THE MUTUAL INCLUSION OF KARAMATA-STIRLING METHODS OF SUMMATION

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1. INTRODUCTION

In [1], R. P. Agnew studied in detail a method for evaluation of sequences and series which seems to have fundamental significance comparable to that of the classical methods of Cesàro, Abel, and others. Lototsky had introduced this method in an article published in 1953 (see [2]), and Agnew named it the Lototsky method. However, Lototsky's method is essentially a special case of a class of summation methods introduced in 1935 by J. Karamata (see [3]). Karamata (see Section 2) gives a fundamental theorem about inclusion of Borel's, Euler's and his own method for positive sequences. Agnew pointed out that, without the restriction to positive sequences, the Borel method does not include the Lototsky method.

The paper of Agnew raises interest in a detailed study of Karamata-Stirling methods. Agnew shows the power of the method; he gives counter-examples and develops, although in a special case, the technique which is applicable also to the larger class. In the present paper we shall study the mutual inclusion of Karamata-Stirling methods.

2. DEFINITIONS

Since the paper [3] of Karamata is now not readily accessible, we give the necessary definitions in full.

For x real (or complex) let $(x)_0 = 0$ and

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \qquad (n = 1, 2, \cdots),$$

and define the numbers $\left[{n \atop \nu}\right]$ and $\left\{{n \atop \nu}\right\}$ in the following manner: $\left[{n \atop \nu}\right] = \left\{{n \atop \nu}\right\} = 0$, except in the case where the pair of integers n, ν satisfies the conditions $n \geq 1$ and $1 \leq \nu \leq n$; in the latter case, define $\left[{n \atop \nu}\right]$ and $\left\{{n \atop \nu}\right\}$ by the conditions

$$(x)_n = \sum_{\nu=1}^{n} \left[{n \atop \nu}\right] x^{\nu},$$

$$x^n = \sum_{\nu=1}^{n} (-1)^{n+\nu} \left\{{n \atop \nu}\right\} (x)_{\nu}.$$

The numbers $\left[{n \atop \nu}\right]$ and $\left\{{n \atop \nu}\right\}$ are Stirling numbers of the first and second kind, respectively, with a slight modification of the definition of Nielsen (see [4, pp. 66-

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71]). One readily obtains the recurrence formulae

$$\begin{bmatrix} n+1 \\ \nu \end{bmatrix} = n \begin{bmatrix} n \\ \nu \end{bmatrix} + \begin{bmatrix} n \\ \nu - 1 \end{bmatrix}, \tag{1}$$

$$\begin{Bmatrix} n+1 \\ \nu \end{Bmatrix} = \nu \begin{Bmatrix} n \\ \nu \end{Bmatrix} + \begin{Bmatrix} n \\ \nu - 1 \end{Bmatrix}, \tag{2}$$

which are valid for $n = 1, 2, \cdots$ and every integer ν. It is also easy to show that from

$$A_0 = B_0, \qquad A_n = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} B_\nu \tag{3}$$

follows

$$B_n = (-1)^n \sum_{\nu=0}^{n} (-1)^\nu \begin{Bmatrix} n \\ \nu \end{Bmatrix} A_\nu \qquad (n = 1, 2, \cdots). \tag{4}$$

DEFINITION 1. *The sequence* $s = \{s_\nu\}$ *is* KS(λ)-*summable to* S *if*

$$S_n^\lambda(s) \overset{\text{def}}{=} \frac{1}{(\lambda)_n} \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} \lambda^\nu s_\nu \to S \qquad (n \to \infty). \tag{5}$$

This is the definition of the class of Karamata-Stirling summability methods. When $\lambda = 1$, Karamata-Stirling summability of the sequence $s_0, s_1, s_2, \cdots$ is equivalent to Lototsky summability of the sequence $s_1, s_2, s_3, \cdots$. The KS(λ)-methods are regular for $\lambda > 0$. This restriction on λ will be made in the sequel.

