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LOGIC WITHOUT TAUTOLOGIES

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In the first edition of *Introduction to Logic* (p. 259), Copi gave a system of natural deduction for sentential calculus, and he included in the second edition of *Symbolic Logic* (pp. 53 ff.) my proof that the system is incomplete. In this paper, I want to show, first, that the matrix used to prove the incompleteness of the system in fact furnishes a decision procedure for it; second, that any "formal" extension of the system is complete; third, that the system contains no "tautologies" or "theorems", though it contains "contradictions"; fourth, that though the system does not permit the deduction of all conclusions from premisses which tautologically imply them, still it does permit the deduction of some tautological equivalent of any non-tautological conclusion tautologically implied by the premisses. Another result may also be of interest. The usual replacement rule does not hold for the system, although a certain "weak" replacement rule does hold. Finally, the system actually worked with, proved equivalent to Copi's, is perhaps interesting in its own right.

1 *The equivalence of Copi's system and* **C**. The rules of Copi's system are here transcribed in the metalinguistic notation that will be used throughout this paper. Thus, instead of "p" and "q" and the like, Roman capitals, with or without subscripts and other affixes, are used as metalinguistic variables. (In one later context, however, "A" and "B" are used as proper names of atomic sentences.) In the presentation of Copi's system, "$\vdash_C$" will mean "yield(s) by Copi's rules" and "$\underset{C}{\leftrightarrow}$" will mean "may, according to Copi's rules, replace or be replaced by".

The first nine of Copi's rules are:

1. Modus Ponens: $S_1 \supset S_2, S_1 \vdash_C S_2$
2. Modus Tollens: $S_1 \supset S_2, \sim S_2 \vdash_C \sim S_1$
3. Hypothetical Syllogism: $S_1 \supset S_2, S_2 \supset S_3 \vdash_C S_1 \supset S_3$
4. Disjunctive Syllogism: $S_1 \vee S_2, \sim S_1 \vdash_C S_2$
5. Constructive Dilemma: $(S_1 \supset S_2) \cdot (S_3 \supset S_4), S_1 \vee S_3 \vdash_C S_2 \vee S_4$
6. Destructive Dilemma: $(S_1 \supset S_2) \cdot (S_3 \supset S_4), \sim S_2 \vee \sim S_4 \vdash_C \sim S_1 \vee \sim S_3$
7. Simplification: $S_1 \cdot S_2 \vdash_C S_1$

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8. Conjunction: $S_1, S_2 \vdash_C S_1 \cdot S_2$
9. Addition: $S_1 \vdash_C S_1 \vee S_2$

Now follow ten replacement rules, many with two parts (duals).

10. De Morgan's Theorems: $\sim(S_1 \cdot S_2) \underset{C}{\Leftrightarrow} S_1 \vee \sim S_2$
10′. $\sim(S_1 \vee S_2) \underset{C}{\Leftrightarrow} \sim S_1 \cdot \sim S_2$
11. Commutation: $S_1 \vee S_2 \underset{C}{\Leftrightarrow} S_2 \vee S_1$
11′. $S_1 \cdot S_2 \underset{C}{\Leftrightarrow} S_2 \cdot S_1$
12. Association: $S_1 \vee (S_2 \vee S_3) \underset{C}{\Leftrightarrow} (S_1 \vee S_2) \vee S_3$
12′. $S_1 \cdot (S_2 \cdot S_3) \underset{C}{\Leftrightarrow} (S_1 \cdot S_2) \cdot S_3$
13. Distribution: $S_1 \cdot (S_2 \vee S_3) \underset{C}{\Leftrightarrow} (S_1 \cdot S_2) \vee (S_1 \cdot S_3)$
13′. $S_1 \vee (S_2 \cdot S_3) \underset{C}{\Leftrightarrow} (S_1 \vee S_2) \cdot (S_1 \vee S_3)$
14. Double Negation: $S \underset{C}{\Leftrightarrow} \sim\sim S$
15. Transposition: $S_1 \supset S_2 \underset{C}{\Leftrightarrow} \sim S_2 \supset \sim S_1$
16. Definition of Material Implication: $S_1 \supset S_2 \underset{C}{\Leftrightarrow} \sim S_1 \vee S_2$
17. Definitions of Material Equivalence: $S_1 \equiv S_2 \underset{C}{\Leftrightarrow} (S_1 \supset S_2) \cdot (S_2 \supset S_1)$
17′. $S_1 \equiv S_2 \underset{C}{\Leftrightarrow} (S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)$
18. Exportation: $(S_1 \cdot S_2) \supset S_3 \underset{C}{\Leftrightarrow} S_1 \supset (S_2 \supset S_3)$
19. Tautology: $S \underset{C}{\Leftrightarrow} S \vee S$

Later editions of *Introduction to Logic*, as well as editions of *Symbolic Logic*, include the dual of 19:

19′. $S \underset{C}{\Leftrightarrow} S \cdot S$

The system **C** contains, from the foregoing set, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 17′, and 19, and in addition one rule not given by Copi:

20. $S_1 \vee (S_2 \cdot \sim S_2) \vdash_C S_1$

When these rules are read as rules of **C**, "$\vdash_C$" will mean "yield(s) according to the rules of **C**", and a similar adjustment will be made for "$\underset{C}{\Leftrightarrow}$".

The system **C** is equivalent to Copi's in the sense that there is a proof for an argument in either system if there is a proof for that argument in the other. Such a statement presupposes the meaning of "proof of an argument in Copi's system" and "proof of an argument in **C**". It is assumed here that these meanings are understood. The statement also presupposes that the sentences of the systems do not differ. The sentences of the system are either atomic or are built up from atomic sentences by signs for negation, conjunction, disjunction, material implication, and material equivalence. More exactly, some infinite stock of atomic sentences being supposed, a sentence S is any member of the intersection of all sets which contain the atomic sentences and which contain $\sim(S)$ if they contain S, and contain $(S_1) \cdot (S_2)$, $(S_1) \vee (S_2)$, $(S_1) \supset (S_2)$, and $(S_1) \equiv (S_2)$ if they contain S_1 and S_2. In the sequel, conventions of punctuation other than those implied by the foregoing definition are used; they are used by Copi and in any case are familiar.

At the outset, a rule is denoted by a numeral, primed or unprimed, to indicate that it is regarded as primitive in Copi's system; to indicate that it is regarded as primitive in C, it is denoted by prefixing the letter "C" to the numeral. But as primitive rules of Copi's system are shown to be derivable in C, the letter "C" becomes part of their denotation, and primes are suppressed.

Since the replacement rules of Copi's system are derivable from the primitive replacement rules of C in quite familiar ways, only perfunctory hints need be given here. Perhaps it is best to begin by using C14 and C10 to show that 10′ is derivable in C. Then, with 10′ available as one of the two rules C10, C10 and C14 may be used in turn with C11, C12, and C13 to show that 11′, 12′, and 13′ are derivable in C. Now use C16, C11, and C14 to show that 15 is derivable in C. Use C10, C15, and C12 to show that 18 is derivable in C. Use C14 and C19 to show that 19′ is derivable in C.

C17′, included here as primitive in C, has a peculiar interest. It is in fact derivable from the remaining primitive rules of C. I have not been able to derive it from the other primitive replacement rules of Copi's system (and hence of course not from the other primitive replacement rules of C), and I believe that it is not thus derivable. The proof of its derivability from the remaining primitive rules of C is therefore delayed. Its presence does not greatly add, at least relatively, to the considerable tedium of the computations that impend, and it is to alleviate such tedium that I have thought it worth while to introduce the "simpler" system C.[1]

Copi's rule 3, Hypothetical Syllogism, is omitted here in favor of C20. The following sketch should suffice to show that 3 is derivable in C. (From this point on, reference to uses of Double Negation, Tautology, Association and Commutation for both conjunction and disjunction are frequently omitted, as are other obvious steps.)

1.	$S_1 \supset S_2$	hyp
2.	$S_2 \supset S_3$	hyp
3.	$(\sim S_1 \vee S_2) \cdot (\sim S_2 \vee S_3)$	C16, C8
4.	$\{(\sim S_1 \cdot \sim S_2) \vee [(\sim S_1 \cdot S_3) \vee (S_2 \cdot S_3)]\} \vee (S_2 \cdot \sim S_2)$	C13
5.	$(\sim S_1 . \sim S_2) \vee [(\sim S_1 \cdot S_3) \vee (S_2 \cdot S_3)]$	C20
6.	$(\sim S_1 \cdot \sim S_2) \vee [S_3 \cdot (\sim S_1 \vee S_2)]$	C13
7.	$(\sim S_1 \cdot \sim S_2) \vee S_3$	C13, C7
8.	$(\sim S_1 \vee S_3) \cdot (\sim S_2 \vee S_3)$	C13
9.	$S_1 \supset S_3$	C7, C16

With C3 available, 5 is derived as follows:

1.	$(S_1 \supset S_2) \cdot (S_3 \supset S_4)$	hyp
2.	$S_1 \vee S_3$	hyp
3.	$\sim S_1 \supset S_3$	C14, C16
4.	$\sim S_2 \supset \sim S_1$	C7, C15
5.	$S_3 \supset S_4$	C11, C7
6.	$\sim S_2 \supset S_4$	C3, C3
7.	$S_2 \vee S_4$	C16, C14

C15 can be applied to each conjunct in the major premiss of a Destructive Dilemma, and C5 can be applied to the result. Thus 6 is derivable in C.

The following sketch shows that 4 is derivable in C.

