CERTAIN FORMULAS EQUIVALENT TO THE AXIOM OF CHOICE

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In this note it will be shown that each of the following four set-theoretical formulas:

$\mathfrak{A}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) = \aleph(\mathfrak{n})$, *then* $\mathfrak{m} = \mathfrak{n}$.

$\mathfrak{B}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) > \aleph(\mathfrak{n})$, *then* $\mathfrak{m} > \mathfrak{n}$.

$\mathfrak{C}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and* $\mathfrak{m} > \mathfrak{n}$, *then* $\aleph(\mathfrak{m}) > \aleph(\mathfrak{n})$.

$\mathfrak{D}$. *For any cardinal numbers* $\mathfrak{m}$, $\mathfrak{n}$, *and* $\mathfrak{p}$, *if* $\mathfrak{m} < \mathfrak{p}$, $\mathfrak{n} < \mathfrak{p}$, *and for any cardinal numbers* $\mathfrak{x}$ *and* $\mathfrak{y}$, *if* $\mathfrak{m} + \mathfrak{x} = \mathfrak{p}$ *and* $\mathfrak{n} + \mathfrak{y} = \mathfrak{p}$, *then* $\mathfrak{x} = \mathfrak{y}$, *then either* $\mathfrak{m} \geqslant \mathfrak{n}$ *or* $\mathfrak{n} > \mathfrak{m}$.

is inferentially equivalent to the axiom of choice.[1]

Concerning these formulas it should be noted: 1) that the formulas $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are related to the following theorem:

T1. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively and* $\mathfrak{m} \leqslant \mathfrak{n}$, *then* $\aleph(\mathfrak{m}) \leqslant \aleph(\mathfrak{n})$.,

which is provable without the aid of the axiom of choice[2] and: 2) that the formula $\mathfrak{D}$ is a weaker formulation of the law of trichotomy for cardinal numbers, since in it the trichotomy is preceded by an antecedent consisting of three conditions.[3]

Proof:

(i) *The axiom of choice implies* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, *and* $\mathfrak{D}$. It is easy to determine that these theorems follow from the axiom of choice. Viz., in order

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to prove formula $\mathfrak{A}$ assume the said axiom and the conditions given in the antecedent of $\mathfrak{A}$. Since we have the axiom of choice, the cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ are certain alephs whose respective indices are, say, the ordinal numbers α and β. Hence, $\mathfrak{m} = \aleph_\alpha$ and $\mathfrak{n} = \aleph_\beta$ and due to our assumption that $\aleph(\mathfrak{m})$ and $\aleph(\mathfrak{n})$ are the least alephs which are not $\leqslant$ $\mathfrak{m}$ and $\mathfrak{n}$ respectively we have obviously $\aleph(\mathfrak{m}) = \aleph_{\alpha+1}$ and $\aleph(\mathfrak{n}) = \aleph_{\beta+1}$. But, since $\aleph(\mathfrak{m}) = \aleph(\mathfrak{n})$, $\aleph_{\alpha+1} = \aleph_{\beta+1}$, which gives at once $\alpha + 1 = \beta + 1$. Since $\alpha + 1$ and $\beta + 1$ are the next ordinal numbers to α and β respectively, then $\alpha = \beta$. Hence $\aleph_\alpha = \aleph_\beta$ and, therefore, $\mathfrak{m} = \mathfrak{n}$. Thus, formula $\mathfrak{A}$ is proved with the aid of the axiom of choice. Evidently, using very similar reasoning, we can prove easily that the axiom of choice implies formulas $\mathfrak{B}$ and $\mathfrak{C}$. Since $\mathfrak{D}$ is a weaker formulation of the law of trichotomy for cardinal numbers, formula $\mathfrak{D}$ follows, obviously, from the axiom of choice.

