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ON THE CONNECTION OF THE FIRST-ORDER FUNCTIONAL CALCULUS WITH MANY-VALUED PROPOSITIONAL CALCULI

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From the results presented in my paper [2] it follows that it is possible to approximate the first-order functional calculus by many valued propositional calculi; in this paper* we shall describe this approximation.

We shall use the terminology of [2] and in particular:

(1) individual variables: $x_1, x_2, \ldots$ [or simply x],

(2) apparent individual variables: $a_1, a_2, \ldots$ [or simply a],

(3) finite number of functional variables: $f_1, \ldots, f_c$,

(4) logical constants: ' (negation), + (alternative), Π (general quantifier),

(5) atomic expressions: $R, R_1, R_2, \ldots$; expressions: $E, F, G, E_1, F_1, G_1, \ldots$[1]

(6) $w(E)$ –the number of different individual [$p(E)$–apparent] variables occurring in the expression E,

(7) $\{i_m\}$ –the sequence $i_1, \ldots, i_m$; $\{i_{w(E)}\}$–all different indices of those and only those individual variables which occur in E,

(8) $n(E) = \max\{w(E) + p(E), \max\{i_{w(E)}\}\}$,

(9) $\bar{n}(E) = n(E)$, if E is an alternative of normal forms, $\bar{n}(E) = \max\{n(E), n(F)\}$, where F is the simplest alternative of normal forms equivalent to E, in the opposite case (we choose an arbitrary alternative),

(10) $\bar{c}$ –maximum of arguments of $f_1, \ldots, f_c$,

(11) $E(u/z)$ –the expression resulting from E by substitution of u for each occurrence of z in E (with usual conditions),

(12) $C(E)$ –the set of all significant parts of the formula E: $H \epsilon C(E)$.[2] $\equiv$. $H = E$ or there exist F, G, H_1 such that: $(H = F) \wedge (E = F') \vee \{(H = F) \vee (H = G)\}\ \ (E = F + G) \vee (\exists i)\{H = H_1(x_i/a)\} \wedge (E = \Pi a H_1)$,

(13) Skt –the set of all formulas of the form $\Sigma a_1 \ldots \Sigma a_i \Pi a_{i+1} \ldots \Pi a_k F$, where F is a quantifierless expression containing no free variables, Πa_j is the sign of the universal quantifier binding the variable a_j and $\Sigma a_j G = (\Pi a_j G')'$, $j = 1, \ldots, k$.[3]

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(14) $S(\{i_m\})$ —the set of all atomic formulas R such that all indices of free variables occurring in R belong to $\{i_m\}$,
(15) $n(E, r) = \max\{n(E_1), \ldots, n(E_r)\}$,
(16) $M, M_1, \ldots$ —functions of all atomic formulas with values 1 and 0; T, $T_1, \ldots$—functions on $S(1, \ldots, t)$, for given t, with values 1 and 0 (we shall name such functions "functions of the rank t"),
(17) (K) —for each K,
(18) $w_1, v_1, \ldots$ —numbers 0 or 1.

The formal proof $E_1, \ldots, E_n$ of the formula E is defined in the usual way, but to the proof of given theorems we must also assume that for each $i = 1, \ldots, n$, E_i is an alternative of significant parts of the formula E; the number n is named the length of this formal proof. The thesis is the last element of a formal proof.

Obviously:

L.0. If the length of a formal proof of the formula E is n, then the length of some formal proof of the formula $E(x/y)$ also is n.

L.1. For each formula E we may write an alternative F of formulas $G \in Skt$ such that E is a thesis if and only if F is a thesis, $E' + F$ is a thesis; we may also assume that $G = \Sigma a_1 \ldots \Sigma a_{m-1} \Pi a_m H$ where H is quantifier-free.

L.1. asserts the existence of Skolem's normal form for theses, see [1].
In the following we shall interpret the signs ' and + as Boolean operations$\urcorner$ (complemention) and $\dot{+}$ (addition) respectively; therefore Π is interpreted as an infinite Boolean multiplication. By this interpretation we have extended the function M, see (16), on all formulas and therefore we shall use the symbol $M\{E\}$ for an arbitrary E.

It is known:

T.1. The formula E is a thesis if and only if for an arbitrary M we have $M\{E\} = 0$.

