SEMIGROUPS AND SUBMODULAR FUNCTIONS

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The structure of open additive semigroups of complex numbers was investigated by Hille and Zorn [2], who obtained a characterization of them in terms of subadditive functions. Rosenbaum [3] extended this to a characterization of open semigroups in Euclidean n-space. That these results are sufficient to give a nearly complete description of the open semigroups of a locally compact abelian group was observed by the author [4]. It is natural to try to extend this characterization to arbitrary real linear topological spaces.

One cannot simply copy the proofs of Hille and Zorn, since these proofs use the local compactness of the plane to select a suitable basis of coordinates. Rosenbaum's characterization in n-space depends on the characterization in (n-1)-space, and thus is highly finite-dimensional. The assumption that 0 is a limit point of the semigroup is explicit in most of the results; when this is true the semigroup is called *angular*. In this note we shall consider open semigroups in a real linear topological space, and we shall consider only those which we shall call *radiant*. The purpose of this assumption is to enable us to have the infinite-dimensional version of Theorem 7.6.4 of [1] at our disposal. This restriction is made at some cost; it is no longer true, even in the plane, that we are discussing all angular semigroups. On the other hand, the restriction also results in some gain: some of the semigroups under discussion are not angular, and, of course, we can obtain some results in the infinite-dimensional case.

Throughout, let E denote a real linear topological space, and let R denote the field of real numbers.

Definition 1. A subset A of E is said to be radially convex if, for $x \in E$ and $\alpha, \beta \in R$, $\alpha x \in A$ and $\beta x \in A$ imply $\gamma x \in A$ for every $\gamma \in R$ satisfying $\alpha < \gamma < \beta$.

Thus A is radially convex if and only if the intersection of A with any one-dimensional subspace of E is either void or is a convex subset of that one-dimensional subspace.

A subset S of E is called a semigroup if $x, y \in S$ implies $x + y \in S$. If S is an open semigroup, then either $S = E$, or else there exists a continuous linear functional f on E such that $S \subset \{x \in E: f(x) > 0\}$; that is, S is contained in a half-space of E [5]. We shall call an open semigroup *proper* if $S \neq E$; the latter condition is equivalent to the assumption that $0 \notin S$.

Definition 2. A semigroup S in E is said to be a radiant semigroup if S is a proper open semigroup in E which is radially convex.

In the real line R, the only radiant semigroups are the semigroups of one of the following forms: (1) $\{\lambda \in R: \lambda > \xi \geq 0\}$, (2) $\{\lambda \in R: \lambda < \xi \leq 0\}$. Hence if S is a radiant semigroup in E and if $x \neq 0$ is any element of E, then either S and the one-dimensional subspace Rx of E do not meet, or else $S \cap Rx = Tx$, where T is a radiant semigroup of the reals.

Definition 3. For any nonvoid subset A of E, we define a function d_A on E as follows: $d_A(x) = \inf\{\alpha \in R: \alpha x \in A\}$. We call d_A the order function of A.

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We adopt the standard convention that $d_A(x) = +\infty$ if the set over which the infimum is taken is empty. Observe that $d_A(0) = -\infty$ if and only if $0 \in A$, and that $d_A(0) = +\infty$ if and only if $0 \notin A$. The remarks above concerning the radiant semigroups of reals yield at once

LEMMA 1. *If* S *is a radiant semigroup in* E, *then for each* $x \in E$ *either* $d_S(x) = -\infty$ *or* $0 \leq d_S(x) \leq +\infty$.

We shall use the notation $\alpha \cup \beta$ to denote the maximum of two extended reals α and β. We shall say that α and β *have the same finitary character* if one of the following is true: (1) $\alpha = \beta = -\infty$, (2) $\alpha = \beta = +\infty$, (3) α and β are both finite; otherwise α and β are said to have different finitary character. In what follows, the restriction that two extended reals have the same finitary character plays the role usually filled by the restriction against expressions of the form $\infty - \infty$.