1.	$S_1 \vee S_2$	hyp
2.	$\sim S_1$	hyp
3.	$\sim S_1 \vee S_2$	C9
4.	$(S_2 \vee S_1) \cdot (S_2 \vee \sim S_1)$	C8
5.	$S_2 \vee (S_1 \cdot \sim S_1)$	C13
6.	S_2	C20

To show that Copi's system is equivalent to C, it suffices to show that C20 is derivable in Copi's system.

1.	$S_1 \vee (S_2 \cdot \sim S_2)$	hyp
2.	$(S_1 \vee S_2) \cdot (S_1 \vee \sim S_2)$	13
3.	$\sim S_1 \supset S_2$	7, 14, 16
4.	$S_2 \supset S_1$	7, 16
5.	$\sim S_1 \supset S_1$	3
6.	S_1	16, 14, 19

Consideration of the course of our proofs would show that C is equivalent to a system differing from C only in the inclusion of C3 in place of C20. The system C, with C20, is preferable again because computations are reduced, but also because it reflects more strikingly the reasoning of much of the sequel, which turns on the adding and dropping of conjuncts and disjuncts.

At any rate, the proof is now complete of:

Theorem 1 C *and the system of natural deduction for sentential calculus given by Copi in the first edition of* Introduction to Logic, *p. 259* (*and elsewhere*, *e.g.*, Symbolic Logic, *second edition*, *pp. 35, 40-41*) *are equivalent.*

In view of Theorem 1, "C" with unprimed numerals will refer to rules of either system; note that either of two duals may be referred to by one such designation.

2 *Further properties of* C. It will be useful to have at hand a number of facts concerning derivability in C. These are grouped in Lemma 1.

Lemma 1.

1. $S \vdash_{\overline{C}} S$

Proof: This rule may be taken to be a consequence of a definition of derivability in C. Since none has been provided, it may be noted that the rules of C suffice to justify the repetition of a line of a proof, for example by successive uses of C19, thus: S, $S \vee S$, S.

2. $(S_1 \cdot \sim S_1) \cdot S_2 \vdash_{\overline{C}} S_3$

Proof:

1.	$(S_1 \cdot \sim S_1) \cdot S_2$	hyp
2.	$S_1 \cdot \sim S_1$	C7

3. $(S_1 \cdot \sim S_1) \vee S_3$ C9
4. S_3 C20

3. *If* $S_1, S_2, \ldots, S_n \vdash_{\mathsf{C}} S_r$ *then* $S_q \vee S_1, S_q \vee S_2, \ldots, S_q \vee S_n \vdash_{\mathsf{C}} S_q \vee S_r$.

Proof: In a proof of S_r from premisses $S_1, S_2, \ldots, S_n$ by the rules of **C**, a line T_k is derived from lines T_i, T_j. (1) If T_k was derived from T_i by a rule of replacement, the same replacement can be made in $S_q \vee T_i$ to obtain $S_q \vee T_k$. (2) If T_k was derived from T_i by C20, then T_i is $T_k \vee (S \cdot \sim S)$. C20 produces $S_q \vee T_k$ from $S_q \vee [T_k \vee (S \cdot \sim S)]$. (3) If T_k was derived from T_i by C7, then T_i is $T_k \cdot S$. By C13 and C7, $S_q \vee T_k$ is derived from $S_q \vee (T_k \cdot S)$. (4) If T_k was derived from T_i and T_j by C8, then T_k was $T_i \cdot T_j$. By C8 and C13, $S_q \vee (T_i \cdot T_j)$ is derived from $S_q \vee T_i$ and $S_q \vee T_j$. (5) If T_k was derived from T_i by C9, then T_k is $T_i \vee S$. By C9 and C12, $S_q \vee (T_i \vee S)$ is derived from $S_q \vee T_i$.

Such expressions as for example "$\bigwedge_{i=1}^{n} S_i$" may be defined inductively ("$\bigwedge_{i=1}^{k+1} S_i = \left(\bigwedge_{i=1}^{k} S_i\right) \cdot S_{k+1}$" would be a clause of the definition) but since we have at hand Association and Commutation for both conjunction and disjunction, "$\bigwedge_{i=1}^{n} S_i$" ("$\bigvee_{i=1}^{n} S_i$") will be used for any conjunction (disjunction) of conjuncts (disjuncts) $S_1, S_2, \ldots, S_n$.

4. *If* $S_1 \vdash_{\mathsf{C}} T_1, S_2 \vdash_{\mathsf{C}} T_2, \ldots, S_n \vdash_{\mathsf{C}} T_n$, *then* $\bigvee_{i=1}^{n} S_i \vdash_{\mathsf{C}} \bigvee_{i=1}^{n} T_i$.

Proof: In a proof by mathematical induction, the inductive hypothesis would be $\bigvee_{i=1}^{k} S_i \vdash_{\mathsf{C}} \bigvee_{i=1}^{k} T_i$ and the hypothesis of the probandum would be $S_{k+1} \vdash_{\mathsf{C}} T_{k+1}$. By Lemma 1.3, $\bigvee_{i=1}^{k} S_i \vee S_{k+1} \vdash_{\mathsf{C}} \bigvee_{i=1}^{k} S_i \vee T_{k+1}$ and $\bigvee_{i=1}^{k} S_i \vee T_{k+1} \vdash_{\mathsf{C}} \bigvee_{i=1}^{k} T_i \vee T_{k+1}$. Q.E.D. by transitivity of $\vdash_{\mathsf{C}}$.

5. $S_1, S_2, \ldots, S_n \vdash_{\mathsf{C}} T_1$; $S_1, S_2, \ldots, S_n \vdash_{\mathsf{C}} T_2$; $\ldots$; $S_1, S_2, \ldots, S_n \vdash_{\mathsf{C}} T_r$ *if and only if* $\bigwedge_{i=1}^{n} S_i \vdash_{\mathsf{C}} \bigwedge_{i=1}^{r} T_i$.

Proof: By C7 and C8 (among other rules).

In the sequel, "α" and "β" with or without subscripts or other affixes will be used as metalinguistic variables whose values are either atomic sentences or the negations of atomic sentences. If α_1 is an atomic sentence and α_2 is the negation of α_1, then α_1 and α_2 are said to be a complementary pair, and each is said to be a complement of the other. Taken together with earlier decisions about notation, these entail that, for example, "$\bigvee_{i=1}^{s} \bigwedge_{j=1}^{r_i} \alpha_{ij}$" will denote a disjunction of s disjuncts, where the i'th disjunct is a conjunction of r_i conjuncts, and where each α_{ij} is an atomic sentence or the negation of an atomic sentence; in short, it denotes a sentence in disjunctive normal form.

6. *For every sentence S of* $\mathbf{C}$, *there are sentences* $S' = \bigvee_{i=1}^{s} \bigwedge_{j=1}^{r_i} \alpha_{ij}$ *and* $S'' = \bigwedge_{u=1}^{p} \bigvee_{v=1}^{q_u} \beta_{uv}$ *such that:*

(i) $S \underset{C}{\Leftrightarrow} S'$ *and* $S \underset{C}{\Leftrightarrow} S''$;
(ii) *exactly the same atomic sentences occur in* S, S', *and* S'';
(iii) *in any given disjunct* $\bigwedge\alpha_i$ *of* S', *no* α *occurs more than once, and in any given conjunct* $\bigvee\beta_u$ *of* S'', *no* β *occurs more than once*;

and

(iv) *if* $i \neq j$, *then one of the disjuncts* $\bigwedge\alpha_i$, $\bigwedge\alpha_j$ *in* S' *contains at least one* α *not contained in the other; and in* S'' *one of the conjuncts* $\bigvee\beta_u$, $\bigvee\beta_v$ *contains at least one* β *not contained in the other, if* $u \neq v$.

(Conditions (iii) and (iv) insure that S' and S'' contain no redundancy.)

Proof: A proof by mathematical induction would note that we have at hand **C**17 for the elimination of the triple bar (note that the second part of the rule is not also needed for this purpose), **C**16 for the elimination of the horseshoe, De Morgan's Theorems, Double Negation, and (two forms each of) Association, Commutation, Tautology, and Distribution.

In the sequel a prime and a double prime will be used to denote sentences related to S as are S' and S''. $\bigwedge\alpha_i$ will be the i'th disjunct of S'; $\bigvee\beta_u$ will be the u'th conjunct of S''; subscripts of course will vary at need. Where context permits, subscripts will be omitted. Note further that Lemma 1.6 claims the existence of disjunctive and conjunctive normal forms (that can replace or be replaced by the sentence of which they are normal forms). It does not claim that these are "distinguished" normal forms (that is, it does not claim that each disjunct (conjunct) of the normal form contains an occurrence of every atomic sentence occurring in the original). But every atomic sentence of the original does occur at least once in the normal form.

7. $S_1, \ldots, S_n \vdash_{C} T_1$; $S_1, \ldots, S_n \vdash_{C} T_2$; $\ldots$, $S_1, \ldots, S_n \vdash_{C} T_m$; *if and only if* $\left(\bigwedge_{i=1}^{n} S_i\right)' \vdash_{C} \left(\bigwedge_{i=1}^{m} T_i\right)''$.

Proof: By Lemma 1.5, 6, definitions of the metalinguistic symbols, and transitivity properties of the relation denoted by "$\vdash_{C}$".

8. *If for every disjunct* $\bigwedge\alpha_i$ *in* S' *that does not contain a complementary pair and every conjunct* $\bigvee\beta_u$ *in* S_2'' *there is some* α *in* $\bigwedge\alpha_i$ *identical with some* β *in* $\bigvee\beta_u$, *then* $S_1' \vdash_{C} S_2''$.