Let $M/s_1, \ldots, s_t/$ be a function on $S(1, \ldots, t)$ such that for an arbitrary $R \in S(1, \ldots, t)$ we have:

$$M/s_1, \ldots, s_t/(R) = M\{R(x_{s_1}/x_1) \ldots (x_{s_t}/x_t)\}.^4$$

L.2. If $k_1, \ldots, k_q \leq t$, then:

$$M/s_1, \ldots, s_t//k_1, \ldots, k_q/ = M/s_{k_1}, \ldots, s_{k_q}/.$$

The proof is immediately.

In the sequel we shall write $\{i_t\}, i$ instead of $i_1, \ldots, i_t, i$ if i is different from $i_1, \ldots, i_t$; $\{i_t\}, i$ instead of $i_1, \ldots, i_t$, if $i = i_j$ for some $j \leq t$; therefore $M/\{i_t\}$ – instead of $M/i_1, \ldots, i_t/$ and $M/\{s_{i_t}\}$ – instead of $M/s_{i_1}, \ldots, s_{i_t}/$.

We shall also consider a Boolean algebra whose elements are n-tuples of numbers 0 and 1 and operations $\neg$ (complemention) and $\dot{+}$ (addition);[5] this Boolean algebra determines a many valued propositional calculus.

Let

$(I)\quad E_1, \dots, E_k, \dots$

be the sequence of all formulas of the considered calculus and let $N(E_k) = k$ —the index of E_k, $k = 1, 2, \dots$; let t be a natural number and Q a function on atomic formulas $R \in S(1, \dots, t)$ whose values are n-tuples of numbers 0 and 1; we shall use the following abbreviation:

$$Q(R) = \begin{pmatrix} w_{1\,N(R)} \\ \vdots \\ w_{n\,N(R)} \end{pmatrix}$$

$D.1.$ $g(t, j, q, \{i_m\}, Q) \,.\equiv.\, (i_1, \dots, i_m \leq t) \wedge (R)\,\{(R \in S(\{i_m\})) \to (w_{j\,N(R)} = w_{q\,N(R)})\}$.

We explain the meaning of $D.1.$:

$$\begin{array}{c|cccccc} & R_1, & \dots, & R_k, & \dots, & R_u \\ \hline 1 & 0 & \dots & w_{1k} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ j & \cdot & \dots & w_{jk} & \dots & \cdot \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ q & \cdot & \dots & w_{qk} & \dots & \cdot \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ n & 1 & \dots & w_{nk} & \dots & 0 \end{array}$$

— all elements of the set $S(\{i_m\})$. The relation $g(t, j, q, \{i_m\}, Q)$ asserts that the lines j and q are equal; on this figure:

$$Q(R_k) = \begin{pmatrix} w_{1\,k} \\ \vdots \\ w_{n\,k} \end{pmatrix}$$

Let Q be the function defined above and V — the function defined in the following way:

$(1d)$ $V\{t, Q, \{i_m\}, R\} = Q(R)$, if R is an atomic formula,

$(2d)$ $V\{t, Q, \{i_m\}, F'\} = V^{\neg}\{t, Q, \{i_m\}, F\}$,

$(3d)$ $V\{t, Q, \{i_m\}, F + G\} = V\{t, Q, \{i_m\}, F\} \dot{+} V\{t, Q, \{i_m\}, G\}$,

$(4d)$ Let $k = N(\Pi a F)$ and $k_r = N\{F(x_r/a)\}$; then: $V\{t, Q, \{i_m\}, \Pi a F\} =$

$$\begin{pmatrix} w_{1\,k} \\ \vdots \\ w_{n\,k} \end{pmatrix} .\equiv.\ (j)\,\{(j \leq n) \to (w_{j\,k} = 1 \,.\equiv.\, (q)\,(r)\,\{(q \leq n) \wedge (r \leq t) \wedge g(t, j, q, \{i_m\}, Q) \wedge V\{t, Q, \{i_m\}, r, F(x_r/a)\} = \begin{pmatrix} v^r_{1\,k_r} \\ \vdots \\ v^r_{n\,k_r} \end{pmatrix} \to (v^r_{j\,k_r} = v^r_{q\,k_r} = 1)\})\}.$$

