A SIMPLIFIED TREATMENT OF THE STRUCTURE OF SEMIGROUPS OF PARTIAL ISOMETRIES

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Preliminaries. One-parameter semigroups of partial isometries were studied in [3] and [4], where a complete structure theorem was obtained for the nilpotent case. In this note, we offer a new and simplified treatment; moreover we are able to dispense with the assumption that the semigroup is nilpotent. We begin with a brief review of our original approach.

Let S_t $(0 \leq t < \infty)$ be a strongly continuous one-parameter semigroup of partial isometries on a separable Hilbert space H. We call S_t *nilpotent* if $S_{t_0} = 0$ for some t_0; we call the smallest such t_0 the *index* of S_t, and we denote it by $i(S_t)$. If K is a separable Hilbert space and $\alpha > 0$, we denote by $L^2(K, \alpha)$ the Hilbert space of measurable K-valued functions on $[0, \alpha]$ with square-integrable K-norm. For $f \in L^2(K, \alpha)$, define

$$R_t f(x) = \begin{cases} 0 & \text{if } x < t, \\ f(x-t) & \text{if } t \leq x \leq \alpha, \end{cases}$$

with the understanding that $R_t \equiv 0$ if $t \geq \alpha$. Then R_t is a nilpotent semigroup of partial isometries, and $i(R_t) = \alpha$. We say that S_t is a *truncated shift* if it is unitarily equivalent to some R_t semigroup. For each operator A, we denote by ran A and ker A the range and null-space, respectively, of A. The statement $B \longleftrightarrow C$ means that B and C commute.

In [3], the following theorem was proved.

THEOREM A. *If* S_t *is a semigroup of partial isometries and* $i(S_t) = \alpha$, *then the following statements are equivalent:*

(a) S_t *is a truncated shift,*

(b) *the von Neumann algebra generated by the* S_t $(0 \leq t \leq \alpha)$ *is a factor,*

(c) *for each* t $(0 \leq t \leq \alpha)$, $\operatorname{ran} S_t = \ker S_{\alpha - t}$.

The difficult step turned out to be the implication (b) $\Rightarrow$ (c). This was effected by a laborious argument based on the structure of the discrete case given in [1], and involving a fairly delicate limiting argument in passing to the continuous case. Using Theorem A and the reduction theory for von Neumann algebras, we proved in [4] that each nilpotent S_t is the direct integral of truncated shifts. In the treatment given below, we avoid the difficult transition from the discrete to the continuous. The argument is almost (but not quite) self-contained. The punch line of the proof is a novel characterization of truncated shifts, which is embodied in the Lemma at the end of the proof. The argument used in the Lemma is disjoint from that used in the rest of the paper, so that the reader can have the denouement at the outset.

THEOREM. *Let* S_t *be a semigroup of partial isometries on* H. *Then*

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$$H = H_0 \oplus H_1 \oplus H_2 \oplus H_3,$$

where each H_i *reduces* S_t *and* $S_t \mid H_1$ *is invertible,* $S_t \mid H_2$ *is purely isometric,* $S_t \mid H_3$ *is purely coisometric, and* H_0 *has a direct integral decomposition relative to which* S_t *decomposes into truncated shifts.*

Proof of the Theorem. For $0 \leq t < \infty$, let E_t and F_t be the projections on the initial and final spaces of S_t, and set K_t and C_t equal to $1 - E_t$ and $1 - F_t$, respectively. Then

$$E_t \longleftrightarrow F_u \text{ for all } t \text{ and } u, \tag{1}$$

$$S_t K_u = \begin{cases} 0 & \text{if } t \geq u, \\ K_{u-t} S_t & \text{if } t < u, \end{cases} \tag{2}$$

$$S_t C_u = C_{t+u} S_t. \tag{3}$$

The relation (1) follows from Lemma 2 of [1]. The proof of (2) is like that of Lemma 3 of [1], but we give it anyway. If $t \geq u$, then

$$S_t E_u = S_t S_u^* S_u = S_{t-u} S_u S_u^* S_u = S_{t-u} S_u = S_t,$$

while if $t < u$,

$$S_t E_u = S_t S_t^* S_{u-t}^* S_{u-t} S_t = F_t E_{u-t} S_t = E_{u-t} F_t S_t = E_{u-t} S_t.$$

The relation (2) follows immediately from these relations, and (3) comes from similar computations.

