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Convexity Properties for Cycle Spaces

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1. Introduction

In this article we study compact q-cycles on a complex reduced analytic space X, mainly in the case where q is the maximal dimension of a compact (irreducible) analytic subset of X.

We first give a result that generalizes a classical result due to Norguet and Siu [18] about finiteness of compact hypersurfaces in a p-convex manifold; it gives a suitable sufficient condition for X to have only finitely many irreducible compact q-cycles.

Theorem 1. *Let X and Y be complex spaces such that X is contained in Y as a locally closed analytic subset. Suppose that*:

(a) $H^q(X, \Omega^q_X)$ *has finite dimension over* $\mathbb{C}$, *say* N; *and*
(b) $H^{q+1}(Y, \mathcal{F}) = 0$ *for every coherent subsheaf* $\mathcal{F} \subset \Omega^q_Y$.

Then X has at most N compact irreducible analytic subsets of dimension q.

We then study the convexity properties of the space of compact q-cycles $\mathcal{C}_q(X)$.

Theorem 2. *Let X be a cohomologically q-complete complex space that is Kählerian and $(q+r)$-convex for some nonnegative integer r. Then $\mathcal{C}_q(X)$ is r-complete with corners.*

This looks like a nice "convexity transfer", but it is quite weak because the r-convexity with corners is not so restrictive for $r > 0$. The method is similar to the one used in [18] but requires us to work with r-plurisubharmonic functions (see Section 3.1 for definitions) and to prove an approximation result by functions that are r-convex with corners.

Theorem 3. *Let Z be a complex space admitting a continuous exhaustion function φ that is q-plurisubharmonic. If Z belongs to $\mathcal{S}_0$, then Z is q-complete with corners.*

Note. $\mathcal{S}_0$ is the class of complex spaces such that, on every relatively compact open subset, there exist continuous strongly plurisubharmonic functions. For instance,

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Z belongs to $\mathcal{S}_0$ if Z is *K-complete* [15] and a fortiori if Z is holomorphically separable (see Lemma 5).

THEOREM 4. *Let Z be a complex space and let φ be a strongly q-plurisubharmonic continuous function on Z. Then, for every $\varepsilon \in C^0(Z, \mathbb{R})$ with $\varepsilon > 0$, there is a function $\tilde{\varphi}$ on Z such that $\tilde{\varphi}$ is q-convex with corners and $|\tilde{\varphi} - \varphi| < \varepsilon$.*

In order to get more information than that stated in Theorem 2—namely, the (relative) 0-completeness (0-complete $=$ Stein) of $\mathcal{C}_q(X)$—we must make a more restrictive hypothesis: we ask X to be k-Stein in the sense of [7] for some integer $k \geq q$. This is slightly more than usual k-completeness, but it gives (strong) convexity information for q-cycles in a situation where they are not necessarily maximal.

THEOREM 5. *Let X be a k-Stein space via the mapping $\pi\colon X \to \mathbb{P}^k$ for some integer $k \geq q$. Then every connected component $\mathcal{Z}$ of $\mathcal{C}_q(X)$ has a continuous exhaustion function that is strongly plurisubharmonic along the fibers of $\pi_\star\colon \mathcal{Z} \to \mathcal{C}_q(\mathbb{P}^k)$, where $\pi_\star$ is the direct image map of q-cycles.*

2. Proof of Theorem 1

2.1. Preliminaries

Let Z be a complex space. The definition of q-convexity used in this paper is that of Andreotti and Norguet [4]: A function $\varphi \in C^2(Z, \mathbb{R})$ is said to be *q-convex* if, for every point $a \in Z$, there is a coordinate patch $(U, \iota, \hat{U})$, where $U \ni a$, $\hat{U}$ is open in some Euclidean complex space, and $\iota\colon U \to \hat{U}$ is a holomorphic embedding such that there exists a $\hat{\varphi} \in C^2(\hat{U}, \mathbb{R})$ with $\hat{\varphi} \circ \iota = \varphi|_U$ and the Levi form of $\hat{\varphi}$ has at most q nonpositive eigenvalues at every point of $\hat{U}$. We say that Z is *q-convex* if there exists a class C^2 exhaustion function φ on Z that is q-convex on $Z \setminus K$, where $K \subset Z$ is a compact set. If we may take K as the empty set, then Z is said to be *q-complete*.

The space Z is said to be *cohomologically q-convex* (resp., *cohomologically q-complete*) if $H^i(X, \mathcal{F})$ has finite dimension as a complex vector space (resp., $H^i(X, \mathcal{F})$ vanishes) for every $i > q$ and every coherent sheaf $\mathcal{F}$ on X. (Cohomologically 0-complete $\equiv$ Stein $\equiv$ 0-complete.)

Also, we recall the definition of the sheaf of germs of holomorphic j-forms Ω_Z^j on Z, $j \in \mathbb{N}$. If Z is an analytic subset of a domain $D \subset \mathbb{C}^n$ then we define

$$\Omega_Z^j := (\Omega_D^j / (\mathcal{I}_Z \Omega_D^j + d\mathcal{I}_Z \wedge \Omega_D^{j-1}))|_Z,$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z in D. In general, by using local embeddings and patching we obtain coherent $\mathcal{O}_Z$-modules Ω_Z^j, $j \in \mathbb{N}$.

