SOME FREE PRODUCTS OF CYCLIC GROUPS

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Let p and q be positive integers. The principal question we wish to consider is that of giving a representation in terms of linear fractional transformations for the free product of a cyclic group of order p and a cyclic group of order q. These include the Hecke groups discussed in [1], which correspond to the choice $p = 2$, $q \geq 3$. In addition we consider two subgroups of the modular group and show that these are free products of cyclic groups. The method of proof we employ is elementary; it is patterned after the proof given by K. A. Hirsch in his appendix to the second volume of Kurŏs's book on group theory [2], that Γ is the free product of a cyclic group of order 2 and a cyclic group of order 3. The referee points out that essentially the same proof is given by H. Rademacher in his paper [5].

We introduce some notation. For a positive integer n, define

$$\lambda_n = 2\cos(\pi/n), \quad A_n = \begin{bmatrix} 0 & 1 \\ -1 & \lambda_n \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 & -1 \\ 1 & \lambda_n \end{bmatrix}.$$

Then the eigenvalues of A_n and B_n are the numbers α_n, β_n, where

$$\alpha_n = \exp(i\pi/n), \quad \beta_n = \exp(-i\pi/n)$$

are primitive $(2n)^{th}$ roots of unity. If $n > 1$, these are distinct and both A_n and B_n are similar to diag (α_n, β_n). Thus the least positive integer k such that $A_n^k = \pm I$, $B_n^k = \pm I$ is $k = n$; and

$$A_n^n = B_n^n = -I.$$

We set $A = A_p$, $B = B_q$,

$$\Delta = \Delta_{p,q} = \{A, B\}$$

($\{A, B\}$ denotes the group generated by A and B) and agree to identify a matrix with its negative. This is equivalent to considering Δ as a group of linear fractional transformations (which, by the way, is a discontinuous group). Then to show that Δ is the free product of a cyclic group of order p and a cyclic group of order q it is only necessary to show that the relations

$$A^p = B^q = 1$$

are the defining relations for Δ. We assume that $p \geq 2$, $q \geq 3$. (The case $p = q = 2$ will be treated separately.) We set

$$a_r = \frac{\alpha_p^r - \beta_p^r}{\alpha_p - \beta_p} = \frac{\sin(r\pi/p)}{\sin(\pi/p)}, \tag{1}$$

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$$b_s = \frac{\alpha_q^s - \beta_q^s}{\alpha_q - \beta_q} = \frac{\sin(s\pi/q)}{\sin(\pi/q)}. \tag{2}$$

LEMMA 1. *For all integers* r *and* s,

$$A^r = a_r A - a_{r-1} I, \quad B^s = b_s B - b_{s-1} I.$$

The proof is an easy consequence of the relationships

$$A^2 = \lambda_p A - I, \quad B^2 = \lambda_q B - I,$$

and we omit it.

If we note that

$$a_r = \lambda_p a_{r-1} - a_{r-2}, \quad b_s = \lambda_q b_{s-1} - b_{s-2},$$

then, forming the product $A^r B^s$, we obtain

LEMMA 2. *For all integers* r *and* s,

$$A^r B^s = \begin{bmatrix} a_r b_s + a_{r-1} b_{s-1} & a_r b_{s+1} + a_{r-1} b_s \\ a_{r+1} b_s + a_r b_{s-1} & a_r b_s + a_{r+1} b_{s+1} \end{bmatrix}.$$

From this lemma, (1), and (2) we deduce

LEMMA 3. *Suppose that* $1 \leq r \leq p - 1$ *and* $1 \leq s \leq q - 1$. *Then* $A^r B^s$ *has nonnegative elements. The diagonal elements are always positive, and at most one off-diagonal element vanishes for any particular pair* r, s.

We are now prepared to prove our first theorem.

THEOREM 1. *Suppose that* $p \geq 2$ *and* $q \geq 3$. *The relationships*

$$A^p = B^q = 1$$

are defining relationships for Δ, *and consequently* Δ *is the free product* $\{A\} * \{B\}$ *of a cyclic group of order* p *and a cyclic group of order* q.

Proof. Any word W of Δ may be written as

$$W = B^{s_n} A^{r_1} B^{s_1} \cdots A^{r_{n-1}} B^{s_{n-1}} A^{r_n},$$

where the r's are all between 1 and $p - 1$ and the s's are all between 1 and $q - 1$, except that r_n and s_n each may be 0. Suppose that $W = 1$. Then $B^{-s_n} W B^{s_n} = 1$, so that

$$A^{r_1} B^{s_1} \cdots A^{r_{n-1}} B^{s_{n-1}} A^{r_n} B^{s_n} = 1. \tag{3}$$

We may assume relationship (3) to be of *shortest* length, which we continue to denote by n. Then $s_n \neq 0$, since s_n being zero leads to the relationship

$$A^{r_1 + r_n} B^{s_1} \cdots A^{r_{n-1}} B^{s_{n-1}} = 1,$$

which is of length at most n - 1. Moreover, $r_n \neq 0$, since r_n being zero leads to the relationship

$$A^{r_1} B^{s_1} \cdots A^{r_{n-1}} B^{s_{n-1}+s_n} = 1,$$

which is also of length at most n - 1.

