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THE COMPLETENESS OF S1 AND SOME RELATED SYSTEMS

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The system S1, although dating back to Lewis and Langford in 1932 [12] has proved singularly recalcitrant to the algebraic and semantic techniques applied so successfully to other modal logics. In this paper* we define S1-algebras (section **2**), use them to prove the finite model property for S1 (section **3**), introduce a semantical definition of S1-validity (section **4**) and make a few remarks about various other systems which seem amenable to the S1 treatment (section **5**).

1 *The system* S1. We use the basis for S1 given by Lemmon in [9, p. 178]. Lemmon takes $\sim$, $\supset$, and L as primitive with the definitions[1]:

Def $\prec$: $(\alpha \prec \beta) =_{df} L(\alpha \supset \beta)$
Def =: $(\alpha = \beta) =_{df} ((\alpha \prec \beta) \,.\, (\beta \prec \alpha))$
Def M: $M\alpha =_{df} \sim L \sim \alpha$

The axioms are:

1.1 $Lp \supset p$
1.2 $(L(p \supset q) \,.\, L(q \supset r)) \supset L(p \supset r)$

and the rules:

1.3 If α is a **PC**-tautology or an axiom then $L\alpha$ is a theorem.
1.4 Uniform substitution for propositional variables.
1.5 Modus Ponens: $\vdash \alpha$, $\vdash \alpha \supset \beta \rightarrow \vdash \beta$
1.6 Substitution of proved strict equivalents.

In view of 1.3 and 1.6 the choice of primitives is immaterial. The following strict equivalences will frequently be tacitly assumed in what follows:

*This paper was written in 1969 before the publication of A. Shukla's work on S1 in [15]. A comparison between his algebras and ours is instructive. I am indebted to Mr. K. E. Pledger of the Victoria University of Wellington Mathematics Department for drawing my attention to some errors in an earlier draft of this paper.

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1.7 $Lp = \sim M \sim p$
1.8 $(p \prec q) = \sim M(p \,.\, \sim q)$

2 *S1-algebras.*[2] A quadruple $\langle K \times - *\rangle$ is an *S1-algebra* iff $\langle K \times -\rangle$ is a Boolean algebra, K is closed with respect to $*$ and for any $a, b \in K$ the following conditions hold:

2.1 $a \subset *a$
2.2 $*a + *b + *0 = *(a + b) + *0$
2.3 $*0 \subset *a + *b$ *provided* $a \times b = 0$
2.4 *If* $*a \subset *0$ *then* $a = 0$

In the presence of $*0 = 0$ (2.5) or even $*0 \subset *a$ (2.6) 2.3 becomes redundant and 2.2 reduces to

2.7 $*a + *b = *(a + b)$

Conversely given 2.7, 2.6 can be proved.

Lemma 1. *In any* S1-*algebra if* $a \subset b$ *then* $*a \subset (*b + *0)$

This is an easy consequence of 2.2

Theorem 1. *Conditions* 2.2 *and* 2.3 *in an* S1-*algebra can be replaced by the following condition* (*for any* $a, b, c \in K$):

2.8 $*(a \times c) \subset *(a \times -b) + *(b \times c)$

We first prove 2.8 from 2.2 and 2.3

	(1)	$(a \times c) \subset (a \times -b) + (b \times c)$
(1) lemma 1	(2)	$*(a \times c) \subset *((a \times -b) + (b \times c)) + *0$
(2) 2.2	(3)	$*(a \times c) \subset *(a \times -b) + *(b \times c) + *0$
2.3	(4)	$*0 \subset *(a \times -b) + *(b \times c)$
(3)(4)	(5)	$*(a \times c) \subset *(a \times -b) + *(b \times c)$