If $\pi : X \to Y$ is a holomorphic map of complex spaces, then there is a canonical $\mathcal{O}_Y$-module morphism $\Omega_Y^j \to \pi_* \Omega_X^j$ that induces a map in the cohomology

$H^q(Y, \Omega_Y^j) \to H^q(Y, \pi_*\Omega_X^j)$; furthermore, when composed with the natural map $H^q(Y, \pi_*\Omega_X^j) \to H^q(X, \Omega_X^j)$, this morphism gives

$$u_j\colon H^q(Y, \Omega_Y^j) \to H^q(X, \Omega_X^j).$$

REMARK 1. Let X be a complex space and let $\Gamma \subset X$ be a compact analytic subset of dimension q. By a classical theorem due to Lelong [16], there exists a canonical trace map (see [8] for details)

$$\mathrm{Tr}_\Gamma\colon H^q(X, \Omega_X^q) \to \mathbb{C}, \quad \xi \mapsto \int_\Gamma \xi,$$

given by "integrating cohomology classes" on Γ.

Now suppose that X is contained as an analytic subset of a complex space Y. The natural map

$$H^q(Y, \Omega_Y^j) \to H^q(X, \Omega_X^j),$$

which can also be defined more explicitly via the Čech cohomology and the foregoing trace maps, give the commutative diagram

$$\begin{array}{ccc} H^q(Y, \Omega_Y^q) & \longrightarrow & H^q(X, \Omega_X^q) \\ \big\downarrow & & \big\downarrow \\ \mathbb{C} & = & \mathbb{C}\,, \end{array}$$

where the vertical arrows are given by integrating on Γ.

REMARK 2. Let Z be a compact complex space of dimension n and let $\pi : \tilde{Z} \to Z$ be the normalization map. Then, integrating cohomology classes on Z and $\tilde{Z}$ gives the canonical commutative diagram

$$\begin{array}{ccc} H^n(Z, \Omega_Z^n) & \longrightarrow & H^n(\tilde{Z}, \Omega_{\tilde{Z}}^n) \\ \big\downarrow & & \big\downarrow \\ \mathbb{C} & = & \mathbb{C}\,. \end{array}$$

LEMMA 1. *If X is an analytic subset of a complex space Y with $H^{q+1}(Y, \mathcal{F})$ vanishing for every coherent subsheaf $\mathcal{F} \subset \Omega_Y^j$, then u_j is surjective.*

Proof. Let $\nu\colon X \to Y$ denote the inclusion map. We note that the natural map $\beta_j : \Omega_Y^j \to \nu_*\Omega_X^j$ is a surjection of $\mathcal{O}_Y$-modules; then the hypothesis and an exact cohomology sequence give the surjectivity of $H^q(Y, \Omega_Y^j) \to H^q(Y, \nu_*\Omega_X^j)$. We conclude easily since $H^\bullet(Y, \nu_*\Omega_X^j) \cong H^\bullet(X, \Omega_X^j)$. □

LEMMA 2. *Let X be a complex space and let $\pi : \tilde{X} \to X$ be the normalization map. Let $n := \dim(X)$. Then the natural map $H^n(X, \Omega_X^n) \to H^n(\tilde{X}, \Omega_{\tilde{X}}^n)$ is surjective.*

Proof. Let $\gamma\colon \Omega_X^n \to \pi_*\Omega_{\tilde{X}}^n$ be the canonical map. Because $\operatorname{Ker}\gamma$ and $\operatorname{Coker}\gamma$ are supported on $\operatorname{Sing}(X)$, which has complex dimension less than n, the lemma follows by standard machinery of long exact sequences and [21]. □

LEMMA 3. *Let Z be a normal compact complex space of dimension n. Then the following statements hold.*

(1) *The canonical map $H_c^n(Z_{\mathrm{reg}}, \mathcal{F}) \to H^n(Z, \mathcal{F})$ is bijective for every $\mathcal{F} \in \operatorname{Coh}(Z)$.*

(2) *If Z is connected, then $\operatorname{Tr}_Z\colon H^n(Z, \Omega_Z^n) \to \mathbb{C}$ is an isomorphism.*

Proof. (1) Let $A := \operatorname{Sing}(Z)$; hence $\dim(A) \le n-2$. Then the exact sequence

$$H^{n-1}(A, \mathcal{F}|_A) \to H_c^n(Z_{\mathrm{reg}}, \mathcal{F}) \to H^n(Z, \mathcal{F}) \to H^n(A, \mathcal{F}|_A),$$

where $\mathcal{F}|_A$ is the topological restriction, together with [21] gives the conclusion.

(2) We have a canonical diagram that is commutative; namely,

$$\begin{tikzcd} H_c^n(Z_{\mathrm{reg}}, \Omega_Z^n) \arrow[r] \arrow[d] & H^n(Z, \Omega_Z^n) \arrow[d] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C}. \end{tikzcd}$$

Thus the map in statement (2) of the lemma is not zero; hence it is surjective. Then we conclude easily by statement (1), taking into account that

$$H_c^n(Z_{\mathrm{reg}}, \Omega_Z^n) \simeq (H^0(Z_{\mathrm{reg}}, \mathcal{O}_Z))^\star \simeq \mathbb{C}. \qquad \square$$

LEMMA 4. *Let Y be a complex space and let $Z_1, \dots, Z_m$ be distinct irreducible compact analytic subsets of Y of dimension q. Then the map*

$$H^q(Y, \Omega_Y^q) \ni \xi \mapsto \left(\int_{Z_1} \xi, \dots, \int_{Z_m} \xi\right) \in \mathbb{C}^m$$

is surjective if $H^{q+1}(Y, \mathcal{F}) = 0$ for every coherent subsheaf $\mathcal{F} \subset \Omega_Y^q$.