Proof: For any $\bigwedge\alpha_i$ that does contain a complementary pair, $\bigwedge\alpha_i \vdash_{C} S_2''$ by Lemma 1.2. Consider now any $\bigwedge\alpha_i$ in S_1' that does not contain a complementary pair. For any $\bigvee\beta_u$ in S_2'', there exists by hypothesis an α in $\bigwedge\alpha_i$ and a β in $\bigvee\beta_u$ such that $\alpha = \beta$. By C7, $\bigwedge\alpha_i \vdash_{\mathbf{C}} \alpha$; by C9, $\beta \vdash_{\mathbf{C}} \bigvee\beta_u$. By Lemma 1.5 and C19, $\bigwedge\alpha_i \vdash_{\mathbf{C}} S_2''$. Q.E.D. by Lemma 1.4 and C19.

3 *A decision procedure for validity in* **C**. Consider now the matrix $\mathfrak{M} = \langle K, D, -, +, \times, \rightarrow, \leftrightarrow\rangle$ where K is the set $\{0, 1, 2\}$, D (the set of designated elements of K) is the set $\{0\}$, and the singulary operator - and the binary operator + are defined by the tables:

+	0	1	2	-
0	0	0	0	2
1	0	1	1	1
2	0	1	2	0

Tables for the remaining operators are constructed by setting

$$x \times y = -(-x + -y)$$
$$x \rightarrow y = -x + y$$
$$x \leftrightarrow y = (x \rightarrow y) \times (y \rightarrow x)$$

The tables so constructed are:

×	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

→	0	1	2
0	0	1	2
1	0	1	1
2	0	0	0

↔	0	1	2
0	0	1	2
1	1	1	1
2	2	1	0

The mappings considered in the sequel are mappings with domain the set of sentences constructible from the atomic sentences that occur in some set of sentences and range a subset of K. Context will usually make clear the domain of a mapping, and cursory expressions will be used: thus, "a mapping of S" will mean "a mapping with domain determined by the atomic sentences occurring in S and range a subset of K." These mappings Θ will conform to the following conditions:

1. For n atomic sentences, and a subset of K with m elements, there are m^n mappings.
2. $\Theta(\sim S) = -\Theta(S)$
3. $\Theta(S_i \vee S_j) = \Theta(S_i) + \Theta(S_j)$
4. $\Theta(S_i \cdot S_j) = \Theta(S_i) \times \Theta(S_j)$
5. $\Theta(S_i \supset S_j) = \Theta(S_i) \rightarrow \Theta(S_j)$
6. $\Theta(S_i \equiv S_j) = \Theta(S_i) \leftrightarrow \Theta(S_j)$

The next lemma catalogues some properties of these mappings.

Lemma 2.

1. *If* Θ' *is a subset of* Θ, *then* $\Theta'(S) = \Theta(S)$.
2. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are mutually exclusive sets of atomic sentences, and* Θ *and* Φ *are mappings with domains determined by* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively, then the*

mapping $\Theta \cup \Phi$ *with domain determined by* $\mathscr{A} \cup \mathscr{B}$ *is such that* $(\Theta \cup \Phi)(S) = \Theta(S)$ *if the atomic sentences of S are members of* $\mathscr{A}$.

Proofs: 1 and 2 are proved from set-theoretical considerations alone.

The next seven properties may be verified by inspection of $\mathfrak{M}$ and the conditions imposed on mappings Θ.

3. $\Theta\left(\bigwedge_i S_i\right) = 0$ *if and only if* $\Theta(S_i) = 0$ *for each* S_i *in* $\bigwedge_i S_i$.

4. $\Theta\left(\bigvee_i S_i\right) = 0$ *if and only if* $\Theta(S_i) = 0$ *for some* S_i *in* $\bigvee_i S_i$.

5. $\Theta\left(\bigwedge_i S_i\right) = 2$ *if and only if* $\Theta(S_i) = 2$ *for some* S_i *in* $\bigwedge_i S_i$.

6. $\Theta\left(\bigvee_i S_i\right) = 2$ *if and only if* $\Theta(S_i) = 2$ *for each* S_i *in* $\bigvee_i S_i$.

7. $\Theta\left(\bigwedge_i S_i\right) = 1$ *if and only if* $\Theta(S_i) \neq 2$ *for each* S_i *in* $\bigwedge_i S_i$ *and* $\Theta(S_i) = 1$ *for some* S_i *in* $\bigwedge_i S_i$.

8. $\Theta\left(\bigvee_i S_i\right) = 1$ *if and only if* $\Theta(S_i) \neq 0$ *for each* S_i *in* $\bigvee_i S_i$ *and* $\Theta(S_i) = 1$ *for some* S_i *in* $\bigvee_i S_i$.

9. *If* $\Theta(A_i) = 1$ *for each atomic sentence* A_i *that occurs in* S, *then* $\Theta(S) = 1$; *and if* $\Theta(A_i) \neq 1$ *for each* A_i *that occurs in* S, *then* $\Theta(S) \neq 1$.

10. $\Theta\left(\bigwedge \alpha_i\right) \neq 0$ *for all* Θ, *if and only if* $\bigwedge \alpha_i$ *contains a complementary pair.*

Proof: If $\bigwedge \alpha_i$ contains a complementary pair, then either $\Theta(\alpha) = 2$ for one of the pair, or $\Theta(\alpha) = 1$ for each of the pair; in either case, $\Theta\left(\bigwedge \alpha_i\right) \neq 0$. If $\bigwedge \alpha_i$ does not contain a complementary pair, then no atomic sentence occurs in $\bigwedge \alpha_i$ more than once. By applications of Lemma 2.1, the sum of the mappings which severally send the α's in $\bigwedge \alpha_i$ onto $\{0\}$ sends $\bigwedge \alpha_i$ onto $\{0\}$.

11. $\Theta\left(\bigvee \beta_u\right) \neq 2$ *for all* Θ, *if and only if* $\bigvee \beta_u$ *contains a complementary pair.*

Proof: Similar to proof of 10, it being noted that $\Theta(\beta) = 0$ or $\Theta(\beta) = 1$ for some member of the pair.

12. *There is a* Θ *such that* $\Theta\left(\bigwedge \alpha_i\right) = 0$ *and* $\Theta\left(\bigvee \beta_u\right) \neq 0$ *if and only if there is a* Θ *such that* $\Theta\left(\bigwedge \alpha_i\right) = 0$ *and there is no* α *in* $\bigwedge \alpha_i$ *and* β *in* $\bigvee \beta_u$ *such that* $\alpha = \beta$.

Proof: If $\Theta\left(\bigwedge \alpha_i\right) = 0$ and $\Theta\left(\bigvee \beta_u\right) \neq 0$, then $\Theta(\alpha) = 0$ for all α in $\bigwedge \alpha_i$, by Lemma 2.3. If there are α and β in $\bigwedge \alpha_i$ and $\bigvee \beta_u$ respectively such that $\alpha = \beta$, then $\Theta(\beta) = 0$ and thus $\Theta\left(\bigvee \beta_u\right) = 0$ by Lemma 2.4, contrary to the hypothesis. If $\Theta\left(\bigwedge \alpha_i\right) = 0$ and there is no pair α, β such that $\alpha = \beta$ (where

α is in $\bigwedge\alpha_i$ and β is in $\bigvee\beta_u$), then every pair α, β is either complementary or its members contain no atomic sentence in common. Let Θ_1 be the submapping of Θ with domain determined by the atomic sentences that occur in $\bigwedge\alpha_i$ and hence in those β's (if any) in $\bigvee\beta_u$ that are complements of any α in $\bigwedge\alpha_i$. Let the domain of Φ be determined by the remaining atomic sentences (if any) that occur in $\bigvee\beta_u$; let the range of Φ be $\{1\}$. Then by Lemma 2.1, 2, $(\Theta_1 \cup \Phi)\left(\bigwedge\alpha_i\right) = \Theta_1\left(\bigwedge\alpha_i\right) = 0$, and $(\Theta_1 \cup \Phi)\left(\bigvee\beta_u\right) = 2$ if each β is the complement of some α, or $(\Theta_1 \cup \Phi)\left(\bigvee\beta_u\right) = 1$ if some β is not the complement of any α. Then $(\Theta_1 \cup \Phi)\left(\bigwedge\alpha_i\right) = 0$ and $(\Theta_1 \cup \Phi)\left(\bigvee\beta_u\right) \neq 0$.

13. *If there is a mapping Θ onto a subset of $\{0, 2\}$ such that $\Theta(S_1) = 0$ and $\Theta(S_2) = 2$, then there is a mapping Θ onto a subset of K such that $\Theta(S_1) = 0$ and $\Theta(S_2) = 2$.*

Proof: $\{0, 2\} \subset K$.

