))|^p(1-|z|)^{kp-1}\,dA(z).$$

Now, using the conformality at the boundary, by changing variables we have, for $l\le k$,

$$\int_{|z|\le 1} |h^{(l)}(\tau(z))|^p(1-|z|)^{kp-1}\,dA(z) \le \int_Q |h^{(l)}(z)|^p y^{kp-1}\,dx\,dy < \infty.$$

It follows that

$$\int_{|z|<1} |f'(z)|^p(1-|z|)^{p-1}\,dA(z) < \infty.$$

Since $p \le 2$ this implies that $f \in H^p$, and hence the nontangential maximal function of f is in L^p. It follows that h has the same property on I, again because τ is conformal at the boundary. The norm estimate follows from the proof. □

For a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$ we use the usual notations:

$$|\gamma| = \gamma_1 + \cdots + \gamma_n, \qquad D^\gamma = \frac{\partial^{|\gamma|}}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}}.$$

If $f: B^n \to \mathbf{C}$ is continuous, define

$$I_\beta f(z) = \frac{1}{\Gamma(\beta)} \int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} f(tz)\,dt.$$

LEMMA 2.4. *If f is holomorphic in B^n then*

$$|z|^{|\gamma|}(D^\gamma f)(z) = (I_\beta F)(z),$$

where

$$F(z) = |z|^{|\gamma|}(D^\gamma R^\beta f)(z).$$

Proof. It is sufficient to prove this in the case where f is a homogeneous polynomial of (say) degree l. Then

$$\begin{aligned} F(tz) &= t^{|\gamma|}|z|^{|\gamma|}(l+1)^\beta (D^\gamma f)(tz) \\ &= t^{|\gamma|}|z|^{|\gamma|}(l+1)^\beta t^{l-|\gamma|}(D^\gamma f)(z) \\ &= t^l |z|^{|\gamma|}(l+1)^\beta D^\gamma f(z). \end{aligned}$$

The result follows. □

LEMMA 2.5. *Suppose $R^\beta f \in H^p(B^n)$ and k is an integer greater than β. Fix a multi-index γ with $|\gamma| \le k$ and let $g(z) = (1-|z|)^{k-\beta}(D^\gamma f)(z)$. Then if $1 < \alpha < \alpha_1$ there is a constant $C = C(\alpha, \alpha_1)$ such that $M_\alpha g(\zeta) \le C M_{\alpha_1} R^\beta f(\zeta)$.*

Proof. Fix $\alpha_1 > \alpha$. Then there is an $\epsilon = \epsilon(\alpha, \alpha_1) > 0$ such that if $z \in D_\alpha(\zeta)$ *then the polydisc*

$$P = \{w : |w_j - z_j| \le \epsilon(1-|z|^2)\} \subseteq D_{\alpha_1}(\zeta).$$

By Cauchy's formula,

$$R^\beta f(z) = \frac{1}{(2\pi i)^n} \int_T \frac{R^\beta f(w)}{(w_1-z_1)\cdots(w_n-z_n)}\,dw,$$

where T is the distinguished boundary of P. Now if we apply D^γ and make the obvious estimate we obtain

$$|D^\gamma R^\beta f(z)| \le C \frac{M_{\alpha_1} R^\beta f(\zeta)}{(1-|z|)^k}.$$

From Lemma 2.4 we obtain

$$|z|^{|\gamma|}|D^\gamma f(z)| \le C\int_0^1 (1-t)^{\beta-1}(1-t|z|)^{-k}\,dt\cdot M_{\alpha_1}R^\beta f(\zeta)$$
$$\le C(1-|z|)^{\beta-k}M_{\alpha_1}R^\beta f(\zeta).$$