Proof. Let $Z := Z_1 \cup \dots \cup Z_m$. By Lemma 1, it suffices to prove Lemma 4 for $Y = Z$. But this is a straightforward consequence of Lemmas 2 and 3 and the canonical commutative diagram in Remark 2. □

2.2. Proof of Theorem 1 and Consequences

For a complex space Z, let $V_q(Z)$ be the vector space of compact cycles of dimension q with complex coefficients. That is, an element of $V_q(Z)$ is given as a formal sum

$$\sum_{i=1}^m \lambda_i \Gamma_i,$$

where $m \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, and $\Gamma_i \subset Z$ are distinct compact irreducible analytic subsets of dimension q. By integrating cohomology classes, we thus obtain a canonical complex linear map

$$\Phi_q(Z)\colon V_q(Z) \to H^q(Z, \Omega^q_Z)^\star := \mathrm{Hom}_{\mathbb{C}}(H^q(Z, \Omega^q_Z), \mathbb{C}).$$

Let now X and Y be complex spaces such that $X \subset Y$ as a locally closed analytic subset. Then we have a commutative diagram of canonical maps:

$$\begin{array}{ccc} V_q(X) & \longrightarrow & H^q(X, \Omega^q_X)^\star \\ \downarrow & & \downarrow \\ V_q(Y) & \longrightarrow & H^q(Y, \Omega^q_Y)^\star. \end{array}$$

Coming back to the situation of Theorem 1, we deduce by Lemma 4 that $\Phi_q(Y)$ is injective. Thus, the preceding diagram and the (obvious) injectivity of the map $V_q(X) \to V_q(Y)$ imply that $\Phi_q(X)$ is injective, which proves Theorem 1. □

COROLLARY 1. *Let X be a cohomologically $(q-1)$-convex space that is cohomologically q-complete. Then X has finitely many compact irreducible analytic subsets of dimension q, and their number is bounded by* $\dim H^q(X, \Omega^q_X)$.

COROLLARY 2. *Let X be an irreducible complex space of dimension n. Suppose that X is cohomologicaly $(n-2)$-convex and noncompact. Then X has only finitely many compact irreducible hypersurfaces, and their number is bounded by* $\dim H^{n-1}(X, \Omega^{n-1}_X)$.

Proof. Since X is cohomologically $(n-1)$-complete by [22], Corollary 2 follows immediately from Corollary 1. □

REMARK 3. For X smooth, we recover a result in [18].

PROPOSITION 1. *Let X be a complex space such that $H^{q+1}(X, \mathcal{F})$ vanishes for every coherent subsheaf $\mathcal{F} \subset \Omega^q_X$. Then $\mathcal{C}_q(X)$ is K-complete. In particular, on every irreducible component of $\mathcal{C}_q(X)$ there exist smooth strongly plurisubharmonic functions; a fortiori, they are Kählerian.*

Note. By [15] we say that a complex space Z is *K-complete* if, for every point $z_0 \in Z$, there is a holomorphic mapping $F\colon Z \to \mathbb{C}^N$, $N = N(z_0)$, such that z_0 is isolated in its fiber $F^{-1}(F(z_0))$.

REMARK 4. Another possibility for obtaining continuous strongly plurisubharmonic functions on $\mathcal{C}_q(X)$ is to have a (q,q)-form α, smooth of class C^2 on X, such that $\partial\bar{\partial}\alpha \gg 0$ in the sense of Lelong. In particular this holds if X has a Kähler form ω and if $\omega^{q+1} := \omega \wedge \cdots \wedge \omega$ (the product is taken $q+1$ times) is $\partial\bar{\partial}$-exact; see [27].

Proof of Proposition 1. To conclude, by [23, Cor. 6, p. 235] it remains to show that $\mathcal{C}_q(X)$ is K-complete. In order to settle this, we give the following lemma.

LEMMA 5. *Let Z be a holomorphically separable complex space. Then Z is K-complete.*

Proof. First we demonstrate the following.

CLAIM. *Let $a \in Z$ and let $Y \subset Z$ be an analytic subset containing a. Then, for every discrete sequence $\{z_\nu\} \subset Y$ such that $z_\nu \neq a$ for all ν, there exists a holomorphic function f on Z with $f(a) = 0$ and $f(z_\nu) \neq 0$ for all ν. Moreover, if* $\dim_a Y > 0$, *then we may choose f with the additional property that*

$$\dim_a Y \cap \{f = 0\} < \dim_a Y.$$

To see this, we let $E = \{f \in \mathcal{O}(Z);\ f(a) = 0\}$ and $G_\nu = \{f \in E;\ f(z_\nu) \neq 0\}$. Clearly E is a nonempty Fréchet space and each G_ν is a dense open subset of E for every ν. Since the index set is at most countable, by Baire's theorem $\bigcap_\nu G_\nu$ is dense in E and a fortiori is not empty. Then any $f \in \bigcap_\nu G_\nu$ will do the job.