14. *If there is a mapping Θ of S_1' and S_2'' onto a subset of K such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$, then there is a mapping with the same domain onto a subset of $\{0, 2\}$ such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$.*

Proof: If the hypothesis holds for Θ, then there exists $\bigwedge\alpha_i$ in S_1' such that $\Theta(\alpha) = 0$ for each α in $\bigwedge\alpha_i$, by Lemma 2.4, 3. There also exists $\bigvee\beta_u$ in S_2'' such that $\Theta(\beta) = 2$ for each β in $\bigvee\beta_u$ by Lemma 2.5, 6. Hence the set of atomic sentences occurring in $\bigwedge\alpha_i$ and $\bigvee\beta_u$ is mapped by Θ onto a subset of $\{0, 2\}$. Let Θ_1 be the subset of Θ with domain determined by the atomic sentences occurring in $\bigwedge\alpha_i$ and $\bigvee\beta_u$. $\Theta_1\left(\bigwedge\alpha_i\right) = 0$ and $\Theta_1\left(\bigvee\beta_u\right) = 2$ by Lemma 2.1. Let Φ be any mapping with domain determined by the remaining atomic sentences with occurrences in S_1' and S_2'' and range a subset of $\{0, 2\}$. Then $(\Theta_1 \cup \Phi)\left(\bigwedge\alpha_i\right) = \Theta_1\left(\bigwedge\alpha_i\right) = 0$ by Lemma 2.2, so that $(\Theta_1 \cup \Phi)(S_1') = 0$. Further, $(\Theta_1 \cup \Phi)\left(\bigvee\beta_u\right) = \Theta_1\left(\bigvee\beta_u\right) = 2$, so that $(\Theta_1 \cup \Phi)(S_2'') = 2$. $\Theta_1 \cup \Phi$ is the required mapping.

15. *$S_1' \supset S_2''$ is a tautology of the classical sentential calculus if and only if there is no mapping Θ onto a subset of K such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$.*

Proof: The sub-matrix $\mathfrak{M}'$ of $\mathfrak{M}$ in which K' is $\{0, 2\}$, D' is D, and arguments and values of $-'$, $+'$, $\times'$, $\rightarrow'$, and $\leftrightarrow'$ are as in $\mathfrak{M}$ but restricted to K' is (isomorphic to) the two-valued matrix characteristic for classical sentential calculus. (Put **T** for 0, **F** for 2.) Hence, $S_1' \supset S_2''$ is a tautology of the classical sentential calculus if and only if there is no mapping of S_1', S_2'' onto a subset of $K' = \{0, 2\}$ such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$. The probandum follows by Lemma 2.13, 14.

16. *If no* $\bigvee \beta_u$ *in* S_2'' *contains a complementary pair, and if there is a mapping* Θ *of* S_1' *and* S_2'' *onto a subset of* K *such that* $\Theta(S_1') = 0$ *and* $\Theta(S_2'') = 1$, *then there is a mapping* Θ *onto a subset of* $\{0, 2\}$ *such that* $\Theta(S_1') = 0$ *and* $\Theta(S_2'') = 2$.

Proof: If the hypothesis holds for Θ, then there exists $\bigwedge \alpha_i$ in S_1' and $\bigvee \beta_u$ in S_2'' such that $\Theta\left(\bigwedge \alpha_i\right) = 0$ and $\Theta\left(\bigvee \beta_u\right) = 1$. By the proof of Lemma 2.12, for every α in $\bigwedge \alpha_i$ and every β in $\bigvee \beta_u$, $\alpha \neq \beta$. Then every β in $\bigvee \beta_u$ is either complementary to some α in $\bigwedge \alpha_i$ or contains an atomic sentence not occurring in $\bigwedge \alpha_i$; further, at least one β in $\bigvee \beta_u$ contains an atomic sentence not occurring in $\bigwedge \alpha_i$ (otherwise, $\Theta(\beta) = 2$ for every β in $\bigvee \beta_u$ and thus $\Theta\left(\bigvee \beta_u\right) = 2$, contrary to the hypothesis). Let Θ_1 be the subset of Θ with domain determined by the atomic sentences occurring in $\bigwedge \alpha_i$. Let Φ be the mapping such that the domain of Φ is determined by those atomic sentences occurring in $\bigvee \beta_u$ that do not occur in $\bigwedge \alpha_i$ and such that $\Phi(\beta) = 2$ for all β occurring in $\bigvee \beta_u$ but not in $\bigwedge \alpha_i$. (Φ is not null as $\bigvee \beta_u$ contains no complementary pair and hence contains no more than one occurrence of any atomic sentence). Let Ψ map the remaining atomic sentences (if any) occurring in S_1' and S_2'' onto a subset of $\{0, 2\}$. Then the range of $\Theta_1 \cup \Phi \cup \Psi$ is a subset of $\{0, 2\}$, and $(\Theta_1 \cup \Phi \cup \Psi)\left(\bigvee \beta_u\right) = 2 = (\Theta_1 \cup \Phi \cup \Psi)(S_2'')$, while $(\Theta_1 \cup \Phi \cup \Psi)\left(\bigwedge \alpha_i\right) = \Theta_1\left(\bigwedge \alpha_i\right) = 0 \Theta_1 \cup \Phi \cup \Psi(S_1')$.

17. *If there exists* Θ *such that* $\Theta(S_1') = 0$ *and* $\Theta(S_2'') = 1$ *but there exists no* Θ *such that* $\Theta(S_1') = 0$ *and* $\Theta(S_2'') = 2$, *then there exists* $\bigwedge \alpha_i$ *in* S_1' *and* $\bigvee \beta_u$ *in* S_2'' *such that* (i) *if any pair* α, β (α *in* $\bigwedge \alpha_i$ *and* β *in* $\bigvee \beta_u$) *contains a common atomic sentence, the pair is complementary, and* (ii) $\bigvee \beta_u$ *contains a complementary pair the atomic sentence of which does not occur in* $\bigwedge \alpha_i$.

Proof: If the hypothesis holds for Θ, then for some $\bigwedge \alpha_i$ in S_1' and some $\bigvee \beta_u$ in S_2'', $\Theta\left(\bigwedge \alpha_i\right) = 0$ and $\Theta\left(\bigvee \beta_u\right) = 1$. Then no α in $\bigwedge \alpha_i$ is identical with any β in $\bigvee \beta_u$ (otherwise, for some β in $\bigvee \beta_u$, $0 = \Theta(\alpha) = \Theta(\beta)$ and thus $\Theta\left(\bigvee \beta_u\right) = 0$). Hence, (i) if any pair α, β (α in $\bigwedge \alpha_i$ and β in $\bigvee \beta_u$) contains just one atomic sentence, the pair is complementary. If $\bigvee \beta_u$ contains no complementary pair, it contains no more than one occurrence of any atomic sentence. Let Θ_1 be the subset of Θ with domain determined by the atomic sentences occurring in $\bigwedge \alpha_i$ and range a subset of $\{0, 2\}$, and let Φ map remaining atomic sentences of $\bigvee \beta_u$ onto a subset of $\{0, 2\}$ in such a way that for each β in $\bigvee \beta_u$ which is not the complement of some α in $\bigwedge \alpha_i$, $\Phi(\beta) = 2$. Let Ψ map remaining atomic sentences in S_1' and S_2'' onto some subset of $\{0, 2\}$. $(\Theta_1 \cup \Phi)\left(\bigwedge \alpha_i\right) = 0$ and $(\Theta_1 \cup \Phi)\left(\bigvee \beta_u\right) = 2$. Then $(\Theta_1 \cup \Phi \cup \Psi)(S_1') = 0$ and $(\Theta_1 \cup \Phi \cup \Psi)(S_2'') = 2$, contrary to hypothesis. Hence (ii) $\bigvee \beta_u$ contains some complementary pair, which cannot contain an atomic sentence in common with any α in $\bigwedge \alpha_i$ (if it did, one member of the pair would be identical with an α in $\bigwedge \alpha_i$, contrary to what is proved in (i)).

18. *For every* Θ, $\Theta(S') = \Theta(S) = \Theta(S'')$.

Proof: Computation shows that the left and right hand sides of any instance of a primitive replacement rule of C must be assigned the same element of K.

19. *If* $\bigwedge \alpha_i$ *contains a complementary pair, then for every* Θ *with range a subset of* $\{0, 2\}$, $\Theta\left(\bigwedge \alpha_i\right) = 2$; *and if* $\bigvee \beta_u$ *contains a complementary pair, then for every* Θ *with range a subset of* $\{0, 2\}$, $\Theta\left(\bigvee \beta_u\right) = 0$.

Proofs: By Lemma 2.4, 5.

An abbreviation will now be convenient.

Definition 1. $S_1, S_2, \ldots, S_n \therefore S_{n+1}$ is C-valid if and only if there is no mapping Θ (with domain determined by the atomic sentences occurring in the S_i and range a subset of K) such that $\Theta(S_i) = 0$ for all i, $1 \leq i \leq n$, but $\Theta(S_{n+1}) \neq 0$.

C-validity is an attribute of arguments. Lemma 3 collects some facts about this attribute.

Lemma 3.

1. *If* $S_1, S_2, \ldots, S_n \therefore S_{n+1}$ *is* C-*valid, then* $T_1, T_2, \ldots, T_m, S_1, S_2, \ldots, S_n \therefore S_{n+1}$ *is* C-*valid.*
2. *If each of* $S_1, S_2, \ldots, S_n \therefore T_1$; $S_1, S_2, \ldots, S_n \therefore T_2$; $\ldots$; $S_1, S_2, \ldots, S_n \therefore T_m$; *and* $T_1, T_2, \ldots, T_m \therefore S_{n+1}$ *is* C-*valid, then* $S_1, S_2, \ldots, S_n \therefore S_{n+1}$ *is* C-*valid.*

Proofs: 1 and 2 follow from Definition 1.

3. $S_1, S_2, \ldots, S_n \therefore S_{n+1}$ *is* C-*valid if and only if* $\bigwedge_{i=1}^{n} S_i \therefore S_{n+1}$ *is* C-*valid.*

Proof: By Lemma 2.3.