(ii) *Formula* $\mathfrak{A}$ *implies the axiom of choice.* It is well known that the following theorem:

T2. *If the cardinal number* $\mathfrak{m}$ *is not finite, then* $\aleph(\mathfrak{m}^2) = \aleph(\mathfrak{m})$

is provable without the aid of the axiom of choice[4] and that the theorem:

T3. *If the cardinal number* $\mathfrak{m}$ *is not finite, then* $\mathfrak{m} = \mathfrak{m}^2$

is inferentially equivalent to the axiom of choice.[5]

Now, assume that $\mathfrak{m}$ is an arbitrary cardinal number which is not finite. Hence $\mathfrak{m}^2$ is also not finite cardinal, and from **T2** it follows that $\aleph(\mathfrak{m}^2) = \aleph(\mathfrak{m})$. But, in such a case, due to $\mathfrak{A}$, we have $\mathfrak{m}^2 = \mathfrak{m}$; and, therefore, we obtain theorem **T3** which shows that $\mathfrak{A}$ implies the axiom of choice.

(iii) *Formula* $\mathfrak{B}$ *implies the axiom of choice.* Let us assume formula $\mathfrak{B}$ and consider $\mathfrak{m}$ as an arbitrary cardinal number which is not finite. Moreover, assume that $\aleph(\mathfrak{m})$ is the least Hartogs' aleph with respect to $\mathfrak{m}$. Since $\aleph(\mathfrak{m})$ is Hartogs' aleph with respect to $\mathfrak{m}$, then *it is not true that* $\aleph(\mathfrak{m}) \leqslant \mathfrak{m}$. Therefore, we have:

1. *either* $\mathfrak{m} < \aleph(\mathfrak{m})$ *or* $\mathfrak{m}$ *and* $\aleph(\mathfrak{m})$ *are incomparable*

But, the second case is impossible, since the incomparability of $\mathfrak{m}$ and $\aleph(\mathfrak{m})$ gives at once that *it is not true that* $\mathfrak{m} > \aleph(\mathfrak{m})$. Hence, an application of this last formula to $\mathfrak{B}$ (for: $\mathfrak{m} = \aleph(\mathfrak{m})$) implies that it is not true that $\aleph(\mathfrak{m}) < \aleph(\aleph(\mathfrak{m}))$. This gives a contradiction, since the least Hartogs' aleph $\aleph(\aleph(\mathfrak{m}))$ with respect to $\aleph(\mathfrak{m})$ must be greater than $\aleph(\mathfrak{m})$. Therefore, we have:

$$\mathfrak{m} < \aleph(\mathfrak{m}),$$

which shows that $\mathfrak{m}$ is an aleph. Thus, formula $\mathfrak{B}$ implies the axiom of choice.

(iv̈) *Formula $\mathfrak{C}$ implies the axiom of choice.* It is well known, that without the aid of the axiom of choice we can establish that if $\mathfrak{m}$ is an arbitrary cardinal number which is not finite, then *either* $\mathfrak{m} < \mathfrak{m}^2$ *or* $\mathfrak{m} = \mathfrak{m}^2$. But, in virtue of $\mathfrak{C}$ and **T2**, the case that $\mathfrak{m} < \mathfrak{m}^2$ is excluded, since an application of this case to $\mathfrak{C}$ (for: $\mathfrak{m} = \mathfrak{m}^2$) implies that $\aleph(\mathfrak{m}) < \aleph(\mathfrak{m}^2)$, which is rejected by **T2**. Hence, it must be that $\mathfrak{m} = \mathfrak{m}^2$, i.e. it shows that $\mathfrak{C}$ implies theorem **T3**. Thus, the axiom of choice follows from formula $\mathfrak{C}$.

