IMPROPER EMBEDDINGS AND UNKNOTTINGS OF PL MANIFOLDS

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1. INTRODUCTION

We work throughout in the PL (piecewise linear) category, and we consider the following problem. Let M and Q be a PL n-manifold and a PL q-manifold, respectively, with $\partial M \neq \emptyset$, and suppose that $g: M \to Q$ is a map such that $g \mid \partial M$ is an improper PL embedding, in other words, that $g^{-1}(\text{int } Q) \cap \partial M \neq \emptyset$. Under what additional conditions can we assert that there exists a PL embedding $f: M \to Q$ such that $f \mid \partial M = g \mid \partial M$?

In Section 3, we prove the piping lemma; it enables us to modify singularities, and this yields a better dimensional relationship between the dimensions of M and Q. In Section 4 we prove (for the metastable range of dimensions) that if M and Q are sufficiently connected, then any g satisfying the condition above is homotopic rel ∂M to a PL embedding (Theorem 1). Also, we prove an appropriate unknotting statement, rel ∂M. In the proof of Theorem 1, we use the relative general-position approximation theorem to show that the given map g is homotopic rel ∂M to a general-position PL map. Performing piping, we modify the character of the singularities. We then combine the cone technique of R. Penrose, J. H. C. Whitehead, and E. C. Zeeman [6] with C. P. Rourke's theorem on cone-unknotting of spheres in balls [7, p. 305] in order to remove all singularities at once. Theorem 2 in Section 5 specializes Theorem 1 to Euclidean spaces, and it follows immediately from Theorem 1. In Section 5, we present some other consequences and results. Finally, in Section 6 we consider disconnected bounded PL manifolds (it being understood that each component is bounded), and in Theorem 4 we prove that if ∂M_1 is inessential in the complement of ∂M_2, and vice versa, and if M_1 and M_2 have sufficiently high connectivity, then any PL embedding of ∂M extends to a PL embedding of M. We show that the condition is not necessary.

Let us point out that the main difference between our embedding theorems and known embedding theorems for bounded manifolds (see [2] and [6]) is that we keep the boundary fixed. In our proofs, we start with a map that is an improper PL embedding on ∂M, and we construct a PL embedding of M, which turns out to be an extension of the embedding of ∂M.

2. NOTATION AND DEFINITIONS

By E^n, B^n, and S^n we denote Euclidean n-space, the PL n-ball, and the PL n-sphere, respectively. Furthermore, we denote the interior of a manifold M by int M, and the boundary of M by ∂M. We say that M is *bounded* if $\partial M \neq \emptyset$, and that

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M is *closed* if M is compact and $\partial M = \emptyset$. The manifold M is *open without boundary* if M is noncompact and $\partial M = \emptyset$, and M is *open and bounded* if M is noncompact and $\partial M \neq \emptyset$. A manifold is usually understood to be connected as a topological space, unless the contrary is specified.

If a polyhedron X collapses to the subpolyhedron Y, we write $X \searrow Y$, and we say that Y is a *spine* of X.

Let K and L be PL subspaces of the PL manifold M, and let $n = \dim M$. Then K and L are in *general position* if $\dim(K \cap L) \leq \dim K + \dim L - n$.

Corresponding to a map $f: P \to Q$, we define the sets

$$S'_r = \{x \in P \mid f^{-1}f(x) \text{ has at least } r \text{ points}\}$$

and $S_r(f) = \mathrm{cl}\, S'_r(f)$. The set $S_2(f)$ is called the *singular set* of the map f.

If P and Q are PL spaces, P is compact and f is a PL map, then $S'_r(f)$ is a PL subspace of P, and $S_r(f)$ is a closed PL subspace of P with $\dim S_r(f) = \dim S'_r(f)$.

If P and Q are PL spaces of dimension p and q, respectively, we say that a map $f: P \to Q$ is in *general position* provided

1) f is a PL map,

2) for all r, $\dim S'_r(f) \leq rp + (1 - r)q$,

3) $S_\infty(f) = \emptyset$ (that is, f is not degenerate).

A point $x \in P$ is a *branch point* of f if no neighborhood of x is embedded by f. We denote by B(f) the *branch set* (also called the branch locus or the set of branch points) of f.

Let $f: M^n \to N^k$ be an embedding of one manifold in another. The embedding f is *proper* if $f^{-1}(\partial N^k) = \partial M^n$, and *semiproper* if $f^{-1}(\partial N^k) \subset \partial M^n$. If we use no adjective, we do not assume that the embedding is either proper or semiproper. Analogously, we talk about *proper* and *semiproper* maps. Furthermore, if $f: Q \to N^k$ is an embedding of a cone Q in a manifold N^k, we say that f is *proper* if $f^{-1}(\partial N^k)$ is the base of Q, and *semiproper* if $f^{-1}(\partial N^k)$ is contained in the base of Q. A *proper subcone* (or simply a subcone) R of Q is the cone in Q on a subset of the base of Q, and a *semiproper* or *partial* subcone is the cone in Q on any subset of Q.

The set of spaces

$$\{\Delta^{m+n-q} * \partial\Delta^{q-n},\ \Delta^{m+n-q} * \partial\Delta^{q-m},\ \Delta^{m+n-q} * \partial\Delta^{q-n} * \partial\Delta^{q-m}\}$$

will be called the *standard triple* of the type (m, n, q). It consists of an m-ball and an n-ball, each properly embedded in a q-ball. If M and N are PL submanifolds of Q and $x \in M \cap N$, we say that M and N are *transverse* at x if there exists a closed neighborhood B of x in Q such that $\{B \cap M, B \cap N, B\}$ is PL homeomorphic to the standard triple of the type (m, n, q).

Let $f: M \to Q$ be a semiproper, nondegenerate PL map such that $f \mid \partial M$ is a PL embedding. A point $x \in M$ is a *nice double point* of f if $f^{-1}f(x)$ consists of two points x and x' with neighborhoods U and U' such that $f \mid U$ and $f \mid U'$ are embeddings and f(U) and f(U') are transverse in Q at f(x). We denote by H(f) the set $S_2(f)$ minus the set of nice double points.

3. THE PIPING LEMMA

In this section, we prove the "piping" lemma, which enables us to modify singularities.