4. *If* $S_1 \vdash_{\mathsf{C}} S_2$, *then* $S_1 \therefore S_2$ *is* C-*valid.*

Proof: In view of Lemma 3.1, 2, it suffices to show that in the derivation of S_2 from S_1 by the rules of C, if the k'th line was derived i'th and j'th lines by one of the rules of C, that is, if $T_i, T_j \vdash_{\mathsf{C}} T_k$, then $T_i, T_j \therefore T_k$ is C-valid. There are five cases to be considered. (i) T_k was derived from T_i by one of the rules of replacement. A check of these shows that where $S_1 \overset{}{\underset{C}{\leftrightarrow}} S_2$ is a replacement rule of **C**, $\Theta(S_1) = \Theta(S_2)$ for all Θ, so that if $\Theta(T_i) = 0$ then $\Theta(T_k) = 0$. (ii) T_k was derived from T_i by **C**20. Then T_i is $T_k \vee (S \cdot \sim S)$. But $\Theta(T_k) = 0$ if and only if $\Theta[T_k \vee (S \cdot \sim S)] = 0$. (iii) T_k was derived from T_i by **C**7. Then T_i is $T_k \cdot S$. But $\Theta(T_k \cdot S) = 0$ only if $\Theta(T_k) = 0$. (iv) T_k was derived from T_i and T_j by **C**8. Then T_k is $T_i \cdot T_j$. But $\Theta(T_i \cdot T_j) = 0$ if and only if both $\Theta(T_i) = 0$ and $\Theta(T_j) = 0$. (v) T_k was derived from T_i by **C**9. Then T_k is $T_i \vee S$. But $\Theta(T_i) = 0$ only if $\Theta(T_i \vee S) = 0$.

We note here that with this part of Lemma 3, the proof of the incompleteness of **C**, essentially the result reported in Copi, *Symbolic Logic*, second edition, pp. 53 ff., is at hand. The next part of Lemma 3 shows that the matrix $\mathfrak{M}$ furnishes us with a decision procedure for **C**.

5. *If* $S_1' \therefore S_2''$ *is* C-*valid, then* $S_1' \vdash_C S_2''$.

Proof: Suppose $S_1' \therefore S_2''$ is C-valid. There are two cases to consider. (i) There is no mapping of S_1', S_2'' onto a subset of K for which $\Theta(S_1') = 0$. Then there is no $\bigwedge\alpha_i$ in S_1' such that for some Θ, $\Theta\left(\bigwedge\alpha_i\right) = 0$. Then every $\bigwedge\alpha_i$ in S_1' contains a complementary pair, by Lemma 2.10. By Lemma 1.8, $S_1' \vdash_C S_2''$. (ii) There is a Θ such that $\Theta(S_1') = 0$. Then by Lemma 2.10, at least one $\bigwedge\alpha_i$ contains no complementary pair. For any such $\bigwedge\alpha_i$, $\bigwedge\alpha_i \therefore S_1'$ is C-valid, by Lemma 2.4. $S_1' \therefore S_2''$ is C-valid by hypothesis. For any $\bigvee\beta_u$ in S_2'', $S_2'' \therefore \bigvee\beta_u$ is C-valid, by Lemma 2.3. Hence $\bigwedge\alpha_i \therefore \bigvee\beta_u$ is C-valid, by Lemma 3.2. Then there is no Θ such that $\Theta\left(\bigwedge\alpha_i\right) = 0$ and $\Theta\left(\bigvee\beta_u\right) \neq 0$. Hence there is a pair α, β $\left(\alpha \text{ in } \bigwedge\alpha_i \text{ and } \beta \text{ in } \bigvee\beta_u\right)$ such that $\alpha = \beta$ by Lemma 2.12. But $\bigwedge\alpha_i \vdash_C \alpha$ (by **C**7) and $\beta \vdash_C \bigvee\beta_u$ (by **C**9). Hence $\bigwedge\alpha_i \vdash_C \bigvee\beta_u$. But $\bigwedge\alpha_i$ was selected arbitrarily from among those disjuncts of S_1' that do not contain a complementary pair, and $\bigvee\beta_u$ was an arbitrary conjunct of S_2''. By Lemma 1.8, $S_1' \vdash_C S_2''$.

6. $S_1 \therefore S_2$ *is* C-*valid if and only if* $S_1' \therefore S_2''$ *is* C-*valid.*

Proof: By the proof of Lemma 3.4 (i), there is no Θ such that $\Theta(S_1) \neq \Theta(S_1')$ and there is no Θ such that $\Theta(S_2) \neq \Theta(S_2'')$.

Theorem 2. $S_1, S_2, \ldots, S_n \therefore S_{n+1}$ *is* C-*valid if and only if* $S_1, S_2, \ldots, S_n \vdash_C S_{n+1}$.

Proof: $S_1, S_2, \ldots, S_n \therefore S_{n+1}$ is C-valid if and only if $\bigwedge_{i=1}^{n} S_i \therefore S_{n+1}$ is C-valid, by Lemma 2.3; the latter is C-valid if and only if $\left(\bigwedge_{i=1}^{n} S_i\right)' \therefore S_{n+1}''$ is C-valid, by Lemma 2.18; the latter is C-valid if and only if $\left(\bigwedge_{i=1}^{n} S_i\right)' \vdash_C S_{n+1}''$

by Lemma 3.4, 5. $\left(\bigwedge_{i=1}^{n} S_i\right)' \vdash_C S_{n+1}''$ if and only if $\bigwedge_{i=1}^{n} S_i \vdash_C S_{n+1}$ by Lemma 1.6, but the latter is the case if and only if $S_1, S_2, \ldots, S_n \vdash_C S_{n+1}$ by Lemma 1.5 (and **C**19).

Theorem 2 assures us that the matrix $\mathfrak{M}$ affords a decision procedure for **C**. Given an argument in which premisses and conclusion are constructed from atomic sentences by the connectives $\sim$, $\cdot$, $\vee$, $\supset$, and $\equiv$, we may decide whether or not there exists a proof in which every line is either a premiss of the argument or is derived from one or more preceding lines by the rules of **C**. If we cannot assign values from K to the atomic sentences of the argument in such a way that the premisses all have the value 0, or, if there is such an assignment but for every such assignment the conclusion also has the value 0, then such a proof can be constructed. But if there is an assignment for which the calculated value of each of the premisses is 0 but the calculated value of the conclusion is not 0, then such a proof cannot be constructed.

4 *Incompleteness of* **C** *and associated results.*

Definition 2. A system of natural deduction for classical sentential calculus is incomplete if and only if there are sentences S_1 and S_2 such that $S_1 \supset S_2$ is a tautology of the two-valued sentential calculus but such that S_2 is not derivable from S_1 by the rules of the system.

Theorem 2 (Corollary). **C** *is incomplete in the sense of* Definition 2.

Proof: Copi gives an example to prove incompleteness of his system in *Symbolic Logic,* second edition, p. 57. Here are some other interesting facts with the same result. Let S_2 be any sentence and let S_1 be any atomic sentence not occurring in S_2. Let $\Theta(S_1) = 0$ and let $\Theta(A_i) = 1$ for every atomic sentence A_i occurring in S_2. Then $\Theta(S_2) = 1$ by Lemma 2.9. Hence we can say that **C** "contains no theorems", and that it "contains no tautologies", if we mean that there are no sentences of **C** which are derivable from arbitrarily selected premisses. On the other hand, we can say that **C** "contains contradictions", if we mean that there are sentences of **C** from which arbitrarily selected conclusions are derivable; among these are sentences of the form $S \cdot \sim S$. Note, however, that although any two such sentences may be derived each from the other in **C**, they are not "equivalent"; though we may derive either from the other, we cannot necessarily replace either by the other. It appears then that though **C** contains contradictions, one could not define in **C** "the contradictory" or "the false".

The foregoing fact illustrates a peculiarity of **C**. We cannot prove for it a replacement rule of the usual sort; we cannot prove that S_1 may replace or be replaced by S_2 if either may be derived from the other by the rules of **C**. However, a weak replacement rule is provable for **C**.

Theorem 3. *If* $S_1 \vdash_C S_2$, $S_2 \vdash_C S_1$, $\sim S_1 \vdash_C \sim S_2$, *and* $\sim S_2 \vdash_C \sim S_1$, *then* (*where* $S(S_2)$ *is like* $S(S_1)$ *except that* $S(S_2)$ *contains occurrences of* S_2 *at zero or more* *places where* $S(S_1)$ *contains occurrences of* S_1) $S(S_1) \vdash_C S(S_2)$, $S(S_2) \vdash_C S(S_1)$, $\sim S(S_1) \vdash_C \sim S(S_2)$, *and* $\sim S(S_2) \vdash_C \sim S(S_1)$.

Proof: One proof uses the properties of $\mathfrak{M}$. If $S_1 \vdash_C S_2$ and $S_2 \vdash_C S_1$ then from Lemma 3.4 and Definition 1, for every Θ, $\Theta(S_1) = \Theta(S_2) = 0$ or $\Theta(S_1) \neq 0 \neq \Theta(S_2)$. Similarly, if $\sim S_1 \vdash_C \sim S_2$ and $\sim S_2 \vdash_C \sim S_1$, then either $\Theta(\sim S_1) = \Theta(\sim S_2) = 0$ or $\Theta(\sim S_1) \neq 0 \neq \Theta(\sim S_2)$. But $\Theta(S_1) = 2$ if and only if $\Theta(\sim S_1) = 0$; similarly for S_2. Hence $\Theta(S_1) = 2$ if and only if $\Theta(S_2) = 2$. In sum, $\Theta(S_1) = \Theta(S_2)$ for all Θ. Hence $\Theta S(S_1) = \Theta S(S_2)$. Hence $S(S_1) \therefore S(S_2)$ is C-valid, so that $S(S_1) \vdash_C S(S_2)$, and similarly for the other cases mentioned in the theorem.