(v̈) *Lemmas* **A** *and* **B**. In order to be able to show that the axiom of choice is a consequence of formula $\mathfrak{D}$ the following two lemmas have to be proved without the aid of the said axiom:

Lemma **A**. *For any cardinal number* $\mathfrak{m}$ *and an aleph* $\aleph'$ *such that* $\aleph' < \mathfrak{m}$, *if* $\mathfrak{m} = \mathfrak{m}^2$, *then the difference* $\mathfrak{m} - \aleph'$ *exists and is equal to* $\mathfrak{m}$.[6]

Proof: If $\aleph' < \mathfrak{m}$, then there exists such cardinal number $\mathfrak{p}$ that[7]

2. $\mathfrak{m} = \aleph' + \mathfrak{p}$

Since $\mathfrak{m} = \mathfrak{m}^2$, then from *2* follows

3. $\mathfrak{m} = \mathfrak{m}^2 = (\aleph' + \mathfrak{p})^2 = \aleph'^2 + 2\aleph'\mathfrak{p} + \mathfrak{p}^2 \geqslant \aleph'\mathfrak{p} = (\aleph' + 1)\mathfrak{p} = \aleph'\mathfrak{p} + \mathfrak{p} \geqslant \aleph' + \mathfrak{p} = \mathfrak{m}$,

hence *3* says:

4. $\aleph' + \mathfrak{p} = \aleph'\mathfrak{p}$

It is well known[8] that *4* implies:

5. *either* $\aleph' \geqslant \mathfrak{p}$ *or* $\mathfrak{p} > \aleph'$.

But, the first case is excluded, since if $\aleph' \geqslant \mathfrak{p}$, then $\mathfrak{p}$ is an aleph, which gives: $\aleph' = \aleph' + \mathfrak{p}$. Therefore, this case and *2* imply: $\aleph' = \mathfrak{m}$ which contradicts our assumption. Hence, it must be that $\mathfrak{p} > \aleph'$. But, in such a case we have at once:[9] $\mathfrak{p} = \aleph' + \mathfrak{p}$, which shows that $\mathfrak{p} = \mathfrak{m}$. Since the reasoning given above can be applied to any cardinal number $\mathfrak{q}$ such that $\mathfrak{m} = \aleph' + \mathfrak{q}$, the solution $\mathfrak{p} = \mathfrak{m}$ is unique. Hence, the difference $\mathfrak{m} - \aleph'$ exists and is equal $\mathfrak{m}$. This concluded the proof of lemma **A**.

Lemma **B**. *For any cardinal number* $\mathfrak{m}$ *and an aleph* $\aleph'$ *such that* $\aleph' < 2^{\mathfrak{m}}$, *if* $\mathfrak{m} = 2\mathfrak{m}$, *then the difference* $2^{\mathfrak{m}} - \aleph'$ *exists and is equal to* $2^{\mathfrak{m}}$.

The proof of **B** follows, obviously, from lemma **A** and from the fact that if $\mathfrak{m} = 2\mathfrak{m}$, then $2^{\mathfrak{m}} = 2^{2\mathfrak{m}} = (2^{\mathfrak{m}})^2$.

Concerning lemma **A** which was proved above it should be noted that an omission of its condition, viz. that $\mathfrak{m} = \mathfrak{m}^2$, gives the formula:

$\mathfrak{E}$. *For any cardinal number* $\mathfrak{m}$ *and an aleph* $\aleph'$ *such that* $\aleph' < \mathfrak{m}$, *the difference* $\mathfrak{m} - \aleph'$ *exists.*

which is, obviously an instance (obtainable by substitution) of the following theorem:

T4. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{n} < \mathfrak{m}$, *then the difference* $\mathfrak{m} - \mathfrak{n}$ *exists.*

It is known that theorem **T4** is equivalent to the axiom of choice.[10] It is easy to remark that entirely the same deductions which show that **T4** implies the axiom of choice allow us to establish the fact that the said axiom is a consequence of formula $\mathfrak{S}$. Therefore, Theorem **T4**, its particular instance, i.e. formula $\mathfrak{S}$, and the axiom of choice are inferentially equivalent.