The result follows. □

LEMMA 2.6. *Suppose that ν is arclength measure along a transverse curve and $n-\beta p=1$. Then there is a constant C such that*

$$\int M_\alpha G(\zeta)^p d\nu(\zeta) \le C\|R^\beta f\|_{H^p}^p,$$

where

$$G(z)=(1-|z|)|\nabla f(z)|+(1-|z|)^{1/2}|\nabla_T f(z)|.$$

Proof. If $F=R^\beta f\in H^p(B^n)$ then from Theorem 2.1 we have

$$\int M_\alpha H(\zeta)^p d\nu(\zeta) \le \|R^\beta f\|_{H^p}^p.$$

By Lemma 2.4 we have $|z|\nabla f(z)=I_\beta(|z|\nabla R^\beta f)$. Thus if $z\in D_\alpha(\zeta)$ then

$$|\nabla f(z)| \le C\int_0^1 (1-t)^{\beta-1}(1-t|z|)^{-((n-1)/p+1)}\,dt\cdot M_\alpha H(\zeta)$$
$$\le (1-|z|)^{-1}M_\alpha H(\zeta),$$

taking into account the fact that $n-\beta p=1$. A similar argument works for the term involving $\nabla_T f$. Actually, Lemma 2.4 does not directly apply to $\nabla_T f$ because the coefficients are not constant; however, they are homogeneous of degree 0 and so the argument of Lemma 2.4 does apply. □

In the next theorem we assume that φ is actually defined and satisfies the transversality condition on a larger interval $(a-\epsilon, b+\epsilon)$.

THEOREM 2.2. *Suppose that $1<p\le 2$, $n-\beta p=1$, and that ν is arclength on a transverse curve φ. Then there is a constant C such that*

$$\int M_\alpha f(\zeta)^p\, d\nu(\zeta) \le C\|R^\beta f\|_{H^p}^p;$$

in particular, every $f\in H_\beta^p$ has an admissible limit a.e. $d\nu$.

Proof. Let $\Phi\colon Q=(a,b)\times(0,\delta)\to B^n$ be the almost analytic mapping associated to φ described above. Define a measure μ on B^n as follows:

$$\int g\,d\mu=\int_Q g(\Phi(x+iy))y^{n-2}\,dx\,dy.$$

It follows from the fact that φ is a transverse curve that μ is a Carleson measure; see [3, Lemma 4.3] for a very similar result. It follows from this that

$$\int |g|^p\,d\mu \le C\int M_\alpha g(\zeta)^p\,d\sigma(\zeta)$$

for any $g: B^n \to \mathbf{C}$ which is continuous (see [12, p. 236]). In particular, we see from Lemma 2.5 that if $k > \beta$ is an integer then for $|\gamma| \le k$ we have

$$\int_Q (1-|\Phi(z)|)^{(k-\beta)p} |(D^\gamma f)(\Phi(z))|^p y^{n-2}\, dx\, dy \le C\|R^\beta f\|_{H^p}^p.$$

As we have seen, we may assume that $(\partial/\partial\bar{z})(f\circ\Phi)$ is as smooth as we want on $\bar{Q}$, and hence we may find a function u such that

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(f\circ\Phi)$$

and u is as smooth as we want on $\bar{Q}$. This means that $f\circ\Phi = h+u$, where h is holomorphic on Q. If $l \le k$ then

$$h^{(l)}(z) = \frac{\partial^l}{\partial z^l}(f\circ\Phi) - \frac{\partial^l u}{\partial z^l}$$

and

$$\left|\frac{\partial^l}{\partial z^l}(f\circ\Phi)\right| \le C \sum_{|\gamma|\le k} |(D^\gamma f)\circ\Phi|.$$

Noting that $1-|\Phi(z)| \le Cy$ we find that $\int_Q |h^{(l)}(z)|^p y^{kp-1}\, dx\, dy < \infty$ for all $l \le k$. Hence, by Lemma 2.3, we have $\int_a^b Nh(x)^p\, dx < \infty$. To complete the proof we take $x \in (a,b)$ and assume $\zeta = \varphi(x) = (1,0,\ldots,0)$. Note that

$$\Phi(x+it) - (1-t,0,\ldots,0) = O(t)$$

and hence

$$|h(x+it) - f(1-t,0,\ldots,0)| = |f(\Phi(x+it)) - f(1-t,0,\ldots,0)| \le Mt,$$

where M is the maximum of $|\nabla f|$ on the line joining $\Phi(x+it)$ to $(1-t,0,\ldots,0)$. This line lies in a nontangential region with vertex at ζ and hence in $D_\alpha(\zeta)$. Moreover, the distance from this line to S is of the order of t. Hence $M \le M_\alpha G(\zeta)/t$, with G as in Lemma 2.6. It follows that

$$|f(1-t,0,\ldots,0)| \le Nh(x) + M_\alpha G(\zeta).$$

The same argument will show that if z_1 lies in a nontangential region with vertex at 1, then

$$|f(z_1,0,\ldots,0)| \le C[Nh(x) + M_\alpha G(\zeta)].$$

Finally, from Lemma 2.2 we have, for $2 \le j \le n$,

$$\left|\frac{\partial f}{\partial z_j}(z)\right| \le (1-|z|)^{-1/2} M_\alpha G(\zeta) \quad \text{for } z \in D_\alpha(\zeta).$$