LEMMA 1. *If* $f: M \to Q$ *is a continuous map,* M *and* Q *being* PL *manifolds* (M *compact,* $\dim M \leq \dim Q$), *and* $f \mid \partial M$ *is a* PL *embedding, then* $f \simeq g$ (rel ∂M) *for some* PL, *nondegenerate, and semiproper map* g *such that*

$$\dim H(g) \leq 2n - q - 1 \quad \textit{and} \quad \dim(S_2(g) \cap \partial M) \leq 2n - q - 1 .$$

We can obtain this lemma by applying directly the techniques of general-position approximation theory (see for example J. F. P. Hudson [2, Chapter IV], especially the inductive step in the proofs of Lemmas 4.6 and 4.7). Our lemma is almost the same as Hudson's Lemma 2.3 in [1, p. 18]. Because we do not begin with a proper PL map, we allow the possibility that $S_2(g) \cap \partial M \neq \emptyset$; however, in our case these singularities are either branch points or they come from identifications of boundary points of M with some interior points of M.

LEMMA 2. *Let* M *be a compact* PL n-*manifold with* $\partial M \neq \emptyset$, *and let* Q *be a* PL q-*manifold* ($q \geq n + 3$). *Let* $f: M \to Q$ *be a continuous map such that* $f \mid \partial M$ *is a* PL *embedding and*

$$f^{-1}(\text{int } Q) \cap \partial M = R \neq \emptyset .$$

Then $f \simeq g'$ (rel ∂M) *for some* PL *general-position map* g' *such that* g' *is semiproper, and with the property that* $S_2(g')$ *has a* $(2n - q - 1)$-*dimensional spine, say* P, *and* $g'S_2(g') \searrow g'(P)$.

Before proving the lemma, we present a brief geometric description of the idea of the proof. We do it by means of two figures, which are inadequate to the real situation but illustrate the procedure suggestively. Both figures present the configuration in the ambient manifold; the domain is not drawn. In Figure 1, we have drawn a transversal intersection of two sheets, a piece of the boundary $g(\partial M)$, a PL segment $g(L)$ connecting the nice intersection point $g(a)$ with a point of $g(\partial M)$, and a small regular neighborhood $D_b = g(B_b)$ of $g(a)$ inside the horizontal sheet. The n-ball D_b can be considered as a cone over its boundary. If we choose a new vertex of this cone outside of $g(M)$ (say above $g(L)$) and make a cone from this point over

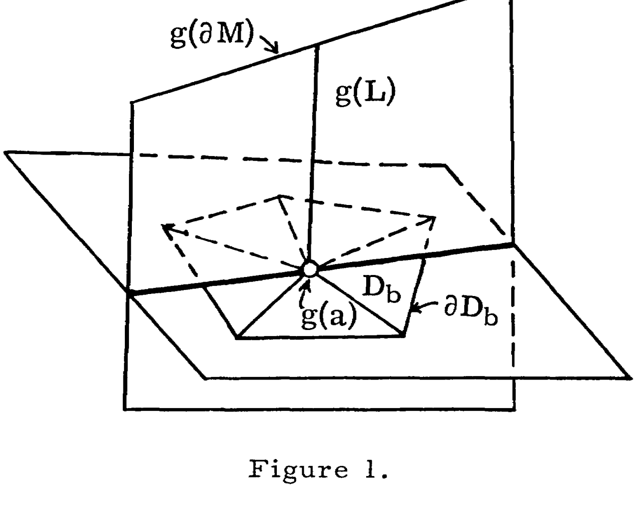

Figure 1.

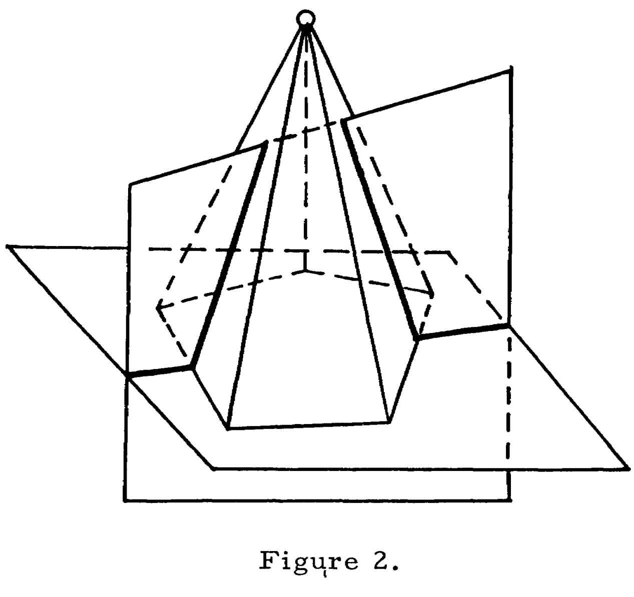

Figure 2.

∂D_b, we get a configuration similar to this in Figure 2. The effect of this step is that we change the intersection between these two sheets, or, roughly speaking, we punch a hole in the intersection part of the sheets. This amounts to punching two holes in the singular set. If we redefine g on $B_b = g^{-1}(D_b)$ in an obvious way, we obtain a new map whose singular set has the additional properties states in the lemma.

Proof of Lemma 2. First we apply Lemma 1 to obtain a map g with the properties described in Lemma 1. If it happens that $\dim S_2(g) \leq 2n - q - 1$, the proof is finished. Therefore, suppose that $\dim S_2(g) = 2n - q$. Consider subdivisions of M and Q such that $g: M \to Q$ is simplicial, and such that $S_2(g)$ and $g(S_2(g))$ are subcomplexes of M and Q. By the properties of g, the sets $H(g)$ and $B(g)$ are contained in the $(2n - q - 1)$-skeleton of $S_2(g)$, and therefore each interior point of a top-dimensional simplex of $S_2(g)$ is a nice double point. Furthermore,

$$\dim (R \cap S_2(g)) \leq 2n - q - 1 \leq (n - 1) - 3,$$

and since $\dim R = n - 1$, this intersection is nowhere dense in R. Denote by $(\Delta_1^1, \Delta_1^2), \cdots, (\Delta_r^1, \Delta_r^2)$ the pairs of $(2n - q)$-dimensional simplexes identified by g, so that $g(\Delta_i^1) = g(\Delta_i^2)$. Notice that $\operatorname{int} \Delta_i^j \subset \operatorname{int} M$ ($j = 1, 2$; $i = 1, 2, \cdots, r$), and that $R - S_2(g) \neq \emptyset$ for each component of R. We now carry out a sequence of modifications that will give the desired effect. We describe one step.