(vi) *Formula* $\mathfrak{D}$ *implies the axiom of choice.* Let us assume formula $\mathfrak{D}$ and consider $\mathfrak{m}$ as an arbitrary cardinal number which is not finite. Put $\mathfrak{n} = \aleph_0\mathfrak{m}$. Hence, the cardinal number $\mathfrak{n}$ possesses the following properties provable in an elementary way: $\mathfrak{a})$ $\mathfrak{n} \geqslant \aleph_0$; $\mathfrak{b})$ $\mathfrak{n} = \mathfrak{n} + 1 = 2\mathfrak{n}$; $\mathfrak{c})$ $2^{\mathfrak{n}} = 2^{\mathfrak{n}} + 1 = (2^{\mathfrak{n}})^2 = 2\cdot 2^{\mathfrak{n}} = 2^{\mathfrak{n}+1}$; $\mathfrak{d})$ $2^{2^{\mathfrak{n}}} = 2^{2^{\mathfrak{n}}} + 1 = 2\cdot 2^{2^{\mathfrak{n}}} = \left(2^{2^{\mathfrak{n}}}\right)^2$; $\mathfrak{e})$ $2^{2^{2^{\mathfrak{n}}}} = \left(2^{2^{2^{\mathfrak{n}}}}\right)^2$.

Since $\mathfrak{n}$ is a cardinal number which is not finite we know,[11] that without the aid of the axiom of choice we can associate with $\mathfrak{n}$ a certain Hartogs' aleph, viz. $\aleph(\mathfrak{n})$ which possesses the following properties:

6. $\aleph(\mathfrak{n})$ *is the least aleph such that* $\aleph(\mathfrak{n})$ *is not* $\leqslant \mathfrak{n}$ *and* $\aleph(\mathfrak{n}) \leqslant 2^{2^{2^{\mathfrak{n}}}}$.

If $\aleph(\mathfrak{n}) = 2^{2^{2^{\mathfrak{n}}}}$, then $\aleph(\mathfrak{n}) > 2^{2^{\mathfrak{n}}} > 2^{\mathfrak{n}} > \mathfrak{n} = \aleph_0\mathfrak{m} \geqslant \mathfrak{m}$. Hence, this case shows that an arbitrary cardinal number $\mathfrak{m}$ is an aleph. Assume, therefore, the second possibility, i.e. $\aleph(\mathfrak{n}) < 2^{2^{2^{\mathfrak{n}}}}$. Since $2^{2^{2^{\mathfrak{n}}}} = \left(2^{2^{2^{\mathfrak{n}}}}\right)^2$, then according to lemma **B** the difference $2^{2^{2^{\mathfrak{n}}}} - \aleph(\mathfrak{n})$ exists and is equal to $2^{2^{2^{\mathfrak{n}}}}$. On the other hand generally we have $2^{2^{\mathfrak{n}}} < 2^{2^{2^{\mathfrak{n}}}}$ and we know that $2^{2^{\mathfrak{n}}} = 2^{2^{\mathfrak{n}}} + 1$, i.e. that $2^{2^{\mathfrak{n}}} \geqslant \aleph_0$. Therefore, from these two facts and the theorem which says

T5. *For any cardinal number* $\mathfrak{m}$, *if* $\mathfrak{m} \geqslant \aleph_0$, *then* $2^{\mathfrak{m}} - \mathfrak{m} = 2^{\mathfrak{m}}$.

and which is provable without the aid of the axiom of choice,[12] we are able to conclude that the difference $2^{2^{2^{\mathfrak{n}}}} - 2^{2^{\mathfrak{n}}}$ exists and is equal to $2^{2^{2^{\mathfrak{n}}}}$. Hence our assumptions and the proved properties of the formulas $2^{2^{2^{\mathfrak{n}}}} - \aleph(\mathfrak{n})$ and $2^{2^{2^{\mathfrak{n}}}} - 2^{2^{\mathfrak{n}}}$ allow us to establish that:

7. *for any cardinal numbers* $\mathfrak{x}$ *and* $\mathfrak{y}$, *if* $\aleph(\mathfrak{n}) + \mathfrak{x} = 2^{2^{2^{\mathfrak{n}}}}$ *and* $2^{2^{\mathfrak{n}}} + \mathfrak{y} = 2^{2^{2^{\mathfrak{n}}}}$, *then* $\mathfrak{x} = \mathfrak{y}$.