Given 2.8 we may prove 2.2 and 2.3

2.8 $0/b$, b/c	(1)	$*(a \times b) \subset *a + *0$
(1) $a + b/a$, a/b	(2)	$*a \subset *(a + b) + *0$
similarly	(3)	$*b \subset *(a + b) + *0$
(2)(3)	(4)	$*a + *b + *0 \subset *(a + b) + *0$
2.8 $a + b/c$, $a + b/a$	(5)	$*(a + b) \subset *((a + b) \times -b) + *(b \times (a + b))$
(5)	(6)	$*(a + b) \subset *(a \times -b) + *b$
2.8 $-b/c$, $0/b$	(7)	$*(a \times -b) \subset *a + *0$
(6)(7)	(8)	$*(a + b) \subset *a + *b + *0$
(4)(8)	(9)	$*a + *b + *0 = *(a + b) + *0$

Thus 2.2. From 2.8 with $-a/c$ we have $*0 \subset *(a \times -b) + *(b \times -a)$ and if $a \times b = 0$ then $a \times -b = a$ and $b \times -a = b$ and so $*0 \subset *a + *b$. QED

Where V is a function from wff of S1 to the members of K in an S1-algebra $\langle K \times - *\rangle$ we shall say that V is a *modal assignment* iff:

2.9 $\vee(\sim\alpha) = -\vee(\alpha)$
2.10 $\vee(\alpha \supset \beta) = -\vee(\alpha) + \vee(\beta)$
2.11 $\vee(M\alpha) = *\vee(\alpha)$

We define a subset D of K as follows:

2.12 $D = \{a \epsilon K\colon\ -*0 \subset a\}$

Clearly D is a filter (additive ideal) in K. We shall say that a wff α of S1 is *S1-valid* iff for every modal assignment $\vee$ to the propositional variables of α in every S1-algebra $\vee(\alpha) \epsilon D$. The following are all easy consequences of 2.9-2.12.

2.13 $\vee(\alpha) \subset \vee(\beta)$ *iff* $\vee(\alpha \supset \beta) = 1$
2.14 $\vee(L\alpha) \epsilon$ D *iff* $\vee(\alpha) = 1$ (2.4 is required here)
2.15 $\vee(\alpha \prec \beta) \epsilon$ D *iff* $\vee(\alpha) \subset \vee(\beta)$
2.16 $\vee(\alpha = \beta) \epsilon$ D *iff* $\vee(\alpha) = \vee(\beta)$

Theorem 2. *If α is a theorem of* S1 *then α is* S1-*valid.*

The proof is by induction on the proof of α in S1. We first show that if α is an axiom then $\vee(\alpha) = 1$.

1.1 By 2.1 we have $a \subset *a$ for every $a \epsilon K$, whence $- * - a \subset a$ whence $\vee(Lp) \subset \vee(p)$ and so by 2.13 $\vee(Lp \supset p) = 1$.
1.2 From 2.8 we have for any $a, b, c \epsilon K$

$$-(*(a \times -b) + *(b \times c)) \subset -*(a \times c)$$

whence with $-c/c$

$$(-*-(-a + b) \times -*-(-b + c)) \subset -*-(-a + c)$$

i.e. where $\vee$ is any modal assignment (by 2.9-2.11)

$$\vee(L(p \supset q)\ .\ L(q \supset r)) \subset \vee(L(p \supset r))$$

whence by 2.13

$$\vee((L(p \supset q)\ .\ L(q \supset r)) \supset L(p \supset r)) = 1$$

1.3 If α is an axiom then $\vee(\alpha) = 1$ and if α is a PC-tautology then $\vee(\alpha) = 1$ for any modal assignment, whence by 2.14 $\vee(L\alpha) \epsilon D$. 1.4 is clearly satisfied and since D is a filter so is 1.5.
1.6 If $\vee(\alpha = \beta) \epsilon D$ then by 2.16 $\vee(\alpha) = \vee(\beta)$.

So every S1 theorem is S1-valid QED.

Theorem 3. *There is a characteristic* S1-*algebra.*

The characteristic algebra is of course the Lindenbaum algebra taken with respect to the relation of strict equivalence, i.e. $|\alpha|$ is the class of all β such that $\vdash_{S1} \alpha = \beta$.