But now Lemma 3 implies that (3) is impossible, and the proof of the theorem is complete.

We next consider two subgroups of the modular group Γ defined in [4]. Here Γ is the multiplicative group of 2×2 rational integral matrices of determinant 1 in which a matrix and its negative are identified. The subgroups we consider are Γ^2, generated by the squares of the elements of Γ, and Γ^3, generated by the cubes of the elements of Γ. It is shown in [4] that $\Gamma^2 = \{R, S\}$ and $\Gamma^3 = \{T, U, V\}$, where $R^3 = S^3 = -I$, $T^2 = U^2 = V^2 = -I$, and

$$R = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix};$$

and it was stated there that Γ^2 is the free product of two cyclic groups of order 3, and that Γ^3 the free product of three cyclic groups of order 2. Here we supply a proof of this statement, which consists of showing that the relations

$$R^3 = S^3 = 1$$

are defining relations for the group Γ^2, and that the relations

$$T^2 = U^2 = V^2 = 1$$

are defining relations for the group Γ^3. The pattern of proof is identical with that of Theorem 1, and we need only prove lemmas analogous to Lemma 3.

By direct calculation, we find that

$$RS = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$RS^2 = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

$$R^2S = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

$$R^2S^2 = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Hence we obtain

LEMMA 4. *Suppose that* $1 \leq r,\ s \leq 2$. *Then* $R^r S^s$ *has nonnegative elements. The diagonal elements are always positive, and at most one off-diagonal element vanishes for any particular pair* $r,\ s$.

This lemma implies

THEOREM 2. *The relationships*

$$R^3 = S^3 = 1$$

are defining relationships for Γ^2, *and consequently* Γ^2 *is the free product* $\{R\} * \{S\}$ *of two cyclic groups of order* 3.

The calculations are somewhat more involved for Γ^3, but of the same type. As in the proof of Theorem 1, if a word of Γ^3 is the identity, we may assume that it begins with T. (It is trivial to prove that T must appear and that

$$\{U, V\} = \{U\} * \{V\}.)$$

Then there is a relationship

$$TX_1 TX_2 \cdots TX_n = 1,$$

where each X is of the form $(UV)^r$, $(VU)^r$, $(UV)^s U$, or $(VU)^s V$. If we define

$$C_r = T(UV)^r, \qquad D_r = T(VU)^r,$$

$$E_s = T(UV)^s U, \quad F_s = T(VU)^s V,$$

then

$$C_r = 3C_{r-1} - C_{r-2}, \quad D_r = 3D_{r-1} - D_{r-2},$$

$$E_s = 3E_{s-1} - E_{s-2}, \quad F_s = 3F_{s-1} - F_{s-2}$$

(since UV and VU have trace 3 and determinant 1), and

$$C_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix},$$

$$E_0 = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} -5 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix},$$

$$F_0 = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

This implies

LEMMA 5. *If* $r \geq 1$, $s \geq 0$ *the matrices* C_r, D_r, E_s, F_s *have nonnegative elements. The diagonal elements are always positive, and at most one off-diagonal element vanishes for a particular pair* $r,\ s$ *in each matrix. These properties are preserved under multiplication.*

As in the proof of Theorem 1, Lemma 5 implies

THEOREM 3. *The relationships*

$$T^2 = U^2 = V^2 = 1$$

are defining relationships for Γ^3, *and consequently* Γ^3 *is the free product*

$$\{T\} * \{U\} * \{V\}$$

of three cyclic groups of order 2.

In conclusion we mention that the omitted case of Theorem 1, corresponding to $p = q = 2$, can be realized by considering the subgroup $\{T\} * \{U\}$ of Γ^3. The groups $\{R\}$ and $\{S\}$ are conjugate subgroups of Γ, and the groups $\{T\}$, $\{U\}$, $\{V\}$ are conjugate subgroups of Γ, as they must be. It is possible to give geometric proofs of Theorems 1, 2, 3 by considering the fundamental regions of the groups and the cycles of vertices, but we do not enter into these discussions here. A geometric discussion of the general situation of the free product of countably many cyclic groups of arbitrary orders has been given by J. Lehner [3].

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