Hence the conditions of formula $\mathfrak{D}$ (for: $\mathfrak{m} = \aleph(\mathfrak{n})$, $\mathfrak{n} = 2^{2^{\mathfrak{n}}}$ and $\mathfrak{p} = 2^{2^{2^{\mathfrak{n}}}}$) are fulfilled and, therefore, we have:

8. *either* $\aleph(\mathfrak{n}) \geqslant 2^{2^{\mathfrak{n}}}$ *or* $2^{2^{\mathfrak{n}}} > \aleph(\mathfrak{n})$.

The first case shows that the cardinal number $\mathfrak{m}$ is an aleph, as $\aleph(\mathfrak{n}) \geqslant 2^{2^{\mathfrak{n}}} > 2^{\mathfrak{n}} > \mathfrak{n} = \aleph_0\mathfrak{m} \geq \mathfrak{m}$. Thus assume the second case, i.e. that $2^{2^{\mathfrak{n}}} > \aleph(\mathfrak{n})$. Now, since $2^{\mathfrak{n}} < 2^{2^{\mathfrak{n}}}$, $2^{2^{\mathfrak{n}}} = \left(2^{2^{\mathfrak{n}}}\right)^2$ and $2^{\mathfrak{n}} = 2^{\mathfrak{n}} + 1$, i.e. that $2^{\mathfrak{n}} \geqslant \aleph_0$, it is evident that lemma **B** and theorem **T5** establish that the differences $2^{2^{\mathfrak{n}}} - \aleph(\mathfrak{n})$ and $2^{2^{\mathfrak{n}}} - 2^{\mathfrak{n}}$ exist and both are equal to $2^{2^{\mathfrak{n}}}$.

Hence we obtain:

9. *for any cardinal numbers* $\mathfrak{x}$ *and* $\mathfrak{y}$, *if* $\aleph(\mathfrak{n}) + \mathfrak{x} = 2^{2^{\mathfrak{n}}}$ *and* $2^{\mathfrak{n}} + \mathfrak{y} = 2^{2^{\mathfrak{n}}}$, *then* $\mathfrak{x} = \mathfrak{y}$.

which together with our assumptions and formula $\mathfrak{D}$ (for: $\mathfrak{m} = \aleph(\mathfrak{n})$, $\mathfrak{n} = 2^{\mathfrak{n}}$ and $\mathfrak{p} = 2^{2^{\mathfrak{n}}}$) gives:

10. *either* $\aleph(\mathfrak{n}) \geqslant 2^{\mathfrak{n}}$ *or* $2^{\mathfrak{n}} > \aleph(\mathfrak{n})$.

If the first case is true, then the cardinal number $\mathfrak{m}$ is an aleph, since $\aleph(\mathfrak{n}) \geqslant 2^{\mathfrak{n}} > \mathfrak{n} = \aleph_0\mathfrak{m} \geq \mathfrak{m}$. If the second case is assumed, then since $\mathfrak{n} < 2^{\mathfrak{n}}$, $2^{\mathfrak{n}} = (2^{\mathfrak{n}})^2$ and $\mathfrak{n} \geqslant \aleph_0$, we can use again lemma **B** and theorem **T4** to prove that:

11. *for any cardinal numbers* $\mathfrak{x}$ *and* $\mathfrak{y}$, *if* $\aleph(\mathfrak{n}) + \mathfrak{x} = 2^{\mathfrak{n}}$ *and* $\mathfrak{n} + \mathfrak{y} = 2^{\mathfrak{n}}$, *then* $\mathfrak{x} = \mathfrak{y}$.