2.17 *K is the set of all such equivalence classes.*

2.18 $-|\alpha|$ *is* $|\sim\alpha|$
2.19 $|\alpha|\times|\beta|$ *is* $|\alpha\ .\ \beta|$
2.20 $*|\alpha|$ *is* $|M\alpha|$

By 1.6 this equivalence relation will generate a congruence. From 1.3 and 1.6 $\langle K\times -\rangle$ is a Boolean algebra.

Lemma 2. (i) *If* $\vdash_{S1} L\alpha$ *then* $|\alpha| = 1$
(ii) *If* $\vdash_{S1} L(\alpha \supset \beta)$ *then* $|\alpha| \subset |\beta|$
(iii) *If* $-*0 \subset |\alpha|$ *then* $\vdash_{S1} \alpha$

Proof: (i) $\vdash_{S1} L\alpha \supset (\alpha = (p \supset p))$ whence if $\vdash_{S1} L\alpha$ then $\vdash_{S1} \alpha = (p \supset p)$ whence $|\alpha| = |p \supset p| = 1$.
(ii) If $\vdash_{S1} L(\alpha \supset \beta)$ then by (i) $|\alpha \supset \beta| = 1$ whence $|\alpha| \subset |\beta|$.
(iii) If $-*0 \subset |\alpha|$ then $*0 + |\alpha| = 1$, i.e., $\vdash_{S1} (M(p\ .\ \sim p) \vee \alpha) = (p \supset p)$ whence $\vdash_{S1} M(p\ .\ \sim p) \vee \alpha$, but $\vdash_{S1} \sim M(p\ .\ \sim p)$ whence $\vdash_{S1} \alpha$.

We show that $\langle K\times - *\rangle$ is an S1-algebra. From 1.1 and 1.3 we may prove $L(p \supset Mp)$ whence by lemma 2(ii) $|\alpha| \subset |M\alpha|$ for any wff α, i.e. $a \subset *a$ and 2.1 holds. 2.2 and 2.3 follow from 2.8 which follows by lemma 2(ii) from $L(M(p\ .\ r) \supset (M(p\ .\ \sim q) \vee M(q\ .\ r))$ (a consequence of 1.2 and 1.3). For 2.4 suppose $*|\alpha| \subset *0$, then $-*0 \subset |\sim M\alpha|$. Whence by lemma 2(iii) $\vdash_{S1} \sim M\alpha$ whence since $\vdash_{S1} \sim Mp \supset (p = (q\ .\ \sim q))$ $\vdash_{S1} \alpha = (q\ .\ \sim q)$, i.e. $|\alpha| = |q\ .\ \sim q| = 0$. Thus $\langle K\times - *\rangle$ is an S1-algebra.

Further $\langle K\times - *\rangle$ is characteristic for S1. For suppose $\dashv_{S1} \alpha$. Then where $\beta_1, \ldots, \beta_n$ are the n wf parts of α let $V(\beta_i) = |\beta_i|$ $(1 \leq i \leq n)$. From 2.18-2.20 it can be seen that 2.9-2.11 are satisfied and that V is a modal assignment and $V(\alpha) = |\alpha|$. Now if $-*0 \subseteq |\alpha|$ then by lemma 2(iii) $\vdash_{S1} \alpha$ contrary to hypothesis. So $-*0 \not\subset |\alpha|$ i.e. $V(\alpha) \notin D$, i.e. α is falsified by $\langle K\times - *\rangle$ QED.