Now, in order to settle the "moreover", let $\{Y_\mu\}_\mu$ be the irreducible components of Y that contain a and are of positive dimension. By adding further points to the sequence $\{z_\nu\}$, we may assume that on each Y_μ there is at least one z_ν. Then we conclude as before.

To finish the proof of the lemma, we let $z_0 \in Z$ and set $Y :=$ the union of all irreducible components of Z containing z_0. Then Y is a neighborhood of z_0 and $n := \dim(Y) < \infty$. The claim readily gives holomorphic functions $f_0, \dots, f_n \in \mathcal{O}(Z)$ such that setting $F := (f_0, \dots, f_n)\colon Z \to \mathbb{C}^{n+1}$ yields $Y \cap F^{-1}(F(z_0)) = \{z_0\}$. This proves the lemma and hence the proposition as well. □

REMARK 5. Let X be a complex space and let

$$\mathrm{AN}_q\colon H^q(X, \Omega_X^q) \to \mathcal{O}(\mathcal{C}_q(X))$$

be the *Andreotti–Norguet transform* obtained by integrating cohomology classes [3; 4], which is well-defined by [8]. If $H^{q+1}(X, \mathcal{F})$ vanishes for every coherent subsheaf $\mathcal{F} \subset \Omega_X^q$, then $\mathrm{Im}(\mathrm{AN}_q)$ separates the points of $\mathcal{C}_q(X)$.

3. *q*-Plurisubharmonic Functions

3.1. Preliminaries

Let X be a complex space. A function $\varphi \in C^0(X, \mathbb{R})$ is said to be *q-convex with corners* [11; 12; 20] if every point of X admits an open neighborhood U on which there are finitely many q-convex functions $f_1, \dots, f_k$ such that

$$\varphi|_U = \max(f_1, \dots, f_k).$$

Denote by $F_q(X)$ the set of all functions q-convex with corners on X.

We say that X is *q-complete with corners* if there exists an exhaustion function $\varphi \in F_q(X)$. (Hence Stein spaces correspond to spaces that are 0-complete with corners.)

This q-convexity with corners is weaker than usual q-convexity. For instance, the intersection of finitely many q-complete open subsets of $\mathbb{C}^n$ is q-complete with corners, but in general it is not q-complete. However, by [11; 12], every complex space X of dimension n that is q-complete with corners is $\tilde{q}$-complete, where

$\tilde{q} = n - [n/(q+1)]$. Notice that, for q with $(n-1)/2 \le q \le n-1$, we have $\tilde{q} = n-1$; this gives no new information, in view of [10] and [19].

Note that, if $\varphi, \psi \in F_q(X)$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\chi' > 0$ and $\chi'' \ge 0$, then $\max(\varphi, \psi)$ and $\chi(\varphi)$ belong to $F_q(X)$. However, if $\{\varphi_\lambda\}_\lambda \subset F_q(X)$ is an arbitrary family and $\varphi := \sup_\lambda \varphi_\lambda$ is (even) continuous, then it may happen that φ does not belong to $F_q(X)$, as simple examples show. To avoid this, we enlarge the set $F_q(X)$ by introducing *q-plurisubharmonic* functions.

An upper semicontinuous function $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ is said to be:

(a) *subpluriharmonic* if, for every $\Omega \Subset X$ and every pluriharmonic function h defined near $\bar{\Omega}$ (i.e., h is locally the real part of a holomorphic function), we have $\varphi \le h$ on Ω when $\varphi \le h$ on $\partial\Omega$;

(b) *q-plurisubharmonic* if, for every open set $G \subset \mathbb{C}^{q+1}$ and holomorphic map $f\colon G \to X$, the function $\varphi \circ f$ is subpluriharmonic on G.

Example [14]. Let X be a complex manifold of pure dimension. Then a function $\varphi \in C^2(X, \mathbb{R})$ is q-plurisubharmonic if and only if the Levi form $L(\varphi)$ has, at every point of X, at most q nonpositive eigenvalues.

Notation. We use $P_q(X)$ to denote the set of all q-plurisubharmonic functions on X, and $SP_q(X)$ denotes the set of all *strongly q-plurisubharmonic* functions on X—that is, those $\varphi \in P_q(X)$ such that, for every $\theta \in C_0^\infty(X, \mathbb{R})$, there exists an $\varepsilon > 0$ with $\varphi + \varepsilon\theta \in P_q(X)$. Obviously we have $F_q(X) \subset SP_q(X) \cap C^0(X, \mathbb{R})$.

Remark 6. $P_0(X)$ and $SP_0(X)$ are precisely the *weakly plurisubharmonic* and *weakly strongly plurisubharmonic* functions (respectively) introduced by Fornæss and Narasimhan [13].

Remark 7. If $D \subset \mathbb{C}^n$ is an open set, then an upper semicontinuous function φ on D is subpluriharmonic if and only if φ is $(n-1)$-plurisubharmonic.

Remark 8. Let $\Omega \subset \mathbb{C}^{n+1}$ be an open set. For every unit vector $v \in \mathbb{C}^{n+1}$ we consider the directional distance $\delta_v\colon \Omega \to \mathbb{R} \cup \{\infty\}$ with respect to v, which is given as follows:

$$\delta_v(z) := \sup\{r > 0;\ z + tv \in \Omega,\ \forall t \in \mathbb{C},\ |t| < r\}, \quad z \in \Omega.$$

If Ω is q-complete with corners, then $-\log \delta_v$ is q-plurisubharmonic [14].