A second proof is more tedious, but makes no use of $\mathfrak{M}$, but only of such properties of **C** as have been proved in Lemma 1. We sketch the main features of representative parts of this proof, by induction on the length of S. For the case where the length of S is 1, S is atomic, and $S(S_1)$ is either S_1 or S does not contain S_1. In either case, if $S(S_2)$ results from replacing no occurrences of S_1 in S by occurrences of S_2, then $S(S_2)$ is identical with $S(S_1)$, and by Lemma 1.1, we have all the cases to be proved. If, however, $S(S_1)$ is S_1 and S_1 is replaced by S_2, then $S(S_2)$ is S_2, and what is to be proved for this case is what is already in the hypothesis, namely that $S_1 \vdash_C S_2$, $S_2 \vdash_C S_1$, $\sim S_1 \vdash_C \sim S_2$, and $\sim S_2 \vdash_C \sim S_1$.

Suppose now that the length of S is greater than 1. There are five cases to be considered. (i) S is a negation, i.e., S is $\sim S^*(S_1)$. Then the length of $S^*(S_1)$ is less than the length of S, and the inductive hypothesis holds; hence $S^*(S_1) \vdash_C S^*(S_2)$, $S^*(S_2) \vdash_C S^*(S_1)$, $\sim S^*(S_1) \vdash_C \sim S^*(S_2)$, and $\sim S^*(S_2) \vdash_C \sim S^*(S_1)$. The last two parts of this hypothesis are the same as the first two parts of what is to be proved. As for the first two parts to be proved, by **C**14, $\sim\sim S^*(S_1) \vdash_C S^*(S_1)$ and $S^*(S_2) \vdash_C \sim\sim S^*(S_2)$. These, with $S^*(S_1) \vdash_C S^*(S_2)$ from the inductive hypothesis, yield $\sim\sim S^*(S_1) \vdash_C \sim\sim S^*(S_2)$, which is $\sim S(S_1) \vdash_C \sim S(S_2)$. $\sim S(S_2) \vdash_C \sim S(S_1)$ is similarly proved. (ii) S is $S^*(S_1) \vee S^{**}(S_1)$. By the inductive hypothesis, $S^*(S_1) \vdash_C S^*(S_2)$, and $S^{**}(S_1) \vdash_C S^{**}(S_2)$. By Lemma 1.4, $S^*(S_1) \vee S^{**}(S_1) \vdash_C S^*(S_2) \vee S^{**}(S_2)$, that is, $S(S_1) \vdash_C S(S_2)$. $S(S_2) \vdash_C S(S_1)$ is similarly proved. Again by the inductive hypothesis, $\sim S^*(S_1) \vdash_C \sim S^*(S_2)$ and $\sim S^{**}(S_1) \vdash_C \sim S^{**}(S_2)$. By Lemma 1.5, $\sim S^*(S_1) \cdot \sim S^{**}(S_1) \vdash_C \sim S^*(S_2) \cdot \sim S^{**}(S_2)$. By **C**10, $\sim[S^*(S_1) \vee S^{**}(S_1)] \vdash_C \sim[S^*(S_2) \vee S^{**}(S_2)]$, which is $\sim S(S_1) \vdash_C \sim S(S_2)$. $\sim S(S_2) \vdash_C \sim S(S_1)$ is similarly proved. Cases (iii) S is $S^* \cdot S^{**}$, (iv) S is $S^* \supset S^{**}$, and (v) S is $S^* \equiv S^{**}$ are also proved by appeal to appropriate replacement rules which allow us to take advantage of Lemma 1.4, 5.

We are now in a position to make good the earlier remark to the effect that one of the rules 17, 17′ may be deleted from the set chosen as equivalent to Copi's system, with preservation of the equivalence. Neither of the rules was cited in deriving Copi's system from **C** (except of course to show that **C** did in fact contain both these rules, but no other derivation appealed to either). Further, only one of the rules is needed to establish case (v) for the weak replacement rule. To show that the other of the rules is derivable from whichever is chosen as primitive, it suffices, in virtue of Theorem 3, to prove that the right hand sides of the two replacement rules

17, 17′ satisfy the conditions in the antecedent of Theorem 3. Such is the point of Lemma 4.

Lemma 4.

1. $(S_1 \supset S_2) \cdot (S_2 \supset S_1) \vdash_C (S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)$
2. $(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2) \vdash_C (S_1 \supset S_2) \cdot (S_2 \supset S_1)$

Proof: Read the following sequence of lines from the top down, with annotations on the right, for a sketch of a proof of 1; for a sketch of a proof of 2, read from the bottom line up, with annotations on the left.

C13, C16	1. $(S_1 \supset S_2) \cdot (S_2 \supset S_1)$	
C13	2. $[(\sim S_1 \vee S_2) \cdot \sim S_2] \vee [(\sim S_1 \vee S_2) \cdot S_1]$	C16, C13
C11, C12	3. $[(\sim S_2 \cdot \sim S_1) \vee (S_2 \cdot \sim S_2)] \vee [(S_1 \cdot S_2) \vee (S_1 \cdot \sim S_1)]$	C13
C9	4. $\{[(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)] \vee [(S_2 \cdot \sim S_2) \vee (S_1 \cdot \sim S_1)]\}$	C11, C12
C9	5. $[(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)] \vee (S_2 \cdot \sim S_2)$	C20
	6. $(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)$	C20

3. $\sim[(S_1 \supset S_2) \cdot (S_2 \supset S_1)] \vdash_C \sim[(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)]$
4. $\sim[(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)] \vdash_C \sim[(S_1 \supset S_2) \cdot (S_2 \supset S_1)]$

Proof: In the following sequence, read down for a proof of 3, up for a proof of 4.

C16	1. $\sim[(S_1 \supset S_2) \cdot (S_2 \supset S_1)]$	
C14	2. $\sim(\sim S_1 \vee S_2) \vee \sim(\sim S_2 \vee S_1)$	C10, C16
C10	3. $\sim(\sim S_1 \vee \sim\sim S_2) \vee \sim(\sim S_2 \vee \sim\sim S_1)$	C14
C14	4. $\sim\sim(S_1 \cdot \sim S_2) \vee \sim\sim(S_2 \cdot \sim S_1)$	C10
C20	5. $(S_1 \cdot \sim S_2) \vee (S_2 \cdot \sim S_1)$	C14
C11, C12	6. $[(S_1 \cdot \sim S_2) \vee (S_2 \cdot \sim S_1)] \vee [(S_1 \cdot \sim S_1) \vee (S_2 \cdot \sim S_2)]$	C9
C13	7. $[(S_1 \cdot \sim S_1) \vee (S_1 \cdot \sim S_2)] \vee [(S_2 \cdot \sim S_1) \vee (S_2 \cdot \sim S_2)]$	C11, C12
C13	8. $[S_1 \cdot (\sim S_1 \vee \sim S_2)] \vee [S_2 \cdot (\sim S_1 \vee \sim S_2)]$	C13
C10, C14	9. $(\sim S_1 \vee \sim S_2) \cdot (S_1 \vee S_2)$	C13
C10	10. $\sim(S_1 \cdot S_2) \cdot (\sim\sim S_1 \vee \sim\sim S_2)$	C10, C14
C10	11. $\sim(S_1 \cdot S_2) \cdot \sim(\sim S_1 \cdot \sim S_2)$	C10
	12. $\sim[(S_1 \cdot S_2) \vee (\sim S_1 \cdot \sim S_2)]$	C10

With Lemma 4, the proof of Theorem 4 is completed.

Theorem 4 *The system of natural deduction for sentential calculus of Copi,* Introduction to Logic, *first edition, p. 259, is equivalent to the system consisting of the following rules:*

C20. $S_1 \vee (S_2 \cdot \sim S_2) \vdash_C S_1$
C7. $S_1 \cdot S_2 \vdash_C S_1$
C8. $S_1, S_2 \vdash_C S_1 \cdot S_2$
C9. $S_1 \vdash_C S_1 \vee S_2$
C10. $\sim(S_1 \cdot S_2) \underset{C}{\Leftrightarrow} \sim S_1 \vee \sim S_2$
C11. $S_1 \vee S_2 \underset{C}{\Leftrightarrow} S_2 \vee S_1$
C12. $S_1 \vee (S_2 \vee S_3) \underset{C}{\Leftrightarrow} (S_1 \vee S_2) \vee S_3$
C13. $S_1 \cdot (S_2 \vee S_3) \underset{C}{\Leftrightarrow} (S_1 \cdot S_2) \vee (S_1 \cdot S_3)$

C14. $S \underset{C}{\Leftrightarrow} \sim\sim S$
C16. $S_1 \supset S_2 \underset{C}{\Leftrightarrow} \sim S_1 \vee S_2$
C17. $S_1 \equiv S_2 \underset{C}{\Leftrightarrow} (S_1 \supset S_2) \cdot (S_2 \supset S_1)$
C19. $S \underset{C}{\Leftrightarrow} S \vee S$

5 *Extensions of* **C**. A system of natural deduction for classical sentential calculus may be said to be complete if it is not incomplete, that is, if S_2 is derivable from S_1 by the rules of the system if $S_1 \supset S_2$ is a tautology. It is known that there are various ways of completing **C**, for example, by adding the rules of Indirect Proof or Conditional Proof (the latter without restrictions imposed by certain directions for writing down proofs, such as occur in *Symbolic Logic*, first edition, pp. 52 ff.), or by the substitution of the rule of Absorption for Destructive Dilemma (as in Copi, *Introduction to Logic*, second edition, p. 277). In fact, the addition to **C** of *any* rule independent of the rules of **C** will result in a system that is complete. This fact can be shown by first showing that a certain addition yields a complete system.