3 *The finite model property.*

Theorem 4. *If* $\langle K\times - *\rangle$ *is an* S1*-algebra and* $\{a_1, \ldots, a_n\}$ *is a finite subset of* K *then there is an* S1*-algebra* $\langle K'\times' -' *'\rangle$ *with* $\overline{\overline{K'}} \leq 2^{2^{n+1}}$ *such that*:

(i) *If* $a, b \in K'$ *then* $-a, a\times b \in K'$ *and* $-'a = -a$ *and* $a \times' b = a\times b$.
(ii) *If* $a, *a \in K'$ *then* $*'a = *a$
(iii) $a_1, \ldots, a_n \in K'$

The proof is an adaptation of McKinsey's proof in [13] for S2 but is a little more complicated because of the extra complexity of S1-algebras. $\langle K'\times' -'\rangle$ is the Boolean subalgebra of $\langle K\times -\rangle$ generated by $\{a_1, \ldots, a_n, *0\}$. Clearly K' will have no more than $2^{2^{n+1}}$ elements. Further (i) and (iii) will be satisfied. We shall show how to define $*'$ so as to satisfy (ii) and so that the result will be an S1-algebra.

We shall say for $a \in K'$, $c \in K$ that c *covers* a iff $a \subset c$ and $(*c + *0) \in K'$. Now there may be an infinite number of c's which cover a but since K' is finite there will be a finite number of them say $c_1, \ldots, c_m$ such that for each c covering a there is some c_i $(1 \leq i \leq m)$ such that $(*c + *0) =$

$(*c_i + *0)$. We shall say that $c_1, \ldots, c_m$ form a finite covering of a. Now either $*a \in K'$ or not. If $*a \in K'$ then let $*'a = *a$. If $*a \notin K'$ then let $*'a = (*c_1 + *0) \times \ldots \times (*c_m + *0)$. Clearly $*'a \in K'$ and further (ii) is satisfied. We have to show that $\langle K' \times' -' *' \rangle$ is an S1-algebra. Since $\langle K' \times' -' \rangle$ is a Boolean algebra and K' is closed with respect to $*'$ we need only show that 2.1-2.4 are satisfied. We note first that $*'0 = *0$ (since $*0 \in K'$).

2.1 First suppose $*a \in K'$. Then $*'a = (*c_1 + *0) \times \ldots \times (*c_m + *0)$ where $c_1, \ldots, c_m$ form a finite covering of a. Now for each c_i $(1 \leq i \leq m)$ $a \subset c_i$, whence by 2.1 (for $*$) $a \subset *c_i$ whence $a \subset *c_i + *0$ whence $a \subset *'a$. And if $*'a \in K'$ $*'a = *a$.

2.2 $(*'a + *'b + *0) = (*'(a + b) + *0)$

First consider the case where none of $*a$, $*b$, $*(a + b) \in K'$. Let $c_1, \ldots, c_m$ form a finite covering of a, $d_1, \ldots, d_k$ of b and $e_1, \ldots, e_h$ of $a + b$. We have to show that,

3.1 $$((*c_1 + *0) \times \ldots \times (*c_m + *0)) + ((*d_1 + *0) \times \ldots \times (*d_k + *0)) + *0 = ((*e_1 + *0) \times \ldots \times (*e_h + *0)) + *0$$

Now each e_i $(1 \leq i \leq h)$ covers $a + b$, whence it covers both a and b and so $(*e_i + *0) = (*c_j + *0)$ for some c_j $(1 \leq j \leq m)$ and $(*e_i + *0) = (*d_p + *0)$ for some d_p $(1 \leq p \leq k)$. Whence $(*c_1 + *0) \times \ldots \times (*c_m + *0) \subset (*e_1 + *0) \times \ldots \times (*e_h + *0)$ and $(*d_1 + *0) \times \ldots \times (*d_k + *0) \subset (*e_1 + *0) \times \ldots \times (*e_h + *0)$ whence $*'a + *'b + *0 \subset *'(a + b) + *0$. Now consider any c_i, d_j $(1 \leq i \leq m,$ $1 \leq j \leq k)$. Since c_i covers a and d_j covers b then $(a + b) \subset (c_i + d_j)$ and $*c_i + *0$, $*d_j + *0 \in K'$ thus by 2.2 $*(c_i + d_j) + *0 \in K'$ and so $(c_i + d_j)$ covers $(a + b)$ and so for some e_p $(1 \leq p \leq h)$ $*(c_i + d_j) + *0 = *e_p + *0$ whence by 2.2 $*c_i + *d_j + *0 = *e_p + *0$ whence $(*e_1 + *0) \times \ldots \times (*e_h + *0) \subset (*c_1 + *d_1 + *0) \times \ldots \times (*c_m + *d_k + *0)$ whence $(*e_1 + *0) \times \ldots \times (*e_h + *0) \subset ((*c_1 + *0) \times \ldots \times (*c_m + *0)) + ((*d_1 + *0) \times \ldots \times (*d_k + *0))$ i.e. $*'(a + b) + *0 \subset *'a + *'b + *0$.