We shall also need the following two lemmas (the first one is obvious).

Lemma 6. *Let $\pi\colon X \to Y$ be a holomorphic map of complex spaces, and let $\psi \in P_q(Y)$. Then $\psi \circ \pi \in P_q(X)$.*

Lemma 7. *Let $\pi\colon X \to Y$ be a finite holomorphic surjective map between pure dimensional complex spaces, and let $\varphi \in P_q(X)$. Let $\psi\colon Y \to \mathbb{R} \cup \{-\infty\}$ be defined by*

$$\psi(y) = \max\{\varphi(x);\ x \in \pi^{-1}(y)\}, \quad y \in Y.$$

If ψ is continuous, then $\psi \in P_q(Y)$.

Proof. Let $A \subset Y$ be a rare analytic set such that $\pi^{-1}(A)$ is rare in X and π induces a locally biholomorphic map between $X \setminus \pi^{-1}(A)$ and $Y \setminus A$. Clearly $\psi_1 := \psi|_{Y \setminus A} \in P_q(Y \setminus A)$. Let ψ_2 denote the upper semicontinuous extension of ψ_1 to Y; since A is locally complete pluripolar, $\psi_2 \in P_q(Y)$ in view of [25, Prop. 6]. But since ψ is continuous, $\psi = \psi_2$, whence the lemma. □

3.2. Proof of Theorem 3

First we quote the following from [25].

THEOREM 6. *Let X be a complex space admitting a continuous exhaustion function Φ that is strongly q-plurisubharmonic. Then X is q-complete with corners.*

Thus, to conclude Theorem 3, we must produce an exhaustion function $\Phi \in SP_q(X) \cap C^0(X, \mathbb{R})$ as in Theorem 6.

For this we proceed as follows. Clearly we may suppose that $\varphi > 0$. (Otherwise, replace φ by $\varphi + C$ for some large constant $C > 0$.) For $n \in \mathbb{N}^\star$ we let $K_n := \{\varphi \leq n\}$ and $D_n = \{\varphi < n+2\}$. Let $\psi_n \in C^0(X, \mathbb{R})$, $\psi_n > 0$, such that ψ_n is strongly plurisubharmonic on D_{n+1}.

Choose constants $a_n > 1$ such that $a_n > \varphi + \psi_n$ on K_{n+2}, and define $h_n : \mathbb{R} \to \mathbb{R}$ by

$$h_n(t) := \max(t, a_n(t - n - 1)), \quad t \in \mathbb{R}.$$

Then h_n is strictly increasing and convex, $h_n(t) = t$ for $t \leq n+1$, and $h_n(n+2) \geq a_n$; thus $h_n(\varphi) > \varphi + \psi_n$ on a neighborhood of the set $\{\varphi = n+2\}$, a fortiori on a neighborhood of ∂D_n. We may thus define $\varphi_n \in C^0(X, \mathbb{R})$ by

$$\varphi_n = \begin{cases} \max(h_n(\varphi), \varphi + \psi_n) & \text{on } D_n, \\ h_n(\varphi) & \text{on } X \setminus D_n. \end{cases}$$

One may easily check that $\varphi_n > 0$, φ_n is exhaustive, $\varphi_n \in P_q(X)$, and $\varphi_n|_{D_{n-1}} \in SP_q(D_{n-1})$ (where it equals $\varphi + \psi_n$). Now, if the sequence $\{\varepsilon_n\}_n$, $\varepsilon_n > 0$, decreases (fast enough) to zero, then we may define $\Phi \in C^0(X, \mathbb{R})$ by

$$\Phi := \varphi + \sum_n \varepsilon_n \varphi_n.$$

Since $\Phi \geq \varphi$, it follows that Φ is exhaustive; we need only check that $\Phi \in SP_q(X)$, which is straightforward owing to the construction of the φ_n. □

Mutatis mutandis, the same proof gives also the following.

COROLLARY 3. *Let $\pi : Z \to Y$ be a holomorphic map of complex spaces such that there exists an exhaustion function $\varphi \in C^0(Z, \mathbb{R})$ that is q-plurisubharmonic along the fibers of π.*

If $Z \in \mathcal{S}_0$, then there exists an exhaustion function $\Phi \in C^0(Z, \mathbb{R})$ that is strongly q-plurisubharmonic along the fibers of π.

COROLLARY 4. *Let X be a complex space that is an increasing union of Stein open subsets. Then X is Stein if and only if X admits a plurisubharmonic exhaustion function $\varphi : X \to \mathbb{R}$.*

3.3. *Proof of Theorem 4*

We give the proof in three steps. The first two deal with a local statement; in the final step, we prove our theorem by a (standard) perturbation procedure.

Let $V \Subset U \Subset X$ be open subsets such that U is holomorphically embedded as an analytic subset of some open set $\hat{U} \subset \mathbb{C}^N$.

Step 1: There exist an open neighborhood D of U in $\hat{U}$ and $\hat{\varphi} \in P_q(D)$ that is locally bounded from below and such that $\hat{\varphi}|_U = \varphi|_U$.