LEMMA 2. *Let* S *be a radiant semigroup in* E *with order function* d_S. *If* $d_S(x)$ *and* $d_S(y)$ *have the same finitary character, then* $d_S(x + y) \leq d_S(x) \cup d_S(y)$.

Proof. If $d_S(x) = d_S(y) = +\infty$, the result is trivial. Suppose then that

$$d_S(x) = d_S(y) = -\infty;$$

then there exist positive reals α, β such that $-\alpha x \in S$, $-\beta x \in S$. Hence

$$-(\alpha \cup \beta)x \in S, \quad -(\alpha \cup \beta)y \in S,$$

and since S is a semigroup, $-(\alpha \cup \beta)(x + y) \in S$. Thus $d_S(x + y) = -\infty$. Finally, let $0 \leq d_S(x) = \alpha < +\infty$, $0 \leq d_S(y) = \beta < +\infty$. Then for any $\varepsilon > 0$, $[(\alpha \cup \beta) + \varepsilon]x \in S$ and $[(\alpha \cup \beta) + \varepsilon]y \in S$, and hence $[(\alpha \cup \beta) + \varepsilon](x + y) \in S$. Thus $(\alpha \cup \beta) + \varepsilon > d_S(x + y)$ for every $\varepsilon > 0$, and therefore $\alpha \cup \beta \geq d_S(x + y)$.

Numerous easy examples show that this inequality will in general fail to hold if $d_S(x)$ and $d_S(y)$ have different finitary character.

We omit the proofs of the next four lemmas, because the proof of Lemma 2 illustrates sufficiently the type of argument which is needed.

LEMMA 3. *The order function of a radiant semigroup is upper semicontinuous.*

LEMMA 4. *If* S *is a radiant semigroup in* E *with order function* d_S, *then* $S = \{x \in E: 0 \leq d_S(x) < 1\}$.

LEMMA 5. *If* S *is a radiant semigroup in* E, *and if* $0 \leq d_S(x) < +\infty$ *and* $0 < \alpha < +\infty$, *then* $\alpha d_S(\alpha x) = d_S(x)$.

We shall call a set C in E a *cone* if C is a semigroup which is closed under multiplication by strictly positive scalars. A cone is called proper if $0 \notin C$.

LEMMA 6. *Let* S *be a radiant semigroup in* E, *with order function* d_S. *Then the set* $\{x \in E: 0 \leq d_S(x) < +\infty\}$ *is the least proper cone in* E *which contains* S.

Now we shall assemble the properties of the order function of a radiant semigroup into a definition.

Definition 4. Let f be a function defined on a real linear topological space, whose range is contained in the set of real numbers with $+\infty$ and $-\infty$ adjoined. The function f is said to be submodular if, whenever $f(x)$ and $f(y)$ have the same finitary character, $f(x + y) \leq f(x) \cup f(y)$. The function f will be called conventional if (i) the set on which it takes finite values is a nonvoid open set, but is not the entire space, and (ii) the function is nonnegative when it is finite. The function f is said to be an order function if the conditions $0 \leq f(x) < +\infty$ and $0 < \alpha < +\infty$ together imply that $\alpha f(\alpha x) = f(x)$.

We use the term "submodular" by analogy with the term "subadditive."

The preceding results can be summarized as follows: the order function of a radiant semigroup in E is an upper semicontinuous conventional submodular order function.

Conversely, let f be an upper semicontinuous conventional submodular function on E. Set $S = \{x \in E: 0 \leq f(x) < 1\}$; S is clearly an open semigroup in E. Also S is neither void nor all of E, since f is conventional. Further, suppose f is an order function. It is then clear that S is radially convex, and hence S is a radiant semigroup. Let d_S be the order function defined by S; we shall compare f and d_S. Suppose that $0 \leq f(x) = \alpha < +\infty$. For any $\varepsilon > 0$,