We now consider the cases where some of $*a$, $*b$, $*(a + b) \in K'$. Each of these has one of the following forms:

3.2 $$*a + ((*d_1 + *0) \times \ldots \times (*d_k + *0)) + *0 = ((*e_1 + *0) \times \ldots \times (*e_h + *0)) + *0$$

3.3 $$((*c_1 + *0) \times \ldots \times (*c_m + *0)) + *b + *0 = ((*e_1 + *0) \times \ldots \times (*e_h + *0)) + *0$$

3.4 $$((*c_1 + *0) \times \ldots \times (*c_m + *0)) + ((*d_1 + *0) \times \ldots \times (*d_k + *0)) = *(a + b) + *0$$

3.5 $$*a + ((*d + *0) \times \ldots \times (*d + *0)) + *0 = *(a + b) + *0$$

3.6 $$((*c_1 + *0) \times \ldots \times (*c_m + *0)) + *b + *0 = *(a + b) + *0$$

3.7 $$*a + *b + *0 = ((*e_1 + *0) \times \ldots \times (*e_h + *0)) + *0$$

3.8 $$*a + *b + *0 = *(a + b) + *0$$

3.3 and 3.6 reduce to 3.2 and 3.5 respectively and 3.8 is trivial. For 3.2 we have as above (3.1) that each $*e_i + *0$ is one of the $(*d_j + *0)$'s. Since each e_i covers $a + b$ then $a \subset e_i$, whence by lemma 1 $*a \subset *e_i + *0$ for each e_i, so $*'a + *'b \subset *'(a + b) + *0$. For the converse $a + b \subset a + d_j$ for each d_j, whence since $*(a + d_j) + *0$ $(= *a + *d_j + *0) \in K'$ $a + d_j$ covers $a + b$ and hence $*(a + d_j) + *0$ is one of the $(*e + *0)$'s. For 3.4 $a + b$ covers both a and b and hence $*(a + b) + *0$ is among the $(*c + *0)$'s and among the $(*d + *0)$'s. Thus $((*c_1 + *0) \times \ldots \times (*c_m + *0)) + ((*d_1 + *0) \times \ldots \times (*d_k + *0)) + *0 \subset *(a + b) + *0$. For the converse each c_i covers a and each d_j covers b so $(a + b) \subset (c_i + d_j)$ whence $*(a + b) \subset *(c_i + d_j) + *0$, whence by 2.2 $*(a + b) \subset *c_i + *d_j + *0$ for each c_i, d_j. For 3.5 $a + b$ covers b and so $*(a + b) + *0$ is among the $(*d + *0)$'s and by 2.2 $*a \subset *(a + b) + *0$ whence $*a + ((*d_1 + *0) \times \ldots \times (*d_k + *0)) \subset *(a + b) + *0$. For the converse each $a + d_j$ covers $a + b$ and so $*(a + b) \subset (*a + *d_j + *0)$ for each d_j. For 3.7 since $*a \in K'$, $*b \in K'$ then by 2.2 $*(a + b) + *0 \in K'$ and $a + b$ covers itself and so is one of the e's. Thus by 2.2 $((*e_1 + *0) \times \ldots \times (*e_h + *0)) \subset *a + *b + *0$. Further since each e_i covers $a + b$ it therefore covers both a and b and hence $*a + *b \subset *e_i + *0$ for every e_i. Whence $*a + *b + *0 \subset ((*e_1 + *0) \times \ldots \times (*e_h + *0))$. Thus $\langle K' \times' -' *' \rangle$ satisfies 2.2.