In order to check this, consider the Hartogs domain U_φ of φ defined by

$$U_\varphi := \{(x,t) \in U \times \mathbb{C};\ |t| < \exp(-\varphi(x))\}.$$

It can be seen that U_φ is q-complete with corners (applying e.g. Theorem 6); then, as an analytic subset of $\hat{U} \times \mathbb{C}$, it admits a neighborhood system of open sets that are q-complete with corners (see [26, Prop. 1, p. 1194]). Therefore, if $\Psi\colon \hat{U} \to \mathbb{R}$ is a continuous function that extends $\varphi|_U$, then there exists an open set $\Omega \subset \hat{U} \times \mathbb{C}$ that is q-complete with corners and such that:

(a) $\Omega \cap (U \times \mathbb{C}) = U_\varphi$;

(b) $\Omega \subset \hat{U}_\Psi := \{(z,t) \in \hat{U} \times \mathbb{C};\ |t| < \exp(-\Psi(z))\}$.

Let $D := \{z \in \mathbb{C}^N;\ (z,0) \in \Omega\}$. Then D is an open subset of $\hat{U}$ that contains U. Let δ denote the boundary distance function of Ω with respect to $v = (0,\dots,0,1) \in \mathbb{C}^{N+1}$ (see Remark 8). Then $-\log\delta$ is q-plurisubharmonic. Define $\hat{\varphi}\colon D \to \mathbb{R}$ by setting, for $z \in D$,

$$\hat{\varphi}(z) = -\log\delta(z,0).$$

Then $\hat{\varphi}$ and D are as desired (e.g., $\hat{\varphi} \geq \Psi|_D$), from which Step 1 follows.

Step 2: For every $c > 0$ there exists $\psi \in F_q(V)$ with $|\psi - \varphi| < c$ on V.

We show this by using the next two lemmas (the first one is quoted from [9]).

Lemma 8. *Let $D \Subset \mathbb{C}^N$ be an open set and let $\varphi \in C^0(D,\mathbb{R}) \cap P_q(D)$. Then, for every $\varepsilon > 0$, there exists a $\tilde{\varphi} \in F_q(D)$ with $|\tilde{\varphi} - \varphi| < \varepsilon$.*

Lemma 9. *Let $\Omega \subset \mathbb{C}^N$ be an open set and let $\varphi \in P_q(\Omega)$ with $\varphi \geq 0$. Then, for every $W \Subset \Omega$, there exists a sequence $\{\psi_\nu\}_\nu \subset F_q(W)$ that decreases pointwise to $\varphi|_W$.*

Proof. By standard arguments, this reduces to the following claim.

Claim. *Let $v \in C^0(\Omega,\mathbb{R})$ with $\varphi < v$ on $\bar{W}$. Then there exists a $\psi \in F_q(W)$ such that $\varphi|_W < \psi < v|_W$.*

In order to show this, consider $W \Subset \Omega$ an open set such that $W \Subset W'$. Then choose $r > 0$ (small enough) such that, for every $\xi \in \mathbb{C}^N$ with $\|\xi\| \leq r$ and $z \in \bar{W}$, one has $\{\xi\} + \overline{W'} \subset \Omega$, $\{\xi\} + \bar{W} \subset W'$, and $\varphi(z+\xi) < v(z)$.

Let $g \in C_0^\infty(\mathbb{C}^N,\mathbb{R})$ be such that $0 \leq g \leq 1$, $g(0) = 1$, and $\operatorname{supp}(g) \subset B(0;r)$ (the ball of radius r in $\mathbb{C}^N$ centered at the origin). Then define $\psi'\colon W \to [0,\infty)$ by setting

$$\psi'(z) = \sup\{\tilde{\varphi}(z+\xi)g(\xi);\ \xi \in \mathbb{C}^N\}, \quad z \in W,$$

where $\tilde{\varphi}$ is the trivial extension of φ to $\mathbb{C}^N$ (i.e., $\tilde{\varphi} = 0$ on $\mathbb{C}^N \setminus \Omega$). Rewriting this definition yields $\psi' = \sup_{\lambda \in \Lambda} \psi_\lambda$, where $\psi_\lambda \colon W \to [0, \infty)$ is given by $\psi_\lambda(z) = \tilde{\varphi}(\lambda)g(\lambda - z)$ for $z \in W$ and $\lambda \in \Lambda := \bar{W} + \overline{B(0;r)}$. Since the family $\{\psi_\lambda\}_\lambda$ consists of smooth functions whose real Hessian is uniformly bounded from below, we deduce that ψ' is continuous; hence ψ' is q-plurisubharmonic.

Apply now Lemma 8 to $\psi' + \varepsilon\theta$, where θ is continuous and strongly plurisubharmonic near $\bar{W}$ with some $\varepsilon > 0$ sufficiently small. The claim follows, whence Lemma 9. □

Now, to conclude Step 2, take $V_1 \Subset D$ an open set with $\bar{V} \subset V_1$. By Lemma 9 there exists a sequence $\{\psi_\nu\}_\nu \subset F_q(V_1)$ decreasing pointwise to $\hat{\varphi}|_{V_1}$. Since $\bar{V}$ is compact and $\hat{\varphi}|_{\bar{V}}$ is continuous, $\psi := \psi_\nu|_V$ (for ν large enough) fulfills our requirements.