Choose $a_i \in \operatorname{int} \Delta_i^1$ and $b_i \in \operatorname{int} \Delta_i^2$ so that $g(a_i) = g(b_i)$ for $i = 1, 2, \cdots, r$. (For example, we can choose the barycenters of the simplexes Δ_i^j). Connect each point a_i by a polygonal arc L_i with some point $c_i \in R$ in such a way that

$$L_i \cap S_2(g) = a_i \quad (i = 1, 2, \cdots, r) \qquad \text{and} \qquad L_i \cap L_k = \emptyset \quad (i \neq k).$$

Furthermore, each $g(L_i)$ is also a polygonal arc. It connects the point $g(a_i) = g(b_i)$ with $g(c_i)$, and $g(L_i) \cap g(L_k) = \emptyset$ for $i \neq k$. Subdivide M and Q again so that g is simplicial and each L_i is a subcomplex. Then the points a_i, b_i, c_i, $g(a_i) = g(b_i)$, and $g(c_i)$ are vertices of the above triangulations of M and Q. Let B_{bi}, B_{Li}, and B_i be simplicial neighborhoods of b_i, L_i, and $g(L_i)$, respectively, in the second barycentric subdivisions of these triangulations. To simplify the notation, we drop the subscript i for the remainder of the proof. Using Whitehead's regular-neighborhood theorem, we see that

(1) B_b and B_L are n-balls, and B is a q-ball;

(2) $g(\partial B_b) \subset \partial B$ and $g(\operatorname{int} B_b) \subset \operatorname{int} B$; that is, B_b is properly embedded in B, and $B_b \cap \partial M = \emptyset$;

(3) $g(\operatorname{Fr}_M B_L) \subset \partial B$ and $g(\operatorname{int} B_L) \subset \operatorname{int} B$; that is, B_L is semiproperly embedded into B,

(4) $B_c = \partial B_L - \operatorname{int}(\operatorname{Fr}_M B_L)$ is an $(n - 1)$-ball, since it is a regular neighborhood of the point c in ∂M.

By the Alexander-Newman theorem, we see that the set $\operatorname{Fr}_M B_L = \partial B_L - \operatorname{int} B_c$ is an $(n - 1)$-ball, and that by (3) it is embedded into ∂B. Furthermore, the intersection $B_b \cap S_2(g) = C_b$ is a $(2n - q)$-ball as the second derived neighborhood of b in Δ^2, which is properly embedded in B_b; analogously, $B_L \cap S_2(g) = C_L$ is a $(2n - q)$-ball, as the second derived neighborhood of the point a in Δ^1, which is properly embedded in B_L, and $g(C_b) = g(C_L)$. Since a is an interior point of Δ^1 and $a \in \operatorname{int} M$, we see that $\partial C_L \subset \operatorname{int} \operatorname{Fr}_M B_L$. Write

$$D_b = g(B_b), \quad D_L = g(B_L), \quad D = g(C_b) = g(C_L), \quad F = g(\mathrm{Fr}_M B_L).$$

Because g is a general-position map and a is a nice double point, we can conclude that

(5) $D = g(C_b) = g(C_L) = B \cap g(S_2(g))$ is a $(2n - q)$-ball properly embedded in B, and

$$D = D_b \cap D_L, \quad \partial D = S^{d-1} = \partial D_b \cap \partial D_L \subset \mathrm{int}\, F.$$

Now we are ready to perform the inductive step. Consider $F \subset \partial B$ as an $(n - 1)$-ball embedded in the $(q - 1)$-sphere ∂B. Since F is link-collapsible on its boundary, we can consider a relative regular neighborhood of F mod ∂F in ∂B (Hudson and Zeeman [3]). Denote by A this neighborhood, which is a $(q - 1)$-ball in ∂B, and observe that F is properly embedded in it. Furthermore, $\mathrm{cl}(\partial B - A)$ is also a $(q - 1)$-ball, by the Alexander-Newman theorem. Since $q - n \geq 3$, we see that $(q - 1) - (n - 1) \geq 3$, and by Zeeman's unknotting theorem for proper ball pairs, the pair (A, F) is an unknotted proper ball pair. Therefore there exists a PL homeomorphism

$$h_0: (A, F) \to (\Delta^{n-1} * \partial\Delta^{q-n}, \Delta^{n-1})$$

onto the standard proper ball pair $(\Delta^{n-1} * \partial\Delta^{q-n}, \Delta^{n-1})$ of type $(q - 1, n - 1)$. The restriction $h_0 \mid \partial A$ maps $\partial A = \partial(\partial B - \mathrm{int}\, A)$ PL homeomorphically onto $\partial(\Delta^{n-1} * \partial\Delta^{q-n})$. Since $\Delta^{n-1} * \partial\Delta^{q-n}$ is a $(q - 1)$-convex linear cell, we consider the $(q - 1)$-linear space E^{q-1} spanned by it. Let u be a point on the straight line E^1 perpendicular to E^{q-1} that intersects E^{q-1} in the barycenter $b\Delta^{n-1}$ of $\Delta^{n-1} \subset \Delta^{n-1} * \partial\Delta^{q-n}$. Let $u * (\Delta^{n-1} * \partial\Delta^{q-n})$ be the cone with vertex u over $\Delta^{n-1} * \partial\Delta^{q-n}$, which is a q-ball. The subcone $u * \partial(\Delta^{n-1} * \partial\Delta^{q-n})$ is a $(q - 1)$-ball. Now, the PL homeomorphism $h_0 \mid \partial A$ extends to a PL homeomorphism

$$h_1: \partial B - \mathrm{int}\, A \to u * \partial(\Delta^{n-1} * \partial\Delta^{q-n}),$$

since $\partial B - \mathrm{int}\, A$ as a $(q - 1)$-ball can be interpreted as a cone over its boundary. Together, h_0 and h_1 define the PL homeomorphism

$$h_2 = h_0 \cup h_1: \partial B \to \Delta^{n-1} * \partial\Delta^{q-n} \cup u * \partial(\Delta^{n-1} * \partial\Delta^{q-n}) = \partial(u * (\Delta^{n-1} * \partial\Delta^{q-n})).$$