The families of projections E_t and F_t are strongly continuous and decreasing. Set $E = \lim E_t$ and $F = \lim F_t$, the limits being taken in the strong operator topology as $t \to \infty$. From the proof of (2) given above, it follows immediately that $S_t \longleftrightarrow E$, so that ran E is a reducing subspace for S_t and so that S_t restricted to ran E is isometric. Likewise, ran F reduces S_t, and S_t on ran F is coisometric. We write

$$H_1 = \text{ran } E \cap \text{ran } F, \qquad H_2 = \text{ran } E \ominus \text{ran } F, \qquad H_3 = \text{ran } F \ominus \text{ran } E.$$

Set $H_0 = H \ominus (H_1 \oplus H_2 \oplus H_3)$. Then $\lim E_t = \lim F_t = 0$ on H_0. For the remainder of the proof, set $H_0 = H$; we may then suppose that $\lim K_t = \lim C_t = 1$. Hence, K_t and C_t describe (continuous) resolutions of the identity.

Now define strongly continuous semigroups L_s and M_s $(s \geq 0)$ by

$$L_s = \int_0^\infty e^{-su}\, dK_u \quad \text{and} \quad M_s = \int_0^\infty e^{-su}\, dC_u.$$

By (2), for each $t \geq 0$,

$$S_t L_s = \left(\int_t^\infty e^{-su}\, dK_{u-t} \right) S_t = e^{-st} L_s S_t. \tag{4}$$

Similarly, (3) gives the equations

$$
\begin{aligned}
S_t M_s &= \left(\int_0^\infty e^{-su}\, dC_{u+t}\right) S_t \\
(5) \qquad &= e^{st}\left(\int_0^\infty e^{-su}\, dC_u = \int_0 e^{-su}\, dC_u\right) S_t = e^{st} M_s S_t,
\end{aligned}
$$

because $C_u S_t = 0$ for $u < t$.

Since $L_s \longleftrightarrow M_s$, the family $N_s = L_s M_s$ forms a strongly continuous semigroup, and by virtue of (4) and (5) we see that $S_t \longleftrightarrow N_s$ for all $s, t \geq 0$. By a theorem of Sz.-Nagy [2, p. 588], we can write $N_s = \int_0^\infty e^{-su} dG_u$, since $0 \leq N_s \leq 1$. Note also that L_t, M_t, S_t, K_t, and C_t all commute with G_u.

Let I be a finite interval $[a, b]$, and set $H' = \operatorname{ran} G(I)$. Then H' reduces L_t, M_t, S_t, K_t, and C_t. If A is an operator leaving H' invariant, set $A' = A \mid H'$. Now

$$L_s' \geq L_s' M_s' = N_s' \geq e^{-bs} \quad \text{and} \quad M_s' \geq e^{-bs},$$

so that $dK_u' = dC_u' = 0$ for $u > b$ (that is, the spectral measures K' and C' are supported in $[0, b]$).

Now define

$$(6) \qquad A = \int_0^b u\, dK_u', \qquad B = \int_0^b u\, dC_u', \qquad R = \int_0^b u\, dG_u'.$$

Since $e^{-sA} e^{-sB} = L_s' M_s' = N_s' = e^{-sR}$, it follows that $A + B = R$. Moreover, since $dK_u' = 0$ for $u > b$, it follows that S_t' is nilpotent and $i(S_t') \leq b$. All of this shows that we can write H as the direct sum of subspaces, each of which reduces S_t, and such that the restriction of S_t to each is nilpotent. To obtain the theorem, we may assume that S_t is nilpotent to begin with, say of index b. We also drop the "prime" notation introduced above.