Definition 3. A system of natural deduction Γ' for classical sentential calculus is a formal extension of Γ if and only if each of the following conditions holds:

1. Γ and Γ' contain the same sentences;
2. every rule of Γ is a rule of Γ';
3. there are sentences S_1 and S_2 such that S_2 is derivable from S_1 in Γ' but not in Γ;
4. for all S_1 and S_2, if all occurrences of an atomic sentence S in S_1 and S_2 are replaced by occurrences of a sentence S^* to obtain S_1^* and S_2^*, then if S_2 is derivable from S_1 in Γ' ($S_1 \vdash_{\Gamma'} S_2$) then S_2^* is derivable from S_1^* in Γ' ($S_1^* \vdash_{\Gamma'} S_2^*$).

Lemma 5. *Let* **C'** *be a formal extension of* **C** *obtained by adding to* **C** *the rule that* $\sim B \vee B$ *is derivable from* A *in* **C'** (*where* A *and* B *are a given pair of atomic sentences of* **C**, **C'**). *Then* $S_1 \vdash_{C'} S_2$ *if and only if* $S_1 \supset S_2$ *is a tautology of the two-valued sentential calculus.*

Proof: From our knowledge of sentential calculus, a system of natural deduction for the two-valued sentential calculus contains all the rules of **C'**, but S_2 is derivable from S_1 in a system of natural deduction for two-valued sentential calculus only if $S_1 \supset S_2$ is a tautology. Hence S_2 is derivable from S_1 by the rules of **C'** only if $S_1 \supset S_2$ is a tautology. Now consider an arbitrary pair of sentences S_1, S_2 such that $S_1 \supset S_2$ is a tautology of the classical two-valued sentential calculus. Let A_i be one of the n atomic sentences that occur in either S_1 or S_2. Then by hypothesis and Definition 3.4, $S_1 \vdash_{C'} (\sim A_i \vee A_i)$. By Lemma 1.5, $S_1 \vdash_{C'} \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)$. By Lemma 1.1, 5, $S_1 \vdash_{C'} S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)$ (since **C'** contains the rules of **C**). Consider now any mapping Θ with domain determined by the atomic sentences in S_1 and S_2 and range a subset of K. If for some A_i, $\Theta(A_i) = 1$, then $\Theta(\sim A_i \vee A_i) = 1$ and $\Theta\left[\bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] \neq 0$, by Lemma 1.3, and $\Theta\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] \neq 0$. Hence for all Θ, if $\Theta\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] = 0$ then $\Theta(A_i) \neq 1$, for all A_i. But the atomic sentences in S_2 are among the A_i. For any Θ, if $\Theta(A_i) \neq 1$ for all A_i, then $\Theta(S_2) \neq 1$, by Lemma 1.9. Hence for all Θ, if $\Theta\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] = 0$ then $\Theta(S_2) = 0$ or $\Theta(S_2) = 2$. If, however, there is a Θ such that $\Theta\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] = 0$ but $\Theta(S_2) = 2$, then $\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] \supset S_2$ is not a tautology, by Lemma 2.15. From sentential calculus, $\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] \supset S_2$ is a tautology if and only if $S_1 \supset S_2$ is a tautology. $S_1 \supset S_2$ is a tautology by assumption. Hence for all Θ, $\Theta\left[S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)\right] = 0$ only if $\Theta(S_2) = 0$. That is, $S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i) \therefore S_2$ is C-valid. By Theorem 2, $S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i) \vdash_{\mathsf{C}} S_2$. Since C' contains the rules of C, $S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i) \vdash_{\mathsf{C}'} S_2$. But we have proved that $S_1 \vdash_{\mathsf{C}'} S_1 \cdot \bigwedge_{i=1}^{n} (\sim A_i \vee A_i)$. Hence, $S_1 \vdash_{\mathsf{C}'} S_2$. (We assume here what it is clear can be proved from a definition of derivability in C', namely that $\vdash_{\mathsf{C}'}$ is transitive.)

The foregoing proof uses properties of the matrix $\mathfrak{M}$. Completeness of the system C' described above may be proved without reference to $\mathfrak{M}$. In view of Copi's results, it will suffice to show that Conditional Proof holds in C', and to do this, it will suffice to show how to construct a proof in C' of a conclusion $S_{n+1} \supset T$ from premisses $S_1, S_2, \ldots, S_n$, given a proof in C' of the conclusion T from premisses $S_1, S_2, \ldots, S_n, S_{n+1}$. From premisses $S_1, S_2, \ldots, S_n$ we derive by C9 (and C11) $\sim S_{n+1} \vee S_1$, $\sim S_{n+1} \vee S_2, \ldots, \sim S_{n+1} \vee S_n$. From any premiss, we derive in C', $\sim S_{n+1} \vee S_{n+1}$. We now continue the proof by adding the lines of the proof of $\sim S_{n+1} \vee T$ from $\sim S_{n+1} \vee S_1$, $\sim S_{n+1} \vee S_2, \ldots, \sim S_{n+1} \vee S_n$, $\sim S_{n+1} \vee S_{n+1}$. Lemma 1.3 shows how to obtain that proof, by the rules of C, from the proof of T from $S_1, S_2, \ldots, S_n, S_{n+1}$. C16 yields $S_{n+1} \supset T$.

Any formal extension C# of C is complete. This fact, proved below, suggests the description of C as *a* "next-strongest" system of natural deduction (not, however, as *the* "next-strongest").[2]

There are two cases to consider, according as the extension C# is or is not "consistent".

Definition 4. A system of natural deduction for sentential calculus is consistent if and only if S_2 is derivable from S_1 by the rules of the system only if $S_1 \supset S_2$ is a tautology of the two-valued sentential calculus.

Theorem 5. *Every formal extension* C# *of* C *is complete.*

Proof: By definition, there exist S_1, S_2 in C and C# such that S_2 is derivable from S_1 by the rules of C# but not by the rules of C. In virtue of Lemma 1.7 and Lemma 3.6, we may confine our attention to S_1' and S_2''. In virtue of Theorem 2, $S_1' \therefore S_2''$ is not C-valid.

(i) Consider first the case in which for all S_1', S_2'' such that S_2'' is derivable from S_1' in C# but not in C there is no Θ (with range a subset of K) such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$. By Lemma 2.15, $S_1' \supset S_2''$ is a tautology of the two-valued sentential calculus. Hence C# is consistent. To see that it is complete, consider S_1', S_2'' such that S_2'' is derivable from S_1' in C# but not in C. Then $S_1' \therefore S_2''$ is not C-valid, and hence there is some Θ such that $\Theta(S_1') = 0$ but $\Theta(S_2'') \neq 0$. Since for all such Θ, $\Theta(S_2'') \neq 2$, by Lemma 2.17 there exist $\bigwedge\alpha_i$ in S_1' and $\bigvee\beta_u$ in S_2'' such that $\bigvee\beta_u$ contains a complementary pair that contains no atomic sentence in common with $\bigwedge\alpha_i$, and such that any atomic sentence common to $\bigwedge\alpha_i$ and $\bigvee\beta_u$ occurs only in a complementary pair α $\left(\text{in } \bigwedge\alpha_i\right)$ and β $\left(\text{in } \bigvee\beta_u\right)$. Since C# contains the rules of C, $\bigwedge\alpha_i \vdash_{C\#} S_1'$ (by C9) and $S_2'' \vdash_{C\#} \bigvee\beta_u$ (by C7). By hypothesis, $S_1' \vdash_{C\#} S_2''$; hence $\bigwedge\alpha_i \vdash_{C\#} \bigvee\beta_u$. Since there is a Θ such that $\Theta\left(\bigwedge\alpha_i\right) = 0$, $\bigwedge\alpha_i$ does not contain a complementary pair, by Lemma 2.10. Now replace the atomic sentences in $\bigwedge\alpha_i$ and $\bigvee\beta_u$ in the following way. If α $\left(\text{in } \bigwedge\alpha_i\right)$ is an atomic sentence, replace it (and any occurrence of it in $\bigvee\beta_u$) by an occurrence of the atomic sentence A. Note that any β $\left(\text{in } \bigvee\beta_u\right)$ affected becomes $\sim A$. If, however, α is the negation of an atomic sentence, replace the atomic sentence in α by an occurrence of $\sim A$. Note that, again, any β affected becomes $\sim A$. Thus every α in $\bigwedge\alpha_i$ has given way to A or to $\sim\sim A$; so far, every β in $\bigvee\beta_u$ that has been affected has given way to $\sim A$. There must be some β in $\bigvee\beta_u$ not yet affected, since $\bigvee\beta_u$ contains a complementary pair distinct in its atomic sentences from the atomic sentences of $\bigwedge\alpha_i$. Replace the atomic sentences in these remaining β's by occurrences of the atomic sentence B. Then at least one β gives way to B, at least one gives way to $\sim B$, and the remainder, if any, give way to B or to $\sim B$. We have by these substitutions obtained $\left(\bigwedge\alpha_i\right)^*$, $\left(\bigvee\beta_u\right)^*$ such that each $(\alpha)^*$ in $\left(\bigwedge\alpha_i\right)^*$ is A or $\sim\sim A$ and each $(\beta)^*$ in $\left(\bigvee\beta_u\right)^*$ is either $\sim A$ or $\sim B$ or B, and where some $(\beta)^*$ is B and some $(\beta)^*$ is $\sim B$. Since $\bigwedge\alpha_i \vdash_{C\#} \bigvee\beta_u$, by Definition 3.4 $\left(\bigwedge\alpha_i\right)^* \vdash_{C\#} \left(\bigvee\beta_u\right)^*$. By C14 (if needed) and C19 (if needed) $A \vdash_{C\#} \left(\bigwedge\alpha_i\right)^*$. Similarly, either $\left(\bigvee\beta_u\right)^* \vdash_{C\#} \sim B \vee B$ or $\left(\bigvee\beta_u\right)^* \vdash_{C\#} \sim A \vee (\sim B \vee B)$. Hence $A \vdash_{C\#} \sim B \vee B$ or $A \vdash_{C\#} \sim A \vee (\sim B \vee B)$. The latter of these (in virtue of C15 and C1) yields $A \vdash_{C\#} \sim B \vee B$. Hence C# contains the rule distinguishing the C′ described in Lemma 5. Hence C# is complete. This concludes the proof of completeness for C# where C# is consistent.