2.3 First suppose that one of $*a$, $*b \notin K'$ say $*a$, then $*'a = ((*c_1 + *0) \times \ldots \times (*c_m + *0))$. In this case clearly $*0 \subset *'a$. And if $*b \notin K'$ then $*0 \subset *'b$. If $*a$, $*b \in K'$ then $*'a = *a$ and $*'b = *b$. In all three cases 2.3 is satisfied.

2.4 Suppose $*'a \subset *0$. Then $*a \subset *0$ for if not then $*a \notin K'$ and $*'a = ((*c_1 + *0) \times \ldots \times (*c_m + *0))$. Now since each c_i covers a, $*a \subset (*c_i + *0)$, whence since $*'a \subset *0$ then $*a \subset *0$, whence $a = 0$.

Given a formula of S1 if α is not a theorem then for some modal assignment V in the characteristic S1-algebra $\mathsf{V}(\alpha) \notin D$. Where $\beta_1, \ldots, \beta_n$ are the n wf parts of α then in the finite subalgebra generated by $\{\mathsf{V}(\beta_1), \ldots, \mathsf{V}(\beta_n), *0\}$, $\mathsf{V}(\alpha) \notin D'$ (since $\mathsf{V}(\alpha)$ and $-*0 \in K'$ and $-*0 \not\subset \mathsf{V}(\alpha)$). I.e. α will be falsified in a finite S1-algebra. S1 thus has the finite model property and is decidable. QED

4 *Semantic models.* We shall use the terminology of [6, pp. 274-276] where S2-models are described.[3] We can consider an S2-model to be an ordered quadruple $\langle W\, N\, \mathbf{R}\, \mathsf{V} \rangle$ where $W = \{w_1, \ldots, w_i, \ldots\}$ is a set of 'worlds', $N \subseteq W$, $\mathbf{R} \subseteq N \times W$ being reflexive over N and for each $w_j \notin N$ having some $w_i \in N$ such that $w_i \mathbf{R} w_j$ and V is an assignment from wff and members of W to $\{1, 0\}$ satisfying the usual truth-functional conditions and for M having, where α is any wff and $w_i \in W$;

4.1 If $w_i \in N$ then $\mathsf{V}(M\alpha, w_i) = 1$ iff for some $w_j \in W$ such that $w_i \mathbf{R} w_j$ $\mathsf{V}(\alpha, w_j) = 1$.
4.2 If $w_i \notin N$ then $\mathsf{V}(M\alpha, w_i) = 1$.

In the case of S1 4.2 requires modification. What we have now to say is that although in any model $\mathsf{V}(M\alpha, w_i)$ *may* be 1 for $w_i \notin N$ it can under some circumstances be 0, i.e. it may, though it need not, be 0 provided:

4.3 If $\mathsf{V}(\alpha, w_i) = 1$ then $\mathsf{V}(M\alpha, w_i) = 1$.

4.4 If for any wff α and β $V((\alpha . \beta), w_j) = 0$ for every $w_j \in W$ then either $V(M\alpha, w_i) = 1$ or $V(M\beta, w_i) = 1$.

It is a consequence of 4.4 that if $V(\alpha, w_j) = 0$ for every $w_j \in W$ then $V(M\alpha, w_i) = 1$ for every $w_i \notin N$. But except in the case of such formulae 4.4 does not give a definite rule for evaluating $M\alpha$ in non-normal worlds. We have to build into the model a way of doing this.