The meaning of (*1d*) - (*3d*) is known; we explain the meaning of (*4d*):

$$
\begin{array}{c|c}
 & R_1 \ldots R_u \\
\hline
1 & 0 \ \ldots \ 1 \\
\cdot & \cdots\cdots \\
j & \cdots\cdots \\
\cdot & \cdots\cdots \\
q & \cdots\cdots \\
\cdot & \cdots\cdots \\
n & 1 \ldots\ 0
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccc}
F(x_1/a) \ldots & F(x_r/a) \ldots & F(x_t/a) \\
\hline
v^1_{1k_1} & \ldots v^r_{1k_r} & \ldots v^t_{1k_t} \\
\cdot & \cdots & \cdots \\
\cdot & \cdots & \cdots \\
\cdot & \cdots & \cdots \\
\cdot & \cdots & \cdots \\
\cdot & \cdots & \cdots \\
v^1_{nk_1} & \ldots v^r_{nk_r} & \ldots v^t_{nk_t}
\end{array}
$$

In the left part of this figure is the figure described above and on the right side we have:

$$V\{t,\ Q,\ \{i_m\},\ r,\ F(x_r/a)\} = \begin{pmatrix} v^r_{1\ k_r} \\ \cdot \\ \cdot \\ \cdot \\ v^r_{n\ k_r} \end{pmatrix},\ r = 1, \ldots, t\,.$$

The definition (*4d*) asserts that $w_{j\ k} = 1$ if and only if for each $q \leq n$, if the lines j and q are equal on the left side, then on the right side of ones we have only 1 (i.e. we have no 0).

D.2. $J(Q,\ t,\ G)\ .\equiv.\ (m)\ (i_1)\ \ldots\ (i_m)\ \{(m + p(G) < t) \wedge (\{i_{w(G)}\} \subset \{i_m\})^6 \rightarrow (j)\ (V\{t,\ Q,\ \{i_m\},\ j,\ G\} = V\{t,\ Q,\ \{i_m\},\ G\})\}$.

We note that $J(Q,\ t,\ G)$ is an invariant relation.

D.3. $F \in P(t,\ Q,\ E)\ .\equiv.\ (\exists G)\ \{(G \in C(E)) \wedge \{J(Q,\ t,\ G) \rightarrow V\{t,\ Q,\ \{i_{w(F)}\},\ F\} = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}\}\}$.

Because the values of M are n-tuples, then in the sequel we shall also write $M = M_n$.

D.4. $F \in P[t,\ E]\ .\equiv.\ (M_n)\ \{(1 \leq n \leq 2^{ct^{\bar{c}}}) \rightarrow (F \in P(t,\ M_n,\ E))\}$.
D.5. $F \in P|E|\ .\equiv.\ (\exists t)\ \{(t \geq n(F)) \wedge (F \in P[t,\ E])\}$.
D.6. $E \in P\ .\equiv.\ E \in P|E|$.

The meaning of D.3. - D.6. is simple; see [2].
We shall prove that P is the class of all true formulas:

D.7. $T \in M[k]\ .\equiv.\ (\exists s_1)\ \ldots\ (\exists s_k)\ \{T = M/\{s_k\}\}$.

$M[k]$ is the set of all functions of the form $M/s_1, \ldots, s_k/$.

D.8. $Q \sim (T_1, \ldots, T_n,\ k)\ .\equiv.\ T_1, \ldots, T_n$ are different functions of the rank k, Q is a function defined on $S(1, \ldots, k)$ whose values are n-tuples of numbers 0, 1 and for each $R \in S(1, \ldots, k)$: $T_j(R) = 1\ .\equiv.\ w_{j\ N(R)} = 1,\ j \leq n$.

D.9. $Q \approx M(T_1, \ldots, T_n, k) \;.\equiv.\; Q \sim (T_1, \ldots, T_n, k)$ and $T_1, \ldots, T_n$ are all elements of $M[k]$.

It is easy to prove:

L.3. If $Q \sim (T_1, \ldots, T_n, k)$, then:

$$g(k, j, q, i_m, Q) \;.\equiv.\; T_j/\{i_m\} = T_q/\{i_m\}\,.$$

L.4. If $g(k, j, q, \{i_m\}, Q)$ and $V\{k, Q, \{i_m\}, E\} = \begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{pmatrix}$, then:

$$w_j = 1 \;.\equiv.\; w_q = 1,\ j, q \leq n\,.$$

The proof of *L.4.* is inductive on the length of the formula E.