By the same argument, h_2 extends to a PL homeomorphism

$$h: B \to u * (\Delta^{n-1} * \partial\Delta^{q-n}).$$

Consider a point v on the perpendicular straight line E^1 between $b\Delta^{n-1}$ and u. The cone $v * \Delta^{n-1}$ is an n-ball (a rectilinear n-simplex), and $v * \Delta^{n-1}$ is a properly embedded cone in the q-ball $u * (\Delta^{n-1} * \partial\Delta^{q-n})$ having Δ^{n-1} as its base. Because of (4), $\mathrm{Fr}_M B_L$ is a face of the n-ball B_L, and B_L can be interpreted as a cone over its face $\mathrm{Fr}_M B_L$. Furthermore, the composition hg restricted to $\mathrm{Fr}_M B_L$ gives a PL homeomorphism

$$hg \mid \mathrm{Fr}_M B_L: \mathrm{Fr}_M B_L \to \Delta^{n-1} \subset \Delta^{n-1} * \partial\Delta^{q-n}$$

onto the simplex Δ^{n-1}, and it extends to a PL embedding

$$h_3: B_L \to v * \Delta^{n-1}$$

onto the n-simplex $v * \Delta^{n-1}$, since B_L can be interpreted as a cone over $\mathrm{Fr}_M B_L$. Now, the map

$$h_4 = h^{-1} h_3 : B_L \to B$$

is another semiproper embedding of B_L into B (the first is $g \mid B_L$), or if we consider B_L as a cone over $\mathrm{Fr}_M B_L$, it is another proper embedding of this cone into the ball B that agrees with g on the base; in other words,

$$h_4 \mid \mathrm{Fr}_M B_L = g \mid \mathrm{Fr}_M B_L,$$

since for $x \in \mathrm{Fr}_M B_L$ we have the relation

$$h_4(x) = h^{-1} h g(x) = g(x).$$

This will enable us to use Lickorish's unknotting theorem for properly embedded cones.

On the other hand, the composition hg restricted to ∂B_b embeds it into the $(q-1)$-sphere $\partial(u * (\Delta^{n-1} * \partial\Delta^{q-n}))$ in such a way that

$$hg(\partial B_b) \cap \Delta^{n-1} = hg(\partial C_b) = h(\partial D) \subset \operatorname{int} \Delta^{n-1}.$$

Since B_b is a cone over its boundary ∂B_b, we can use its cone structure to extend the map $hg \mid \partial B_b$ in an appropriate way. In order to do this, we choose a vertex w on the perpendicular straight line E^1 between v and u. Denote the extension by h_5; then $h_5 : B_b \to u * (\Delta^{n-1} * \partial\Delta^{q-n})$ is a proper embedding of the n-ball B_b. Notice that

$$h_5(B_b) \cap v * \Delta^{n-1} = w * (hg(\partial C_b)) \cap v * \Delta^{n-1}$$

collapses onto its base $hg(\partial C_b)$. Since

$$hg(\partial C_b) = hg(\partial C_L) \subset \operatorname{int} \Delta^{n-1},$$

the polyhedron $w * (hg(\partial C_b)) \cap v * \Delta^{n-1}$ is a truncated cone over $hg(\partial C_b)$, and therefore it is PL homeomorphic to $hg(\partial C_b) \times I$. This implies collapsibility to the base $hg(\partial C_b)$; that is, $(h_5)^{-1}(h_5(B_b) \cap v * \Delta^{n-1})$ collapses to ∂C_b, and analogously $(h_3)^{-1}(h_5(B_b) \cap v * \Delta^{n-1})$ collapses to ∂C_L.

Now our modification of the map is almost obvious. We shall keep g fixed outside $\operatorname{int} B_b$, and modify it in $\operatorname{int} B_b$, using the construction above, which will ensure that the new part of the singular set inside B_b collapses to ∂C_b and the new part of the singular set inside B_L collapses to ∂C_L. Let H_t be an ambient isotopy of B that keeps the boundary ∂B fixed and such that $g \mid B_L = H_1 h_4$. (Such an isotopy exists, by Lickorish's unknotting theorem [5, p. 70] for properly embedded cones.) Define a map $g' : M \to E^q$ by the formula

$$g'(x) = \begin{cases} g(x) & \text{for } x \in M - \operatorname{int} B_b, \\ H_1 h^{-1} h_5(x) & \text{for } x \in \operatorname{int} B_b. \end{cases}$$

Finally, let $g' : M \to Q$ be the map obtained after r steps. The map g' has the desired properties. From the construction of g' it is obvious that we have changed g only inside $\operatorname{int} M$, and therefore g is homotopic to g' (rel ∂M). Furthermore, in each step $S_2(g') \cap B_b$ is PL homeomorphic to $\partial C_b \times I$, and analogously $S_2(g') \cap B_L$ is PL homeomorphic to $\partial C_L \times I$, so that

$$S_2(g') \cap (B_b \cup B_L) \searrow \partial C_b \cup \partial C_L .$$

Since Δ^2 - int C_b is PL homeomorphic to $\partial\Delta^2 \times I$, by the regular-neighborhood annulus theorem (Hudson [2]), we deduce that it collapses to $\partial\Delta^2$. Analogously, Δ^1 - int C_L collapses to $\partial\Delta^1$. Similarly, $S_2(g')$ has a $(2n - q - 1)$-dimensional spine that is in fact a $(2n - q - 1)$-skeleton, say P, of $S_2(g)$. From the construction it is obvious that $g'(S_2(g')) \searrow g'(P)$. Since we perform a finite number of punchings, each time inside a q-ball in Q, every new map is homotopic to the previous, keeping ∂M fixed. This proves the lemma.