Step 3: End of Proof of Theorem 4.

Choose open sets $V_i \Subset U_i \Subset W_i \Subset X$, $i \in \mathbb{N}$, such that $\{V_i\}_i$ is a covering of X, each W_i embeds holomorphically into some open subset of $\mathbb{C}^{N_i}$, and $\{W_i\}_i$ is locally finite. Select $\rho_i \in C_0^\infty(X, \mathbb{R})$ such that $-1 \le \rho_i \le 1$, $\rho_i \equiv 1$ on U_i, $\rho_i = -1$ on ∂V_i, and $\operatorname{supp}(\rho_i) \subset W_i$.

Consider $\varepsilon_i > 0$ to be constants sufficiently small that $\varphi + \varepsilon_i\rho_i \in SP_q(X)$ and

$$3\varepsilon_i < 2 \inf_{V_i} \varepsilon. \tag{$\star$}$$

Applying Step 2, there exist $\varphi_i \in F_q(U_i)$ with

$$|\varphi_i - \varphi - \varepsilon_i\rho_i| < \varepsilon_i/2 \ \text{ on } \bar{V}_i. \tag{$\sharp$}$$

Now, for every $x \in X$ set $I(x) := \{i \in I;\ V_i \ni x\}$; then define $\tilde{\varphi} \colon X \to \mathbb{R}$ by

$$\tilde{\varphi}(x) = \sup_{i \in I(x)} \varphi_i(x), \quad x \in X.$$

By ($\star$) and ($\sharp$) we obtain that $|\tilde{\varphi} - \varphi| < \varepsilon$ and every point x_0 of X has an open neighborhood $B \subset \bigcap_{i \in I(x_0)} V_i$ such that

$$\tilde{\varphi}|_B = \max_{i \in I(x_0)} \varphi_i|_B,$$

hence $\tilde{\varphi} \in F_q(X)$. □

4. Proof of Theorem 2

4.1. Some General Considerations

Let q be a nonnegative integer and let $\varphi \colon X \to [0, \infty)$ be a continuous function. Consider $\Phi \colon \mathcal{C}_q(X) \to [0, \infty)$ defined by setting

$$\Phi(\Gamma) := \max\{\varphi(x);\ x \in |\Gamma|\}, \quad \Gamma \in \mathcal{C}_q(X),$$

where $\Gamma = n_1\Gamma_1 + \cdots + n_k\Gamma_k$ for $n_i \in \mathbb{N}^\star$ and where the Γ_i are distinct irreducible compact analytic subsets of X of dimension q; as usual, we put $|\Gamma| := \Gamma_1 \cup \cdots \cup \Gamma_k$, the support of Γ. As in [18], we check easily that Φ is continuous.

REMARK 9. If, moreover, φ is proper and X is Kählerian, then the restriction of Φ to every connected component of $\mathcal{C}_q(X)$ is proper. (For Kählerian metrics on complex spaces we refer the reader to [17] and [28].)

LEMMA 10. *Let r be a nonnegative integer such that φ is $(q+r)$-convex on $X \setminus \{\varphi = 0\}$. Then Φ is r-plurisubharmonic on $\mathcal{C}_q(X) \setminus \{\Phi = 0\}$.*

Proof. We proceed as in [18, pp. 213–214] and consider (for the sake of clarity) only the case where X is smooth. Recall that $\mathcal{C}_q(X)$ is a complex space and that the incidence set

$$G := \{(x, \Gamma) \in X \times \mathcal{C}_p(X);\ x \in |\Gamma|\}$$

is an analytic subset of $X \times \mathcal{C}_q(X)$. Denote by π_1 and π_2 the natural projections from G into X and $\mathcal{C}_q(X)$, respectively. Notice that π_2 is proper.

Now let $\Gamma_0 \in \mathcal{C}_q(X)$ with $\Phi(\Gamma_0) > 0$. We show that Φ is r-plurisubharmonic on a suitable neighborhood W of Γ_0 in $\mathcal{C}_q(X)$. For this we let λ be a real number with $0 < \lambda < \Phi(\Gamma_0)$ and set

$$K := |\Gamma_0| \cap \{\varphi \geq \lambda\}.$$

Clearly K is a compact subset of X. Take $x \in K$ arbitrarily. Then there exists a local chart (U, τ, Ω) with $U \ni x$, $\Omega \subset \mathbb{C}^N$ open, and $0 \in \Omega$ such that:

(a) $\tau(x) = 0$;
(b) $\Delta^n \Subset \Omega$, where Δ is the open unit disc in $\mathbb{C}$;
(c) $(\partial\Delta^{n-q} \times \bar{\Delta}^q) \cap \tau(|\Gamma_0| \cap U) = \emptyset$; and
(d) the restriction of φ to $U \cap \tau^{-1}(\Delta^{n-q} \times \{\xi\})$ is r-convex for every $\xi \in \Delta^q$.

Let $D_x = \tau^{-1}(\Delta^n)$. Choose an open neighborhood W_x of Γ_0 in $\mathcal{C}_q(X)$ such that

(i) $\tau(|\Gamma| \cap U) \cap (\partial\Delta^{n-q} \times \overline{\Delta^q}) = \emptyset$ for $\Gamma \in W_x$, and
(ii) $\tau(|\Gamma| \cap U) \cap \Delta^n \neq \emptyset$ for $\Gamma \in W_x$.