To be able to state Karamata's theorem, we recall

DEFINITION 2. *The sequence* $s = \{s_\nu\}$ *is* E(λ)-*summable to* S *if*

$$E_n^\lambda(s) \overset{\text{def}}{=} \frac{1}{(1+\lambda)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} \lambda^\nu s_\nu \to S \qquad (n \to \infty).$$

It is B-*summable to* S *if*

$$B_y(s) \overset{\text{def}}{=} e^{-y} \sum_{\nu=0}^{\infty} \frac{s_\nu}{\nu!} y^\nu \to S \qquad (y \to \infty).$$

3. THEOREMS

Karamata's fundamental result is contained in

THEOREM K. *For positive sequences and every* $\lambda > 0$,

$$E(\lambda) \subset KS(\lambda) \subset B;$$

that is, for positive sequences the Borel method includes each regular Karamata-Stirling method, and each of these includes the corresponding Euler method.

We shall prove

THEOREM 1. *For every* α $(0 < \alpha < 1)$ *and each* λ $(\lambda > 0)$,

$$\mathrm{KS}(\lambda) \subset \mathrm{KS}(\alpha\lambda);$$

that is, the $\mathrm{KS}(\lambda)$-summability of a sequence implies its $\mathrm{KS}(\tau)$-summability for every $0 < \tau < \lambda$ to the same limit, (but not conversely).

Before the proof of Theorem 1, we quote also

THEOREM 2. *If the sequence* $s = \{s_\nu\}$ *is* $\mathrm{KS}(\lambda)$-*summable for* $\lambda > 0$, *then there exists a constant* M, *independent of* n, *such that*

$$|s_n| < M\frac{\Gamma(\lambda + n)}{(\lambda \log 2)^n}.$$

We do not give the proof of this theorem, because it follows the same lines as Agnew's proof in [1] for $\lambda = 1$.

Proof of Theorem 1. Suppose that, for a fixed $\lambda > 0$,

$$S_n^\lambda = \frac{1}{(\lambda)_n}\sum_{\nu=0}^{n}\begin{bmatrix} n \\ \nu \end{bmatrix}\lambda^\nu s_\nu \to S \qquad (n \to \infty), \tag{6}$$

and for $0 < \alpha < 1$, let

$$S_n^{\alpha\lambda} = \frac{1}{(\alpha\lambda)_n}\sum_{\nu=0}^{n}\begin{bmatrix} n \\ \nu \end{bmatrix}\alpha^\nu\lambda^\nu s_\nu. \tag{7}$$

In view of (3) and (4), it follows from (6) that

$$\lambda^\nu s_\nu = (-1)^\nu\sum_{i=0}^{\nu}(-1)^i\left\{\begin{matrix} \nu \\ i \end{matrix}\right\}(\lambda)_i S_i^\lambda.$$

Therefore

$$S_n^{\alpha\lambda} = \frac{1}{(\alpha\lambda)_n}\sum_{\nu=0}^{n}\begin{bmatrix} n \\ \nu \end{bmatrix}\alpha^\nu(-1)^\nu\sum_{i=0}^{\nu}(-1)^i\left\{\begin{matrix} \nu \\ i \end{matrix}\right\}(\lambda)_i S_i^\lambda = \frac{1}{(\alpha\lambda)_n}\sum_{\nu=0}^{n}S^\lambda(\lambda)_\nu\tau_\nu^n(\alpha), \tag{8}$$

where

$$\tau_\nu^n(\alpha) = \begin{cases} (-1)^\nu\displaystyle\sum_{i=\nu}^{n}(-1)^i\begin{bmatrix} n \\ i \end{bmatrix}\left\{\begin{matrix} i \\ \nu \end{matrix}\right\}\alpha^i, \\ \\ 0 \qquad \text{for negative } \nu \text{ and for } \nu > n. \end{cases} \tag{9}$$

From (9) and the recurrence formulae for $\left[{n \atop \nu} \right]$ and $\left\{ {n \atop \nu} \right\}$, or from