From this it finally follows that

$$M_\alpha f(\zeta) \le C[Nh(x) + M_\alpha G(\zeta)].$$

It remains only to prove the norm estimate. The estimate for $M_\alpha G(\zeta)$ follows from Lemma 2.6. So we need only show that

$$\int_a^b Nh(x)^p\, dx \le C\|R^\beta f\|_{H^p}^p.$$

From Lemma 2.3 we have

$$\int_a^b Nh(x)^p\,dx \le C\sum_{l\le k}\int_Q |h^{(l)}(z)|^p y^{kp-1}\,dx\,dy.$$

Now

$$|h^{(l)}(z)| \le C\sum_{|\gamma|\le k} |(D^\gamma f)(\Phi(z))| + \left|\frac{\partial^l u}{\partial z^l}(z)\right|.$$

The first term is handled by Lemma 2.5 and the argument given in the first part of this proof. We can deal with $\partial^l u/\partial z^l$, $l\le k$, by a trivial pointwise estimate. We can solve the equation

$$\frac{\partial u}{\partial \bar z} = \frac{\partial}{\partial \bar z}(f\circ\Phi)$$

with

$$\left\|\frac{\partial^l u}{\partial z^l}\right\|_\infty \le C\left\|\frac{\partial^{l+1}}{\partial z^l\partial \bar z}(f\circ\Phi)\right\|_\infty, \quad l\le k.$$

Recall that

$$\frac{\partial}{\partial \bar z}(f(\Phi(z)) = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(\Phi(z))\frac{(iy)^N}{N!}\varphi_j^{(N+1)}(x)$$

and hence

$$\left|\frac{\partial^{l+1} u}{\partial z^l\partial \bar z}(f\circ\Phi)(z)\right| \le C\sum_{|\gamma|\le k+1} |(D^\gamma f)(\Phi(z))|y^{N-k}, \quad l\le k.$$

If $R^\beta f\in H^p$, then $f\in H^p$ and hence

$$\begin{aligned}|D^\gamma f(\Phi(z))| &\le (1-|\Phi(z)|)^{-(n/p+k+1)}\|R^\beta f\|_{H^p}\\ &\le y^{-(n/p+k+1)}\|R^\beta f\|_{H^p}, \quad |\gamma|\le k+1.\end{aligned}$$

Thus, for sufficiently large N we have

$$\left\|\frac{\partial^l u}{\partial z^l}\right\|_\infty \le C\|R^\beta f\|_{H^p}. \qquad \square$$

Added in proof: We want to point out that the method of proof of Theorems 1.1 and 1.2 gives immediately a characterization of the Carleson measures for $H^p_\beta(B^n)$, $0<p\le 1$. More precisely,

THEOREM. *Suppose $0<p\le 1$, $n-\beta p = m>0$, and μ is a positive Borel measure on B^n; then there is a constant C such that $\int |f|^p\,d\mu \le C\|R^\beta f\|_p^p$ for all $f\in H^p_\beta(B^n)$ if and only if*

$$(*)\qquad \mu(T(B(\zeta,\delta)))\le C\delta^m$$

for some constant C and for all $\zeta\in S$, $\delta>0$.

Proof. To prove the sufficiency of the condition $(*)$ we take $f\in H^p_\beta(B^n)$. As in the proof of Theorem 1.1, we see that $|f(z)|^p\le C\sum \lambda_k\chi_k\delta_k^{-m}\chi_k(z)$, where now $\chi_k(z)$ is the characteristic function of $T(B(\zeta_k,\delta_k))$. Now integrate on μ to obtain the result.

In the other direction, just test μ against functions of the form $(1-r\langle z, \zeta\rangle)^{-a}$ and argue as in the proof of Theorem 1.2. $\square$

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