4. IMPROPER EMBEDDINGS IN THE METASTABLE RANGE

THEOREM 1. *Let* M *be a compact,* $(2n - q - 1)$-*connected* PL n-*manifold with* $\partial M \neq \emptyset$, *and let* Q *be a* $(2n - q)$-*connected* PL q-*manifold* $(q \geq n + 3)$. *Let* $f: M \to Q$ *be a continuous map such that* $f \mid \partial M$ *is a* PL *embedding and*

$$f^{-1}(\text{int } Q) \cap \partial M = R \neq \emptyset .$$

If $q > 3n/2$, *then* $f \simeq f'$ (rel ∂M), *where* f' *is a semiproper* PL *embedding.*

Furthermore, let $f_{0,1}: M \to Q$ *be a pair of embeddings as above such that* $f_0 \simeq f_1$ (rel ∂M), M *is* $(2n - q)$-*connected, and* Q *is* $(2n - q + 1)$-*connected* $(q \geq n + 3)$. *If* $q \geq 3n/2 + 1$, *then* f_0 *and* f_1 *are isotopic keeping* ∂M *fixed.*

Proof of the embedding part. By Lemma 2, $f \simeq g$ (rel ∂M), where g is a PL map such that

$$g(\text{int } M) \subset \text{int } Q, \qquad \dim S_2(g) \leq 2n - q,$$

$$S_2(g) \searrow P, \qquad g(S_2(g)) \searrow g(P), \qquad \dim P < 2n - q - 1 .$$

We now combine the Penrose-Whitehead-Zeeman cone technique with Rourke's cone-unknotting theorem for spheres in balls. Since M is $(2n - q - 1)$-connected, the inclusion map $i: P \to M$ is inessential. Therefore it extends to a continuous map $i: C \to M$ of a cone C over P. By a general-position argument, we can consider $i: C \to M$ as a semiproper PL embedding, since

$$2(2n - q) - n = 3n - 2q < 0 .$$

Denoting i(C) by C again, we get the result that $X = C \cup S_2(g) \searrow C \searrow 0$ is a collapsible polyhedron.

In Q, we can embed a cone D semiproperly over $Y = g(X)$, by the same argument, since Q is $(2n - q)$-connected and

$$2(2n - q + 1) - q \leq 3n - 2q - 1 < 0 .$$

Furthermore, we can ensure that $D \cap g(M) = Y$. Fulfillment of this condition is not an immediate consequence of the general-position argument, because, on account of the relation

$$2n - q + 1 + n - q = 3n - 2q + 1 \leq 0,$$

we get zero-dimensional intersections. Fortunately, we can remove intersections by piping them over points in R. (See a description of this procedure in the remark following this proof.)

We choose subdivisions of M and Q with respect to which g is simplicial and C and D are subcomplexes, and we denote by A and B the simplicial neighborhoods of C and D, respectively, in the second barycentric subdivisions of these triangulations. Then

(1) A is an n-ball and B is a q-ball,

(2) ∂A is embedded into B by g, since $g \mid \partial M$ is an embedding,

(3) $g^{-1}(B) = A$,

(4) $g \mid (M - \text{int } A)$ is an embedding.

The condition $q > 3n/2$ ensures the applicability of Rourke's theorem [7, p. 305] on the cone unknotting of an (n - 1)-sphere in a q-ball, which says that *any embedding of* S^n *in* B^k *is cone-unknotted* if $k > 3(n + 1)/2$. Since cone-unknottability implies spannability, we conclude that $g(\partial A)$ is spanned by an n-ball $A_1 \subset B$. Since A and A_1 are cones over their boundaries, we can extend the mapping $g \mid \partial A: \partial A \to A_1$ to a PL homeomorphism $\phi: A \to A_1$, and define a new map $f': M \to Q$ by

$$f'(x) = \begin{cases} g(x) & \text{for } x \in M - \text{int } A, \\ \phi(x) & \text{for } x \in \text{int } A. \end{cases}$$

We see that f' is an embedding, since we have introduced no new selfintersections and removed all the old ones. Furthermore, $f' \mid \partial M = f \mid \partial M$, since ∂M has been kept fixed at each step. Since we have changed the map g inside an n-ball, keeping it fixed outside the interior of this ball, with values in a q-ball, we get the relation $f \simeq f'$ (rel ∂M), and obviously $f'(\text{int } M) \subset \text{int } Q$. This proves the embedding part of the theorem. The theorem is dimensionally the best possible. In order to see this, we can use Rourke's example [7, p. 325] of an embedding of S^5 in B^9 that is not spannable by a 6-ball in B^9. Obviously, this embedding of S^5 in B^9 can not be extended to an embedding of B^6.

Remark 1. If $q \geq 2n$, we do not need the condition concerning codimension 3 in Theorem 1, since we can pipe eventual 0-dimensional singularities over R. For $n \geq 3$, Lemma 2 immediately gives us an embedding. The case $n = 1$ is obvious, and in the case $n = 2$ we can pipe the singularities, using the procedure described by Penrose, Whitehead, and Zeeman [6]. For the sake of completeness, we describe this procedure briefly. We start in the same way as in the proof of Lemma 2, getting B_b, B_L, and B with the same properties (1) to (4). Now proceed as follows: Consider the second barycentric subdivision of B. Denote by A the second derived neighborhood of $g(B_L)$ in B. Using again Whitehead's regular-neighborhood theorem, we see that A is a 2n-ball and that A intersects B in a common face F, which is the second derived neighborhood of $g(\text{Fr}_M B_L)$ in ∂B. Consider $C = \text{cl}(B - A)$, which is again a 2n-ball, by the Alexander-Newman theorem. Since B and C are cones over ∂B and ∂C, we can extend $g \mid \partial B_b$ to an embedding $\gamma: B_b \to C$. Now we define $g': M \to Q^{2n}$ by the formula

$$g'(x) = \begin{cases} g(x) & \text{for } x \in M - \operatorname{int} B_b, \\ \gamma(x) & \text{for } x \in \operatorname{int} B_b. \end{cases}$$

The map g' is obviously a well-defined PL map, the points a and b are now nonsingular points with respect to the map g', and no new singular points have been introduced. After a finite number of such steps, we obtain a PL map $f'\colon M \to Q$, which is obviously an embedding; moreover, $f' \mid \partial M = f \mid \partial M$. This completes the extension part of the proof in the case $q \geq 2n$.

Proof of the isotopy part. In order to prove the isotopy part, we shall construct a concordance.