(ii) Suppose on the other hand that there are S_1', S_2'' such that S_2'' is derivable from S_1' in **C**# but not in **C**, and such that for some Θ, $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$. By Lemma 2.4, 5, there exist $\bigwedge\alpha_i$ in S_1' and $\bigvee\beta_u$ in S_2'' such that $\Theta\left(\bigwedge\alpha_i\right) = 0$ and $\Theta\left(\bigvee\beta_u\right) = 2$. By Lemma 2.12, $\bigwedge\alpha_i$ and $\bigvee\beta_u$ contain an atomic sentence in common only if the pair in which it occurs is complementary. Further, by Lemma 2.11, $\bigvee\beta_u$ does not contain a complementary pair. We have $\bigwedge\alpha_i \vdash_{C\#} \bigvee\beta_u$ by reasoning like that in case (i). Now replace every α in $\bigwedge\alpha_i$ that does not contain a curl by A. If any β in $\bigvee\beta_u$ is affected it gives way to $\sim A$. Replace every atomic sentence in every α in $\bigwedge\alpha_i$ that contains a curl by $\sim A$. If any β in $\bigvee\beta_u$ is affected, it gives way to $\sim A$. For any β in $\bigvee\beta_u$ so far unaffected, replace its atomic sentences by $\sim A$ or A according as β does not contain or does contain a curl. The result of this replacement is $\left(\bigwedge\alpha_i\right)^*$, $\left(\bigvee\beta_u\right)^*$, where each $(\alpha)^*$ in $\left(\bigwedge\alpha_i\right)^*$ is A or $\sim\sim A$ and each $(\beta)^*$ in $\left(\bigvee\beta_u\right)^*$ is $\sim A$. By an argument like that in case (i), $A \vdash_{C\#} \sim A$. Since **C**# is a formal extension of **C**, we have in general $S \vdash_{C\#} \sim S$ for any S. Let S_3, S_4 be sentences such that $S_3 \supset S_4$ is not a tautology. We have then $S_3 \vdash_{C\#} \sim S_3$, $\sim S_3 \vdash_{C\#} \sim S_3 \vee S_4$ (**C**9), $\sim S_3 \vee S_4 \vdash_{C\#} S_3 \supset S_4$ (**C**16), S_3, $S_3 \supset S_4 \vdash_{C\#} S_4$ (**C**1). Hence $S_3 \vdash_{C\#} S_4$, so that **C**# is not consistent.

6 *A sufficient condition for derivability in* **C**. A certain sufficient condition for derivability in **C** is perhaps of philosophical interest.

Theorem 6. *If no* $\bigvee\beta_u$ *in* S_2'' *contains a complementary pair, then if* $S_1 \supset S_2$ *is a tautology of the sentential calculus, then* $S_1 \vdash_C S_2$.

Proof: Since the sentential calculus contains the rules of **C**, $S_1 \supset S_2$ is a tautology if and only if $S_1' \supset S_2''$ is a tautology. Suppose then that $S_1 \supset S_2$ is a tautology. By Lemma 2.15, there is no Θ with range a subset of K such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 2$. Suppose further that no $\bigvee\beta_u$ in S_2'' contains a complementary pair. By Lemma 2.16, there is no Θ with range a subset of K such that $\Theta(S_1') = 0$ and $\Theta(S_2'') = 1$. Hence, for every Θ, if $\Theta(S_1') = 0$ then $\Theta(S_2'') = 0$; that is, $S_1' \therefore S_2''$ is **C**-valid. By Theorem 2, $S_1' \vdash_C S_2''$. Then by Lemma 1.6, $S_1 \vdash_C S_2$.

7 *Conclusion.* The foregoing results may be of philosophical interest. It is conceivable that a philosopher, chary of certainties, may want a canon of inference which does not allow the demonstration of any sentence as "true no matter what". At the same time, he may wish to be able to infer conclusions from premisses if the premisses tautologically imply the conclusions, so long as the conclusions are not themselves "certain" but are at best "contingent". **C** satisfies the first of these desires and goes some way toward satisfying the second. It does not go all the way, for it is not the case that for all "contingent" S_1 and S_2, S_2 is derivable from S_1 by the rules of C if $S_1 \supset S_2$ is a tautology. But if S_2 is such that there is by the rules of C an S_2'' which contains no "tautologous" conjunct, then S_2 is derivable from S_1 by the rules of C. Now, if S_2 is contingent, there is some sentence that is tautologically equivalent to S_2, which is in conjunctive normal form, and which contains no tautologous conjunct. Our philosopher, chary of certainties, may in some sense "say the same thing" as the man who says S_2. Though he will not be able in all cases to use the rules of C to show that he is "saying the same thing", he knows that the man who accepts classical sentential calculus will admit the sameness of what is said, and he himself, restricted to the rules of C, may yet at the outset fix upon the tautological equivalent as a satisfactory way of describing the same state of affairs as is described by S_2.

It must be conceded that someone using C is hobbled when he wants to take what seem very natural and permissible steps. He may not appeal to the rules of C, for example, to explicate the set of possible outcomes described by "either it is raining or it is cold" as the set described by "either it is both raining and cold or it is raining but not cold or it is cold but not raining". Such a limitation would seem to disqualify C as a handy canon for reasoning about probabilities. Still, if he is willing to add to his contingent premisses the following instances of the law of excluded middle:

either it is raining or it is not raining,
either it is cold or it is not cold,

he will be able to make the replacement. (The proof of Lemma 5 can be developed to yield this conclusion.)

A word ought to be said about the matrix $\mathfrak{M}$, characteristic for the set of rules C. The attention of this paper has been throughout on syntactic matters, and the matrix has served as a means for proving conclusions about syntax. Yet a comparison with other three-valued logics and consideration of the isomorphism of $\mathfrak{M}'$ with the usual truth tables must suggest that K could be regarded as a set of three "truth values", with 0 as the true, 2 as the false, and 1 as perhaps the doubtful, or the doubtfully significant. If we speak of a three-valued logic here, it is with this difference from some other many-valued logics: the permissible ways of assigning truth values to sentences has the result that not only is the "law of excluded middle" denied to be a "law of logic", but so also is every other candidate for the title.

It is therefore pertinent to observe that this logic might be expected to be of interest to whomever wants to avoid confrontation with what are supposed to be paradoxes entailed by acceptance of the law of excluded middle. Comparisons with intuitionist logic are therefore in order, and we note that in this logic, a proof of $\sim\sim S$ yields a proof of S.

One matter remains uninvestigated in this paper, the question of the functional completeness of C. It is clear from Lemma 2.9 that C is not functionally complete with respect to functions from Cartesian products of K to K. No sentence of C can express the always -0, the always -1, the always -2 functions.

One may hope to increase the expressive powers of **C** by the addition of new connectives, and indeed I believe that the addition of a single singulary connective to the language of **C** will result in a language functionally complete in the sense of the preceding paragraph. Perhaps certain notions behaving somewhat like the familiar modal operators may then be defined. But I suspect that if **C** has any philosophical importance, it will turn out that it can be made part of a first order logic with identity, a logic having characteristics analogous to those of **C**. It is perhaps not hard to imagine ways of extending **C** to a reasonably powerful logic of quantification. As to identity, we would not want to be able to prove any identities from arbitrary premisses. Systems of natural deduction for first-order logic with identity often include as one of two rules governing identity the permission to write down any formulas of the form $x = x$. We should want to deny such permission, but, in the spirit of **C**20, we might grant permission to write down S, given that a sentence of the form $S \vee x \neq x$ has already been written. Some experiment suggests to me that in place of the other usual rule, which (under certain conditions) allows "replacement of equals by equals" we should want rules allowing us to infer such sentences as those of the form $x = y \supset S(y)$ from $S(x)$ and sentences of the form $S(x) \supset S(y)$ from $x = y$. It appears that we should need these stronger rules because an unrestricted rule of conditional proof would not be available in the system.

NOTES

1. **C**17' is in fact independent of the remaining primitive replacement rules of **C**. (Added August, 1973).
2. There is another next strongest system differing in a very few rules. A decision procedure for it can be obtained from the Bocvar table for disjunction. (Added August, 1973).

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