$$(\alpha + \varepsilon) f[(\alpha + \varepsilon)x] = f(x) = \alpha,$$

so that $f[(\alpha + \varepsilon)x] = \alpha(\alpha + \varepsilon)^{-1} < 1$, and hence $(\alpha + \varepsilon)x \in S$. Hence $\alpha + \varepsilon > d_S(x)$ for any $\varepsilon > 0$, and therefore $f(x) \geq d_S(x)$. If $\alpha \neq 0$, then $f(\alpha x) = 1$, so $\alpha x \notin S$, and hence $d_S(x) \geq \alpha = f(x)$. Thus $d_S(x) = f(x)$ if $f(x) \neq 0$. If $f(x) = 0$, then $x \in S$, so that $d_S(x) \geq 0 = f(x)$. Thus $d_S(x) = f(x)$ whenever $f(x)$ is finite. Similarly this holds whenever $d_S(x)$ is assumed to be finite. The reader will be able to convince himself that, given a suitable radiant semigroup S, the function $d_S(x)$ can be modified, on the set on which it is infinite, to yield a different upper semicontinuous submodular function defining the same semigroup. This leads us to make the following normalization.

Definition 5. A conventional function f is said to be a normalized function provided $f(-x) = -\infty$ if and only if $f(x)$ is finite.

The discussion above then establishes the following result.

THEOREM I. *In any real topological linear space* E, *there exists a one-to-one correspondence between the set of radiant semigroups of* E *and the set of upper semicontinuous normalized submodular order functions on* E. *In particular, such a correspondence is determined by the relations*

$$S = \{x \in E: 0 \leq d_S(x) < 1\}, \quad d_S(x) = \inf\{\alpha \in R: \alpha x \in S\},$$

where $d_S(x)$ *denotes the function associated with the radiant semigroup* S.

As indicated in [1], angular semigroups are of considerable importance. We can give necessary and sufficient conditions that a radiant semigroup S be angular, in terms of the order function defining S.

Definition 6. Let f be any function defined on E which assumes either real values or the values $+\infty$ and $-\infty$, and let F be the set of points at which f is finite-valued. If x_0 is any point in E, by the restricted limit inferior of f at x_0 we shall mean $\sup_U \inf_{x \in F \cap U} f(x)$, where U runs over the neighborhoods of x_0.

THEOREM II. *Let* S *be a radiant semigroup in a real linear topological space* E. *A necessary and sufficient condition that* S *be angular is that the restricted limit inferior* μ *of* d_S *at* 0 *satisfy* $\mu \leq 1$.

Proof. In any case $\mu \geq 0$. If S is angular, then given any neighborhood U of 0; there exists an element $x_1 \in U \cap S \subset U \cap F$, so that $d_S(x_1) < 1$, $\inf_{x \in U \cap F} d_S(x) < 1$, and hence $\mu \leq 1$. Conversely, if $\mu \leq 1$, then $\phi(U) = \inf_{x \in U \cap F} d_S(x)$ satisfies $\phi(U) \leq 1$. If $U_1 \subset U_2$, then $\phi(U_1) \geq \phi(U_2)$, so that S will be angular if we show that

$\phi(U) < 1$ for any U. Let V be a neighborhood of 0 such that $2V \subset U$. Since $\phi(V) \leq 1$, there exists an element $x_1 \in V \cap F$ such that $d_S(x_1) < 1 + \frac{1}{2}$. Then $2x_1 \in S$, by properties of one-dimensional radiant semigroups, and hence $d_S(2x_1) < 1$. Since $2x_1 \in U \cap F$, we have $\phi(U) < 1$, which completes the proof.

The connection between this characterization and the characterizations in [1,2,3] is not at all clear. For example, one can not say that the set $\{x \in E: d_S(x) = 1\}$ is the graph of the Hille-Zorn subadditive function describing S, as might be expected. In fact, this set may be empty. In connection with this, however, it is easily seen that the following is true:

THEOREM III. *A necessary and sufficient condition that a radiant semigroup* S *be a cone is that* d_S *take on only the values* $-\infty$, 0, $+\infty$; *or equivalently, that the set* $\{x \in E: d_S(x) = 1\}$ *be empty.*

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