(10) $$(\alpha x)_n = \sum_{\nu=0}^{n} (x)_\nu \tau_\nu^n(\alpha),$$

one gets the recurrence formula

(11) $$\tau_\nu^{n+1}(\alpha) = (n - \alpha\nu)\tau_\nu^n(\alpha) + \alpha\tau_{\nu-1}^n(\alpha),$$

valid for $n = 1, 2, \cdots$ and for every integer ν. We have, for instance,

(12) $$\begin{aligned} &\tau_0^0(\alpha) = 0; \\ &\tau_0^1(\alpha) = 0, \quad \tau_1^1(\alpha) = \alpha; \\ &\tau_0^2(\alpha) = 0, \quad \tau_1^2(\alpha) = \alpha(1 - \alpha), \quad \tau_2^2(\alpha) = \alpha^2; \end{aligned}$$

and generally

(13) $$\tau_1^n(\alpha) = \alpha(1 - \alpha)\cdots(n - 1 - \alpha), \quad \cdots, \quad \tau_n^n(\alpha) = \alpha^n.$$

From (11) and (12) follows by induction

(14) $$|\tau_\nu^n(\alpha)| = \tau_\nu^n(\alpha).$$

Also,

$$\tau_\nu^1(\alpha) \le 0!, \quad \tau_\nu^2(\alpha) \le 1!,$$

and by induction

(15) $$\tau_\nu^n(\alpha) \le (n - 1)! \quad \text{for } n = 1, 2, \cdots.$$

Therefore, the triangular matrix $((p_{n,\nu}))$ with

$$p_{n,\nu} = \{(\lambda)_\nu \tau_\nu^n(\alpha)\} / (\alpha\lambda)_n$$

is positive, $\Sigma p_{n,\nu} = 1$, and for every fixed ν

$$p_{n,\nu} = O\left\{\frac{(n-1)!}{n!\, n^{\alpha\lambda-1}}\right\} \to 0 \quad (n \to \infty).$$

It follows, by Toeplitz' theorem, that $((p_{n,\nu}))$ is regular, that is,

$$S_n^{\alpha\lambda} \to \lim_{n\to\infty} S_n^\lambda = S \quad (n \to \infty).$$

To complete the proof, we have to show that for each $\lambda > 0$ there exist sequences that are KS(λ)-summable, but are not KS($\lambda + \varepsilon$)-summable for $\varepsilon > 0$.

LEMMA. *The sequence* $s = \{s_\nu\}$ $(\nu = 0, 1, \cdots)$ *with*

$$s_0 = 0 \qquad \text{(or any constant)},$$

$$s_\nu = \frac{(-1)^\nu(\nu - 1)!}{x^\nu} = \int_0^\infty \frac{e^{-t}}{t} \left(-\frac{t}{x}\right)^\nu dt \qquad (\nu = 1, 2, \cdots; x > 0)$$

is KS(λ)-*summable only for* $x \geq \lambda \log 2$.

(For the convenience of the reader we give the proof in some detail, although the treatment is similar to that of Agnew, [3, pp. 109-111].) The S^λ_{n+1}-transform of our sequence is

$$S^\lambda_{n+1} = \frac{1}{(\lambda)_{n+1}} \int_0^\infty \frac{e^{-t}}{t} \sum_{\nu=1}^{n+1} \begin{bmatrix} n+1 \\ \nu \end{bmatrix} \lambda^\nu \left(-\frac{t}{x}\right)^\nu dt$$

$$= \frac{1}{(\lambda)_{n+1}} \int_0^\infty \frac{e^{-t}}{t} \left(-\frac{\lambda t}{x}\right)\left(-\frac{\lambda t}{x} + 1\right) \cdots \left(-\frac{\lambda t}{x} + n\right) dt$$

$$= \frac{(-1)^{n+1}}{(\lambda)_{n+1}} \int_0^\infty e^{-\frac{x}{\lambda}u} (u - 1)(u - 2) \cdots (u - n)\, du.$$

We split S^λ_{n+1} into two parts:

$$J_n = \frac{(-1)^{n+1}}{(\lambda)_{n+1}} \int_0^n \{\ \}\, du, \qquad I_n = \frac{(-1)^{n+1}}{(\lambda)_{n+1}} \int_n^\infty \{\ \}\, du.$$

For J_n we have

$$|J_n| \leq \frac{1}{(\lambda)_{n+1}} \sum_{\nu=1}^{n} \int_{\nu-1}^{\nu} e^{-\frac{x}{\lambda}u} |(u - 1) \cdots (u - n)|\, du.$$