Let $f\colon M \times I \to Q \times I$ be defined by the equation $f(x, t) = (f_t(x), t)$, where f_t is the assumed homotopy between f_0 and f_1. We notice that $f \mid \partial(M \times I)$ is a PL embedding. Moreover, we have the relations

(1) $f \mid M \times 0 = f_0$, $\quad f(M \times 0) \subset Q \times 0$,

(2) $f \mid M \times 1 = f_1$, $\quad f(M \times 1) \subset Q \times 1$,

(3) $f \mid \partial M \times I = f_t \mid \partial M \times \mathrm{id} = f_0 \mid \partial M \times \mathrm{id} = f_1 \mid \partial M \times \mathrm{id}$,

(4) $f(R \times I) \subset \operatorname{int}(Q \times I)$.

Because of (4), Lemma 2 applies, and we get the mapping $g\colon M \times I \to Q \times I$, where $f \simeq g$ (rel $\partial(M \times I)$), and $g(\operatorname{int}(M \times I)) \subset \operatorname{int}(Q \times I)$, with $S(g) \searrow P$ and $\dim P \leq 2n - q$. We can now proceed as in the part of the proof about the embedding, getting a concordance between f_0 and f_1. We omit the repetition of this argument. In order to prove the last statement about isotopy, we apply Hudson's Concordance Theorem [10, Theorem 9.8, p. 197]. This completes the proof of the theorem.

5. SOME COROLLARIES AND OTHER RESULTS

THEOREM 2. *Let* M *be a compact,* k-*connected, bounded* PL n-*manifold* ($n > 2k + 2$ *and* $n - k \geq 4$). *Then every* PL *embedding* $g\colon \partial M \to E^{2n-k-1}$ *extends to a* PL *embedding of* M. *Furthermore,* M *unknots* (rel ∂M) *in* E^{2n-k}.

Proof. The assumption $n - k \geq 4$ implies the condition concerning codimension 3, and the assumption $n > 2k + 2$ implies the inequality $4n > 2k + 3n + 2$, that is, $2n - k - 1 > 3n/2$. Since $\partial E^q = \emptyset$, we see that $R = \partial M$, and therefore Theorem 2 is a special case of Theorem 1. The proof of the unknotting part is obvious.

Concerning the existence of an embedding, we may state the next obvious corollary.

COROLLARY 1. *If* M *satisfies the conditions in Theorem* 2 *and* ∂M *can be embedded into* E^{2n-k-1}, *then* M *can also be embedded into* E^{2n-k-1}.

Comparing this corollary with Theorem 1.2, case (a) of Penrose, Whitehead, and Zeeman, we see that in their theorem we can drop the assumption that $\partial M \times I$ is embedded into E^{2n-k-1}.

COROLLARY 2. *Let* M *be a compact,* 1-*connected, bounded* PL n-*manifold* ($n \geq 5$). *Then* M *can be embedded into* E^{2n-2} *and unknotted* (rel ∂M) *in* E^{2n-1}.

Proof. That M can be embedded into E^{2n-2} is a consequence of Corollary 1; for by the theorem of Penrose, Whitehead, and Zeeman, ∂M can be embedded into E^{2n-2}. The unknotting part is a corollary of Theorem 2.

COROLLARY 3. *Let* M *be a compact* k-*connected* PL n-*manifold* ($n > 2k + 2$, $n - k \geq 4$) *such that each component of* ∂M *is* $(k - 1)$-*connected. Then* M *can be embedded in* E^{2n-k-1}.

Proof. Each component of ∂M is a $(k - 1)$-connected, closed PL $(n - 1)$-manifold, and since $2(n - 1) - (k - 1) = 2n - k - 1$, it can be embedded in E^{2n-k-1}, by the theorem of Penrose, Whitehead, and Zeeman [6, Theorem 1.1]. Now we start with any embedding of ∂M and apply Corollary 1.

We notice that the statement above gives the Penrose-Whitehead Theorem [6, Theorem (1.2), case (a)], the only difference being that we do not obtain the case $n = 2k + 2$.

We present two more sufficient conditions, applicable beyond the metastable range, for the extensibility of an embedding $g: \partial M \to E^q$ to an embedding of the whole manifold. The first condition employs further restrictions on the connectivities of M and ∂M, while the second replaces the condition on connectivity of ∂M by a restriction on the embedding g.

The following theorem was proved by several authors. For the proof, see Hudson [2, p. 181].

THEOREM 3 [Dancis, Edwards, Horvatić, Hudson, Tindell, Wall]. *If* M *is a compact* PL n-*manifold with* $\partial M = \emptyset$ *and the pair* $(M, \partial M)$ *is* k-*connected* ($k \leq n - 4$), *then* M *can be embedded in* E^{2n-k-1} *and unknotted in* E^{2n-k}.

PROPOSITION 1. *Let* M *be a compact* PL n-*manifold with connected, nonempty boundary. Let* M *and* ∂M *both be* k-*connected* ($k \leq n - 4$). *Then every* PL *embedding* $g: \partial M \to E^{2n-k-1}$ *can be extended to a* PL *embedding of* M.

Proof. Using the homotopy exact sequence of the pair $(M, \partial M)$, we see that the pair is also k-connected. Therefore Theorem 3 applies and ensures the embeddability of M in E^{2n-k-1}. Furthermore, ∂M is a compact, closed $(n - 1)$-manifold, and by assumption, it is k-connected. Since

$$k \leq n - 4 = (n - 1) - 3,$$

Irwin's embedding-unknotting theorem (see Zeeman [8, Chapter 8, Corollary 2]) implies that ∂M can be embedded in E^{2n-k-2} and unknotted in E^{2n-k-1}. Therefore we can begin with a prescribed embedding $g: \partial M \to E^{2n-k-1}$ and any embedding $h: M \to E^{2n-k-1}$. By Irwin's unknotting theorem, there exists an ambient isotopy between $h \mid \partial M$ and g. Finally, the map

$$f = H_1 h: M \to E^{2n-k-1}$$

is simultaneously a PL embedding and an extension of g. This completes the proof.