For every $\xi \in \Delta^q$, the map

$$\sigma_\xi : (\tau \circ \pi_1|_U)^{-1}(\Delta^{n-q} \times \{\xi\}) \cap \pi_2^{-1}(W_x) \to W_x$$

is an analytic (branched) covering (of some finite degree), where σ_ξ is induced by π_2. It follows that, for $\xi \in \Delta^q$, the function Ψ_ξ on W_x given by

$$\Psi_\xi(\Gamma) := \max\{(\varphi \circ \pi_1)(y);\ y \in \sigma_\xi^{-1}(\Gamma)\}, \quad \Gamma \in W_x,$$

is *continuous* (here we use the topology of $\mathcal{C}_q(X)$!) and, by Lemmas 6 and 7, $\Psi_\xi \in P_r(W_x)$. Define $\Phi_x : W_x \to [0, \infty)$ by setting

$$\Phi_x(\Gamma) := \sup\{\Psi_\xi(\Gamma);\ \xi \in \Delta^q\}, \quad \Gamma \in W_x.$$

Since K is compact, there exist $x_1, \ldots, x_m \in K$ such that $K \subset \bigcup_{j=1}^m D_{x_j}$. Since $\Phi(\Gamma_0) > \lambda$ and φ is less than λ on the compact set $|\Gamma_0| \setminus \bigcup_{j=1}^m D_{x_j}$, there exists an open neighborhood W of Γ_0, $W \subset \bigcap_{j=1}^m W_{x_j}$, with the following properties:

(a) the infimum of Φ on W is $> \lambda$; and
(b) the supremum of φ on $\pi_1(\pi_2^{-1}(W)) \setminus \bigcup_{j=1}^m D_{x_j}$ is $< \lambda$.

These imply easily that, on W,

$$\Phi = \max(\Phi_{x_1}, \dots, \Phi_{x_m}).$$

Consequently, since Φ is continuous and since each Φ_{x_j} is a supremum of a family of r-plurisubharmonic functions, it follows that $\Phi \in P_r(W)$. □

This lemma and Remark 9 give the next proposition.

PROPOSITION 2. *Let X be a $(q+r)$-convex space for some nonnegative integers q and r. Then there is a continuous function $\Phi\colon \mathcal{C}_q(X) \to [0, \infty)$ that is r-plurisubharmonic.*

If, moreover, X is Kählerian, then we may choose Φ such that its restriction to every connected component of $\mathcal{C}_q(X)$ is proper.

Proof. Take Φ_1 as in Lemma 10. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ be increasing and convex, so that $\{\chi = 0\} = (-\infty, 1]$. Then $\Phi := \chi(\Phi_1)$ is as desired. □

4.2. Proof of Theorem 2

The hypotheses of Theorem 2 and Proposition 1 show that $\mathcal{C}_q(X)$ belongs to $\mathcal{S}_0$. On the other hand, Proposition 2 gives a continuous r-plurisubharmonic function Φ on $\mathcal{C}_q(X)$ that is exhaustive on every connected component of $\mathcal{C}_q(X)$. Theorem 2 now follows easily from Theorem 3. □

5. Proof of Theorem 5

Let X be a k-Stein space via the holomorphic map $\pi : X \to \mathbb{P}^k$. We refer the reader to [7] for definitions and further properties. Because π has Stein fibers, we obtain a canonical map $\pi_\star\colon \mathcal{C}_q(X) \to \mathcal{C}_q(\mathbb{P}^k)$ that is holomorphic [5, Thm. 6, p. 109]. By [7] again, there exists a function $\varphi\colon X \to [0, \infty)$ that is proper and of class C^2 such that, for every compact set $K \subset X$, there exists a constant $C_K > 0$ such that

$$i\partial\bar{\partial}\varphi + C_K \pi^\star(\omega)$$

is positive definite on K, where ω is the Kähler form of the Fubini–Study metric on $\mathbb{P}^k$. Consider now the function

$$\Psi_K : \mathcal{C}_q(X) \to [0, \infty)$$

given by

$$\Psi_K(\Gamma) = \int_\Gamma \varphi\beta_K, \quad \Gamma \in \mathcal{C}_q(X),$$

where β_K is the d-exact (q, q)-form that follows from the equation

$$(C_K\pi^\star(\omega) + i\partial\bar{\partial}\varphi_2)^{q+1} = C_K^{q+1}\pi^\star(\omega^{q+1}) + i\partial\bar{\partial}\varphi \wedge \beta_K.$$

Then, using [6], we have that Ψ_K is continuous and strongly plurisubharmonic on the cycles contained in the interior of K. Therefore, $\mathcal{C}_q(X) \in \mathcal{S}_0$.

On the other hand, X is Kählerian by [24] and so the volume is constant on connected components of $\mathcal{C}_q(X)$. Hence the method of Lemma 10 (with the function φ described in this section) produces a continuous function

$$\Phi\colon \mathcal{C}_q(X) \to [0, \infty),$$

whose restriction to every connected component of $\mathcal{C}_q(X)$ is proper and whose restriction to the fibers of $\pi_\star$ becomes plurisubharmonic.

The conclusion of the theorem now follows immediately by Corollary 3 and Remark 6. $\square$

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