T.2. If E is an alternative of formulas belonging to Skt, $F \in C(E)$, $M\{E\} = 0$, $k \geq n(E)$, $Q \approx M(T_1, \ldots, T_n, k)$, then:

(1) If $m + p(F) \leq k$, $F \in S(\{i_m\})$, $M/\{s_{i_m}\} = T_j/\{i_m\}$, $\{i_{w(F)}\} \subset \{i_m\}$,

$$M\{F(x_{s_{i_1}}/x_{i_1}) \ldots (x_{s_{i_m}}/x_{i_m})\} = 0 \text{ and } V\{k, Q, \{i_m\}, F\} = \begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{pmatrix},$$

then $w_j = 0$.

(2) If E is also an alternative of formulas of the form $\Sigma a_1 \ldots \Sigma a_{r-1} \Pi a_r G$, for some quantifierless G, then for each $F \in C(E)$ we have $J(Q, k, F)$ and therefore $E \,\bar{\in}\, P$.

Proof: —First of all we notice that the proof in general case is analogous to the proof in the case $E \in Skt$ and (2) is a simple conclusion from (1) (in view of the form of E).

The proof of (1) is inductive on the number of quantifiers occurring in F and is analogic to the proof of *T.2.* from [2]; we use here *L.3.*

T.3'. If $E_1, \ldots, E_r$ is a formalized proof of the formula E, then for each $k \geq n(E, r)$ we have $E_j \in P[k, E]$, $j = 1, \ldots, r$.

Proof: —By using the proof rules given in [2] or [3] it is easy to prove by induction on $j \leq r$ that for each $k \geq n(E, r)$:

(*1°*) $E_j \in P[k, E]$; therefore $E \in P$.

(*2°*) $E_j + F \in P[k, E]$ for every F such that $C(F) \subset C(E)$ and $k \geq n(F)$.

The proof of (*1°*) and (*2°*) is analogous to the proof of *T.3'.* from [2]; we prove ones simultaneously, see [2]; we use *L.0.*, *L.2.*, *L.3.* and *L.4.*

T.3. If E is a thesis, then $E \in P$ (follows from *T.3'.*).

L.5. There exists Skolem's normal form F of the formula E such that F is an alternative of formulas of the form $\Sigma a_1 \ldots \Sigma a_{m-1} \Pi a_m G$, for some quantifierless G, $\bar{n}(E) = \bar{n}(F)$ and if $E \in P$, then $F \in P$.

To the proof of *L.5.* we use *T.3.*, the deduction theorem and the usual Skolem's method of constructing normal forms.

T.4. The formula E is a thesis if and only if $E \epsilon P$.

T.4. follows from *T.1.*, *T.2.*, *T.3.*, *L.1.* and *L.5.*; to the proof of *T.4.* in the left-hand side we choose F which satisfies *L.1.* and *L.5.*; the whole proof is analogic to the given in [2].

T.4. asserts that P is the class of all true formulas.

If we replace *D.3.* by:

$$D.3'. \quad F \epsilon P(t, Q, E) \, . \equiv . \, J(Q, t, E) \rightarrow V\{t, Q, \{i_{w(F)}\}, F\} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

then *T.4.* remains true for normal forms.

T.4. proves the possibility of approximation of the first-order functional calculus by many valued Boolean propositional calculi; in this approximation the quantifier Π is interpreted as a finite operator, see (*4d*).

The examples we shall give in [4].

NOTES

1. The expression we define in the usual way; the expression in which an apparent variable a belong to the scope of two quantifiers Πa is not a formula; if a does not occur in E, then $\Pi a E$ is not a formula.
2. The dots separate more strongly than parentheses.
3. There are Skolem's normal forms for theses; alternatives of these formulas we also name Skolem's normal forms.
4. We may here replace the indices $1, \ldots, t$ by $i_1, \ldots, i_{w(R)}$.
5. We use the same denotation, because the operations are analogously to the given above.
6. The sign $\subset$ is the inclusion.

REFERENCES

[1] D. Hilbert-P. Bernays: *Grundlagen der Mathematik* II, Berlin 1939.

[2] J. Reichbach: On the definitions of satisfiability, sent to *American Journal of Mathematics.*

[3] J. Reichbach: On characterizations of the first-order functional calculus, *Notre Dame Journal of Formal Logic*, vol. II (1961), pp. 1-15.

[4] J. Reichbach: Some examples of different methods of formal proofs, in preparation.

[5] J. Reichbach: On the connection of the first-order functional calculus with many valued propositional calculi (An abstract). *The Bulletin of the Research Council of Israel.* Section F: Mathematics and Physics, Vol. 10 F1 (1961).

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