PROPOSITION 2. *Let* M *be a compact,* $(2n - q)$-*connected* PL n-*manifold* ($n \leq q - 3$), *and let* $g: \partial M \to E^q$ *be a* PL *embedding. If* g *extends to a* PL *embedding of a cone* $C(\partial M)$ *over* ∂M, *then* g *extends to a* PL *embedding of* M.

Proof. Let K be a subdivision of M such that $g: \partial K \to E^q$ is simplicial and ∂K is a full subcomplex of K. Consider the diagram

$$\begin{tikzcd} & \partial M \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "g"] & \\ M \arrow[r, "\phi"'] & C(\partial M) \arrow[r, "g_1"'] & E^q \end{tikzcd},$$

where i and j are the inclusion maps, g is the preassigned embedding of ∂M, and g_1 is the preassigned extension of g on $C(\partial M)$. Since ∂K is full in K, we define ϕ as follows. Let $\phi \mid \partial M = j$; that is, let $\phi \mid \partial M$ be an inclusion. If v is a vertex in int M, let $\phi(v)$ be the vertex of $C(\partial M)$, and extend ϕ linearly. The composition $g_1 \phi: M \to E^q$ is a PL map such that $g_1 \phi \mid \partial M = g$. Since g_1 is an embedding and ϕ is surjective, it follows that

$$g_1 \phi(M) = g_1 C(\partial M);$$

that is, the image by $g_1 \phi$ is an embedded cone $C(\partial M)$; it is a collapsible polyhedron, and simultaneously it is link-collapsible onto its base $g(\partial M)$. Therefore the relative regular neighborhood of $g\phi(M)$ mod $g(\partial M)$ is a PL q-ball. Denote this ball by B. Then the map $g_1 \phi: M \to B$ is a proper map, and we can apply Irwin's theorem [4, Theorem 1.1], which says that g_2 is homotopic (rel ∂M) to an embedding. This completes the proof.

6. DISCONNECTED, BOUNDED PL n-MANIFOLDS IN E^q

Here we present a sufficient condition for the extensibility to M of an embedding of ∂M into E^q, for the case where M is disconnected. By a disconnected, bounded PL n-manifold M we mean a manifold M such that $\partial M \neq \emptyset$ and each component of M is bounded.

LEMMA 3. *Let* $M^n \subset E^q$ *be a bounded* PL *n-manifold. Then* $E^q - M$ *is* $(q - n - 1)$*-connected.*

Proof. Since every continuous map $f: S^{q-n-1} \to E^q - M$ is homotopic to a PL map, we can assume that f is PL. But every map in E^q is nullhomotopic. Therefore there exists a PL extension $F: D^{q-n} \to E^n$ of f. By a general-position argument, we can assume that F is a map in general position with respect to M. Thus we get 0-dimensional intersections between M and F(D). As usual, we notice that these intersections, as top-dimensional intersections, are transversal. Therefore the intersections can be piped over the boundary of M. Thus we obtain a map $g: D^{q-n} \to E^q - M$ that is an extension of f, and therefore f is nullhomotopic in $E^q - M$.

THEOREM 4. *Let* $M = M_1 \cup M_2$ *be a compact, bounded* PL *n-manifold. Assume that* M_1 *is* $(k+1)$*-connected and* M_2 *is* κ*-connected* $(2k + 2 < n,\ k \leq n - 4)$. *Let* $g: \partial M \to E^{2n-k-1}$ *be a* PL *embedding. If the embeddings*

$$g_1 = g \mid \partial M_1: \partial M_1 \to E^{2n-k-1} - g(\partial M_2)$$

and

$$g_2 = g \mid \partial M_2: \partial M_2 \to E^{2n-k-1} - g(\partial M_1)$$

are inessential, then g *extends to a* PL *embedding of* M.

Proof. Let $W_i = E^{2n-k-1} - g(\partial M_i)$. Each W_i is an open PL n-manifold without boundary. Furthermore, by a general-position argument, each W_i is $(n - k - 2)$-connected. Since $g_1: \partial M_1 \to W_2$ is inessential, it extends continuously to a map of M_1 into W_2. We are now in the situation of Theorem 2. The condition $2k + 2 < n$ implies $q = 2n - k - 1 > 3n/2$. Furthermore, we need $(2n - q - 1)$-connectedness of M_1 and $(2n - q)$-connectedness of W_2. The condition $2n - (2n - k - 1) - 1 = k$ implies that

$$2n - (2n - k - 1) = k + 1 \leq n - k - 2,$$

and we see that both conditions are satisfied. Therefore $g_1: \partial M_1 \to W_2$ extends to a PL embedding, say $g_1: M_1 \to W_2$. Analogously, $g_2: \partial M_2 \to W_1$ extends to a PL embedding $g_2: M_2 \to W_1$. The embeddings g_1 and g_2 together give a PL map $g: M \to E^{2n-k-1}$ such that $g \mid M_i = g_i$, which is in general a PL immersion of M. By a general-position argument, we can assume that $\dim S_2(g) \leq k + 1$, and from the construction we see that $S_2(g) \subset \operatorname{int} M$. Since by assumption M_1 is $(k + 1)$-connected and $k + 1 \leq n - 3$, we can engulf $S_2(g) \cap M_1$ by a PL n-ball B such that $B \subset \operatorname{int} M_1$. Let A be a PL arc that connects a boundary point of ∂B with some point in ∂M_1, and let A be properly embedded in $M_1 - \operatorname{int} B$. Then $B \cup A$ is a collapsible polyhedron, since the ball collapses to any one of its points. Subdivide M, and choose a triangulation of E^{2n-k-1} such that g is simplicial and $B \cup A$ is a subpolyhedron of M. Since $g \mid M_1 = g_1$, we see that $B \cup A$ is embedded into E^{2n-k-1}. Therefore we can look at the second derived neighborhood N of $g(B \cup A)$ in E^{2n-k-1}, subdividing M and E^{2n-k-1} twice barycentrically. By the usual argument, we deduce that

(1) N is a $(2n - k - 1)$-ball;

(2) $N_1 = g^{-1}(N) \cap M_1$ is an n-ball (since it is the second derived neighborhood of $B \cup A$ in M_1);

(3) $N_1 \cap \partial M_1$ is a face of N_1, that is, an $(n - 1)$-ball (since it is the second derived neighborhood of the point $A \cap \partial M_1$);

(4) N_1 is semiproperly embedded in N; in fact, $g(\operatorname{Fr}_M N_1) \subset \partial N$, and $\operatorname{Fr}_M N_1$ is an $(n - 1)$-ball;

(5) $N_2 = g^{-1}(N) \cap M_2$ is a PL n-manifold (since it is the second derived neighborhood of $S_2(g) \cap M_2$); furthermore, $N_2 \subset \operatorname{int} M_2$, and g embeds N_2 properly into N.

