SINGULAR INNER FUNCTIONS WITH DERIVATIVE IN B^p

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For $0 < p < 1$, the space B^p is the class of all functions $f(z)$ analytic in $D = \{z: |z| < 1\}$ for which

$$\|f\|_p = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})| (1 - r)^{1/p-2} d\theta\, dr < \infty .$$

For basic properties of B^p, see [4].

A singular inner function is a function of the form

$$S_\mu(z) = \exp \int \frac{z + e^{it}}{z - e^{it}} d\mu(e^{it}) ,$$

where μ is a positive measure on the unit circle, singular with respect to Lebesgue measure. For a discussion of inner functions, see [3] or [5].

In [1], J. G. Caughran and A. L. Shields asked whether there exists a singular inner function with derivative in the Hardy class $H^{1/2}$. M. R. Cullen [2] conjectured that no singular inner function has derivative in the larger space $B^{1/2}$. In this paper, we disprove the conjecture of Cullen but leave open the question of Caughran and Shields.

THEOREM. *Let the measure* μ *consist of discrete masses* a_j *such that the sequence* $\{a_j\}_{j=1}^\infty$ *belongs to some space* $\ell^{1/q}$ $(1 < q < \infty)$, *and let*

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad \gamma < \frac{2p}{4p - 1}.$$

Then $S'_\mu \in B^\gamma$. *In particular,* $S'_\mu \in B^{1/2}$.

Proof. The formula $S_\mu(z) = \exp \sum_{j=1}^\infty a_j \frac{z + e^{it_j}}{z - e^{it_j}}$ implies that

$$S'_\mu(z) = S_\mu(z) \sum_{j=1}^\infty \frac{-2a_j e^{it_j}}{(z - e^{it_j})^2}.$$

Since

$$\Re \frac{z + e^{it_j}}{z - e^{it_j}} = 1 - 2 \frac{1 - r\cos(\theta - t_j)}{|z - e^{it_j}|^2} \qquad (z = re^{i\theta}),$$

we have the formula

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$$|S_\mu(z)| = \exp\left(\sum_{j=1}^{\infty} a_j \left\{1 - 2\,\frac{1 - r\cos(\theta - t_j)}{|z - e^{it_j}|^2}\right\}\right)$$

$$= K \prod_{j=1}^{\infty} \exp\left\{-2a_j\,\frac{1 - r\cos(\theta - t_j)}{|z - e^{it_j}|^2}\right\},$$

where $K = \exp \sum_{j=1}^{\infty} a_j$. Since $1 - r\cos(\theta - t_j) > 0$ for each index j, we see that

$$|S_\mu(z)| < K \exp\left\{-2a_j\,\frac{1 - r\cos(\theta - t_j)}{|z - e^{it_j}|^2}\right\} \qquad (j = 1, 2, \cdots).$$

Hence

$$|S'_\mu(z)| \le K \sum_{j=1}^{\infty} \frac{2a_j}{|z - e^{it_j}|^2} \exp\left\{-2a_j\,\frac{1 - r\cos(\theta - t_j)}{|z - e^{it_j}|^2}\right\}.$$

Now, since $e^{-x} < x^{-1/p}$ for $x > 0$, we have the inequality

$$|S'_\mu(z)| \le 2K \sum_{j=1}^{\infty} \frac{a_j^{1/q}}{[1 - r\cos(\theta - t_j)]^{1/p}} \cdot \frac{1}{|z - e^{it_j}|^{2/q}}.$$

Using the identity

$$\int_0^{2\pi} \frac{d\theta}{1 - r\cos(\theta - t_j)} = \frac{2\pi}{\sqrt{1 - r^2}},$$

and applying Hölder's inequality to the integral

$$\int_0^{2\pi} \left(\frac{1}{1 - r\cos(\theta - t_j)}\right)^{1/p} \left(\frac{1}{|z - e^{it_j}|^2}\right)^{1/q} d\theta,$$

we obtain the bound

$$\|S'_\mu\|_\gamma < 4K\pi \left(\int_0^1 (1 - r)^{\frac{1}{\gamma} + \frac{1}{2p} - 3}\, dr\right) \sum_{j=1}^{\infty} a_j^{1/q}.$$

This proves the theorem.

The following result is an easy consequence of our theorem.

COROLLARY. *Let* $S_\mu(z)$ *be a singular inner function whose associated singular measure* μ *has finite support* $\{e^{it_j}\}_{j=1}^{n}$ *with* $\mu(e^{it_j}) = a_j$. *Then* $S'_\mu(z) \in B^p$ *for all* $p < 2/3$.

Remark 1. Consider the function

$$I_a(z) = \exp\left(a\,\frac{z+1}{z-1}\right) = \sum_{n=0}^{\infty} c_n z^n \qquad (a > 0).$$

D. J. Newman and H. S. Shapiro [6, p. 253] have shown that

$$nc_n = O(n^{1/4}).$$

Also, it follows from a result of P. L. Duren, B. W. Romberg, and A. L. Shields [4, Theorem 4, p. 41] that $I_a' \in B^p$ for all $p < \frac{2}{2\alpha + 3}$ provided

$$nc_n = O(n^{\alpha}).$$

Hence $I_a' \in B^p$ for all $p < 4/7$. We note that this example already disproves the aforementioned conjecture of Cullen.

Remark 2. The following example shows that our theorem cannot be extended to arbitrary singular inner functions whose associated singular measures are purely atomic. Let

$$f(z) = \exp \sum_{j=2}^{\infty} a_k \frac{z^{n_k}+1}{z^{n_k}-1},$$

where $a_k = k^{-1}(\log k)^{-3/2}$. G. Piranian [7] has shown that $f'(z) \notin B^{1/2}$ provided $n_k \to \infty$ sufficiently fast. Furthermore, in view of the facts that

$$\Re\, \frac{z^n+1}{z^n-1} < 0$$

for $|z| < 1$ and

$$\Re\, \frac{z^n+1}{z^n-1} = 0$$

for all z on the unit circle except the nth roots of unity, we see that $f(z)$ is a singular inner function.

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