But for $\nu - 1 \leq u \leq \nu$,

$$|(u - 1) \cdots (u - n)| \leq n!$$

(Agnew [3, p. 110], and therefore

$$|J_n| \leq \frac{n!}{(\lambda)_{n+1}} \int_0^n e^{-\frac{x}{\lambda}u} du \to 0 \qquad (n \to \infty)$$

for every $\lambda > 0$ (since $(\lambda)_{n+1} \sim n!\, n^\lambda$). For the second part I_n, we have

$$|I_n| \le \frac{1}{(\lambda)_{n+1}} \int_n^\infty e^{-\frac{x}{\lambda}u} (u-1)\cdots(u-n)\,du$$

$$\le \frac{e^{-\frac{x}{\lambda}n}}{(\lambda)_{n+1}} \sum_{\nu=0}^\infty \int_\nu^{\nu+1} e^{-\frac{x}{\lambda}t} (n+t-1)\cdots(n+t-n)\,dt$$

$$\le \frac{e^{-\frac{x}{\lambda}n}}{(\lambda)_{n+1}} \sum_{\nu=0}^\infty e^{-\frac{x}{\lambda}\nu} (n+\nu)\cdots(\nu+1)$$

$$\le \frac{n!\, e^{-\frac{x}{\lambda}n}}{(\lambda)_{n+1}} \sum_{\nu=0}^\infty \binom{n+\nu}{\nu} e^{-\frac{x}{\lambda}\nu}$$

$$\le \{n!\, e^{x/\lambda}\}/\{(\lambda)_{n+1} (e^{x/\lambda} - 1)^{n+1}\},$$

and this expression tends to zero for $x \ge \lambda \log 2$.

Take now the case $0 < x < \lambda \log 2$. We still have

$$J_n = o(1), \qquad (n \to \infty),$$

but we shall show that I_n does oscillate, in this case. Let

$$H_n = \frac{1}{(\lambda)_{n+1}} \int_n^\infty e^{-\frac{x}{\lambda}u} (u-1)\cdots(u-n)\,du,$$

so that $I_n = (-1)^{n+1} H_n$. For $n \ge 2$,

$$H_n \ge \frac{(n-1)}{(\lambda)_{n+1}} \int_n^\infty e^{-\frac{x}{\lambda}u} (u-2)\cdots(u-n)\,du$$

$$\ge \frac{(n-1)}{(\lambda)_{n+1}} e^{-\frac{x}{\lambda}n} \sum_{\nu=0}^\infty \int_\nu^{\nu+1} e^{-\frac{x}{\lambda}t} (t+n-2)\cdots(t+n-n)\,dt$$

$$\ge \frac{(n-1)}{(\lambda)_{n+1}} e^{-\frac{x}{\lambda}n} \sum_{\nu=1}^\infty e^{-\frac{x}{\lambda}(\nu+1)} (n-1+\nu-1)\cdots(\nu+1)\,\nu$$

$$\ge \frac{(n-1)}{(\lambda)_{n+1}} e^{-\frac{x}{\lambda}n} \sum_{\mu=0}^\infty e^{-\frac{x}{\lambda}(\mu+2)} (n-1+\mu)\cdots(\mu+1)$$

$$\geq \frac{(n-1)!\,(n-1)\,e^{-\frac{x}{\lambda}n-\frac{2x}{\lambda}}}{(\lambda)_{n+1}} \sum_{\mu=0}^{\infty} \binom{n-1+\mu}{\mu} e^{-\frac{x}{\lambda}\mu}$$

$$\geq \frac{(n-1)!\,(n-1)\,e^{-\frac{2x}{\lambda}}}{(\lambda)_{n+1}} \cdot \frac{1}{(e^{x/\lambda}-1)^n},$$

and for $0 < x < \lambda \log 2$, $1/\{(e^{x/\lambda} - 1)\} > 1$, so that $H_n \to \infty$ as $n \to \infty$, and I_n oscillates between $-\infty$ and $+\infty$. This proves the lemma.

To complete the proof of Theorem 1, take s_ν as in the lemma. For

$$\lambda \log 2 \leq x < \lambda \log 2 + \varepsilon \log 2,$$

this sequence is not $KS(\lambda + \varepsilon)$-summable, although it is $KS(\lambda)$-summable.

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