We now take a direct integral decomposition that diagonalizes the bounded operator R. That is, we write

$$H = \int \oplus H(\lambda)\, d\mu(\lambda) \quad \text{and} \quad R = \int \oplus \phi(\lambda)\, d\mu(\lambda),$$

where $\phi(\lambda)$ is a scalar multiple of the identity on $H(\lambda)$, and where $0 \leq \phi(\lambda) \leq b$. Since S_t is nilpotent, we can write

$$S_t = \int \oplus S_t(\lambda)\, d\mu(\lambda),$$

where each $S_t(\lambda)$ is a semigroup of partial isometries on $H(\lambda)$ with $i(S_t(\lambda)) \leq b$ (see Theorem 2 of [4]). Moreover, for each $t \geq 0$, the fields $\lambda \to K_t(\lambda)$ and $\lambda \to C_t(\lambda)$ are measurable, and since $i(S_t(\lambda)) \leq b$, we see that for each λ, $d_t K_t(\lambda) = 0$ for $t \geq b$. We can then write

$$K_t = \int \oplus K_t(\lambda)\, d\mu(\lambda) \quad \text{and} \quad C_t = \int \oplus C_t(\lambda)\, d\mu(\lambda), \tag{7}$$

since $K_t(\lambda) = 1 - S_t^*(\lambda) S_t(\lambda)$ and $C_t(\lambda) = 1 - S_t(\lambda) S_t^*(\lambda)$. Now define

$$A(\lambda) = \int u\, dK_u(\lambda) \quad \text{and} \quad B(\lambda) = \int u\, dC_u(\lambda). \tag{8}$$

Since

$$A(\lambda) = \text{strong-lim} \sum u_i\, \Delta K_{u_i}(\lambda) \quad \text{for each } \lambda \quad (|\Delta u_i| \to 0)$$

and

$$\int \oplus \left(\sum u_i\, \Delta K_{u_i}(\lambda)\right) d\mu(\lambda) = \sum u_i\, \Delta K_{u_i} \to A,$$

we see that $A = \int \oplus A(\lambda)\, d\mu(\lambda)$ and likewise $B = \int \oplus B(\lambda)\, d\mu(\lambda)$. Since $A + B = R$, we get the relation

$$A(\lambda) + B(\lambda) = \phi(\lambda) \quad \text{d.c.}(\mu). \tag{9}$$

The theorem is now a consequence of the following result.

LEMMA. *Let* S_t *be a nilpotent semigroup of partial isometries. Set*

$$A_0 = \int u\, dK_u \quad \text{and} \quad B_0 = \int u\, dC_u.$$

If $A_0 + B_0 = \xi 1$, *then* S_t *is a truncated shift and* $i(S_t) = \xi$.

Proof. Since $A_0 \geq 0$ and $B_0 \geq 0$, we see that $dK_u = dC_u = 0$ for $u \geq \xi$. Hence $i(S_t) = \eta \leq \xi$. But then

$$\text{ran } S_t \subseteq \ker S_{\eta - t} \quad \text{for } 0 \leq t \leq \eta.$$

In other words

$$K_{\eta - t} + C_t \geq 1.$$

If $0 \leq t \leq \eta$, then $K_{\xi - t} \geq K_{\eta - t}$ and therefore $K_{\xi - t} + C_t \geq 1$. If $\eta \leq t \leq \xi$, then $C_t = 1$ and therefore $K_{\xi - t} + C_t \geq 1$. Hence, for all t $(0 \leq t \leq \xi)$, we have the relation

$$K_{\xi - t} + C_t \geq 1. \tag{10}$$

Integrating by parts, we obtain the relations

$$(A_0 x, x) = \int_0^\xi t\, d(K_t x, x) = \xi (K_\xi x, x) - \int_0^\xi (K_t x, x)\, dt \tag{11}$$

$$= \xi \|x\|^2 - \int_0^\xi (K_{\xi - t} x, x)\, dt$$

and

$$(B_0 x, x) = \xi \|x\|^2 - \int_0^{\xi} (C_t x, x)\, dt . \tag{12}$$

Adding (11) and (12), we see that

$$\int_0^{\xi} ((K_{\xi-t} + C_t) x, x)\, dt = \xi \|x\|^2 . \tag{13}$$

But then (10) implies that for all t $(0 \leq t \leq \xi)$,

$$K_{\xi-t} + C_t = 1$$

and therefore

$$\text{ran } S_t = \ker S_{\xi-t} .$$

It now follows from the implication (c) $\Rightarrow$ (a) of Theorem A (see p. 748 of [3]) that S_t is a truncated shift of index ξ. This concludes the proof of the theorem.

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