Hence the application of formula $\mathfrak{D}$ (for: $\mathfrak{m} = \aleph(\mathfrak{n})$ and $\mathfrak{p} = 2^{\mathfrak{n}}$) to the above conditions shows that:

12. *either* $\aleph(\mathfrak{n}) > \mathfrak{n}$ *or* $\aleph(\mathfrak{n}) = \mathfrak{n}$ *or* $\mathfrak{n} > \aleph(\mathfrak{n})$.

But the last two cases of *12* are excluded, as they contradict the assumed properties of $\aleph(\mathfrak{n})$. Hence we have: $\aleph(\mathfrak{n}) > \mathfrak{n} = \aleph_0\mathfrak{m} \geq \mathfrak{m}$, which means that $\mathfrak{m}$ is an aleph. Thus, any possible connections between $\aleph(\mathfrak{n})$ and $\mathfrak{m}$ leads to the conclusion that the arbitrary cardinal number $\mathfrak{m}$ is an aleph. This shows that formula $\mathfrak{D}$ implies the axiom of choice. Thus, our proof is completed.

NOTES

1. In [1], Hartogs proved that with any cardinal number $\mathfrak{m}$ which is not finite we can associate an aleph $\aleph(\mathfrak{m})$ such that $\aleph(\mathfrak{m})$ is not $\leqslant \mathfrak{m}$. Later, it was established that if $\aleph(\mathfrak{m})$ is the least such aleph in respect to $\mathfrak{m}$, then: 1) $\aleph(\mathfrak{m}) \leqslant 2^{2^{2^{\mathfrak{m}}}}$ and in the same time $\aleph(\mathfrak{m}) \leqslant 2^{2^{\mathfrak{m}^2}}$; 2) if $\mathfrak{m} = \aleph_\alpha$, then $\aleph(\mathfrak{m}) = \aleph_{\alpha+1}$. *Cf.*, e.g., [2], pp. 311 - 312, [3], pp. 407 - 409 and [9], pp. 28 - 30. The existence of Hartogs' aleph and its properties are provable without the aid of the axiom of choice. The proofs which are given below are established within the general set theory, i.e. the set theory from which the axiom of choice and all its consequences otherwise unprovable have been removed. It is well known that if we base a so defined general set theory on an axiomatic system in which the notions of the cardinal and ordinal numbers cannot be defined, we have to introduce these concepts into the system by means of special axioms.

2. *Cf.* [2], p. 311, theorem **78.**

3. In [7], p. 58, theorem **B**, I presented another example of a weaker formulation of the law of trichotomy for cardinal numbers which is equivalent to the axiom of choice.

4. It is proved by Tarski, *cf.* [9], p. 30, lemma 6, and [2], p. 311, theorem **77**.

5. It is proved by Tarski, *cf.* [8], pp. 147 - 151. *Cf.*, also, [3], pp. 419 - 421.

6. The difference of two cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ is defined as follows: The difference $\mathfrak{m} \dot{-} \mathfrak{n}$ exists if and only if there exists one and only one cardinal number $\mathfrak{p}$ such that $\mathfrak{m} = \mathfrak{n} + \mathfrak{p}$. *Cf.* [2], p. 306, position **47**, and [3], p. 158. Lemma **A** was announced without proof by Tarski in [2], p. 307, theorem **54**.

7. *Cf.*, e.g., [3], p. 144, theorem **1**.

8. *Cf.* [8], p. 148, lemme **1**.

9. *Cf.*, e.g., [3], p. 413, exercise 1.

10. This theorem was announced without proof by Tarski in [2], p. 307, theorem **56**. Sierpiński gave a proof in [5], p. 125, and [3], p. 416 - 417. It should be noted an observation of Sierpiński that for any cardinal number which is not finite the difference $(\mathfrak{m} + \aleph(\mathfrak{m})) \dot{-} \aleph(\mathfrak{m})$ does not exist. *Cf.* [6], p. 8.

11. *Cf.* note 1.

12. This theorem was announced without proof by Tarski in [9], p. 312, theorem **82**. Sierpiński published a proof of this theorem in [4] and [3], pp. 168 - 170.

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