Consider now the second derived neighborhood K of $g(N_1)$ in N. We see immediately that

(6) K is a $(2n - k - 1)$-ball, $\partial K \cap \partial N$ is a common face, and therefore $L = \operatorname{cl}(N - K)$ is a $(2n - k - 1)$-ball such that

(7) $L \cap g(M_1) = \emptyset$ and

(8) g embeds ∂N_2 into ∂L.

Now we restrict our attention to N_2 and L. From the construction it follows that no point of $M - N_2$ is mapped into L, and that $g \mid M - \operatorname{int} N_2$ is an embedding. Since L is a ball, and since g embeds ∂N_2 into ∂L, this implies that the map

$$g \mid \partial N_2: \partial N_2 \to E^{2n-k-1} - g(M_1)$$

is inessential. Extend this map to N_2. Together with the map $g \mid (M_2 - \operatorname{int} N_2)$, the extension gives a continuous extension of the embedding

$$g \mid \partial M_2 \colon \partial M_2 \to E^{2n-k-1} - g(M_1) = W .$$

W is an open PL $(2n - k - 1)$-manifold without boundary, and furthermore, W is $(n - k - 2)$-connected, by Lemma 3. The condition $2k + 2 < n$ implies that $n - k - 1 > k + 1$, so that $n - k - 2 \geq k + 1$. Since M_2 is k-connected, Theorem 2 is again applicable, and therefore this map is homotopic to a PL embedding that keeps ∂M_2 fixed. This completes the proof of the theorem.

Remark 2. We point out that the conditions of Theorem 4 are not necessary. Let the manifold M be a union of two disjoint n-annuli $M_1 = S^{n-1} \times I$ and $M_2 = S^{n-1} \times I$. It is possible to embed M into E^{2n-1} in a linked fashion so that ∂M_1 is not inessential in $E^{2n-1} - \partial M_2$ and ∂M_2 is not inessential in $E^{2n-1} - \partial M_1$. For the case where M has more than two components, we can state the following obvious corollary.

COROLLARY 4. *Let* $M = \bigcup_{i=1}^{r} M_i$ *be a compact* PL n-*manifold with* $\partial M_i \neq \emptyset$ *for each* i. *Assume that each* M_i *is* $(k + 1)$-*connected* (*we can obviously allow that* M_r *be only* k-*connected*) $(2k + 2 < n,\ k \leq n - 4)$. *Let* $g\colon \partial M \to E^{2n-k-1}$ *be a* PL *embedding. If*

$$g_i = g \mid \partial M_i \colon \partial M_i \to E^{2n-k-1} - g\Big(\bigcup_{j \neq i} \partial M_j\Big)$$

is inessential for each i, *then* g *extends to a* PL *embedding of* M.

Proof. Suppose that g has been extended to an embedding g of $\bigcup_{j<i} M_j$. Since

$$g_i \colon \partial M_i \to E^{2n-k-1} - \bigcup_{j \neq i} g(\partial M_j)$$

is inessential, it extends to an embedding of M_i, by the argument used in the proof of Theorem 4. We can remove the intersection with $\bigcup_{j<i} g(M_j)$, considering one component M_j at a time, in the same way as in the proof of Theorem 4. Thus we obtain a continuous extension of g_i onto M_i into

$$E^{2n-k-1} - \bigcup_{j<i} g(M_j) - \bigcup_{j>i} g(\partial M_j) ,$$

which by Theorem 1 is homotopic to an embedding that keeps ∂M_i fixed. This completes the inductive step and therefore the proof of the corollary.

To obtain a concordance between two embeddings as described in Corollary 4, we need an additional restriction.

PROPOSITION 3. *Suppose that the manifold* $M = M_1 \cup M_2$ *satisfies the conditions in Theorem* 4. *Suppose further that*

$$f_0 , f_1 \colon M_1 \cup M_2 \to E^{2n-k}$$

are extensions of the same embedding $g\colon \partial M \to E^{2n-k}$ *and that*

$$f_0 \mid M_1 \simeq f_1 \mid M_1 \ (\mathrm{rel}\ \partial M_1) \quad \text{in } W_2 = E^{2n-k} - g(\partial M_2)$$

and

$$f_0 \mid M_2 \simeq f_1 \mid M_2 \ (\mathrm{rel}\ \partial M_2) \quad \text{in } W_1 = E^{2n-k} - g(\partial M_1).$$

Then f_0 *and* f_1 *are concordant and keep* ∂M *fixed.*

Proof. Let h_t' and h_t'' be the assumed homotopies (rel ∂M_1) and (rel ∂M_2), respectively. We define $h: M \times I \to E^{2n-k} \times I$ by the condition

$$h(x, t) = \begin{cases} (h_t'(x), t) & \text{if } x \in M_1, \\ (h_t''(x), t) & \text{if } x \in M_2. \end{cases}$$

It follows immediately that

$$h \mid (M_1 \times I): M_1 \times I \to (E^{2n-k} \times I) - h(\partial(M_2 \times I)),$$

$$h \mid (M_2 \times I): M_2 \times I \to (E^{2n-k} \times I) - h(\partial(M_1 \times I)),$$

and that

$$h \mid (\partial M \times I) = g \times \mathrm{id}, \quad h \mid (M \times 0) = f_0, \quad h \mid (M \times 1) = f_1.$$

Now the proof of Theorem 4 applies, and therefore h is homotopic (rel $\partial(M \times I)$) to a semiproper PL embedding. In the proof of Theorem 4, we use Lemma 3, which applies to the present case in an obvious way. Interior intersections between the embedded $M_1 \times I$ and $M_2 \times I$ can be removed in the same way over $\partial M_1 \times I$. This gives the desired concordance.

We point out that the additional assumption used in Proposition 3 is in fact necessary.

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