An S1-model is a quintuple $\langle W N \mathbf{R} \mathbf{R'} V\rangle$ where W, N and $\mathbf{R}$ are as in an S2-model and $\mathbf{R'} \subseteq W \times \mathcal{P} W$ (i.e. a relation between members of W and subsets of W) such that for no disjoint subsets A and B of W do we have $w_i\mathbf{R'}A$ and $w_i\mathbf{R'}B$ for any $w_i \in W$, and V is an assignment satisfying the truth-functional conditions and:

4.5 If $w_i \in N$ then $V(M\alpha, w_i) = 1$ iff $V(\alpha, w_j) = 1$ for some $w_j \in W$ such that $w_i\mathbf{R}w_j$.
4.6 If $w_i \notin N$ then $V(M\alpha, w_i) = 0$ provided that $V(\alpha, w_i) = 0$ and $w_i\mathbf{R'}\ \{w \in W: V(\alpha, w) = 1\}$. Otherwise $V(M\alpha, w_i) = 1$.

An S2-model will be an S1-model in which $\mathbf{R'}$ is empty. Since we have already defined S1-validity in terms of S1-algebras we shall say that a wff α is *S1-model-valid* iff $V(\alpha, w_i) = 1$ for every $w_i \in N$ in every S1-model. We shall show that S1-validity and S1-model-validity coincide.

This semantics is somewhat messy and it would be nice to have a simpler condition to replace 4.6 but it is extremely difficult to see how otherwise the almost completely random nature of the assignment to $M\alpha$ in non-normal worlds can be precisely expressed.[4] It should be clear that an S1-model satisfies 4.3 and 4.4.

At this point it might be worth exhibiting the 4-valued matrix (Group V [12, p. 494]) used by Lewis to distinguish S2 from S1 as an S1-model which is not an S2 model. $W = \{w_1, w_2\}$, $\mathbf{R} = \{\langle w_1, w_2\rangle, \langle w_1, w_1\rangle\}$, $\mathbf{R'} = \{\langle w_2, \{w_1\}\rangle\}$. $V(p, w_1) = 1$, $V(p, w_2) = V(q, w_1) = V(q, w_2) = 0$. With this assignment we have $w_2\mathbf{R'}\ \{w \in W: V(p, w) = 1\}$ but not $w_2\mathbf{R'}\ \{w \in W: V((p . q), w) = 1\}$. Whence since $w_2 \notin N$ and $V(p, w_2) = 0$ then $V(M(p . q), w_2) = 1$ and $V(Mp, w_2) = 0$, thus $V((M(p . q) \supset Mp), w_2) = 0$ and $V(L(M(p . q) \supset Mp), w_1) = 0$ whence since $w_1 \in N$, $L(M(p . q) \supset Mp)$ is not S1-model-valid. This model becomes the 4-valued matrix if we let;

$$V(\alpha) = 1 \text{ iff } V(\alpha, w_1) = 1,\ V(\alpha, w_2) = 1$$
$$V(\alpha) = 2 \text{ iff } V(\alpha, w_1) = 1,\ V(\alpha, w_2) = 0$$
$$V(\alpha) = 3 \text{ iff } V(\alpha, w_1) = 0,\ V(\alpha, w_2) = 1$$
$$V(\alpha) = 4 \text{ iff } V(\alpha, w_1) = 0,\ V(\alpha, w_2) = 0$$

The table for M then works out as

p	Mp
1	1
2	2
3	1
4	3

We wish now to show that each S1-model corresponds to an S1-algebra together with a modal assignment and *vice versa*. It is convenient to distinguish the assignment from the rest of the model and call the rest a 'model structure' (m.s.) as Lemmon, following Kripke, does. This part of the paper will follow Lemmon in spirit (particularly [10, pp. 56-62]) and therefore only as much detail will be given as necessary.

Given an S1 m.s. $\langle W\, N\, \mathbf{R}\, \mathbf{R}' \rangle$ we construct an algebra on it as follows: $K = \mathcal{P}\, W$ and the Boolean operations $\times$ and $-$ are the set theoretic operations of intersection and complementation with respect to W. For $A \subseteq W$ we define $*A$ as:

4.7 $\{w \in W\colon (w \in N.\ (\exists x)(x \in A.\ w\mathbf{R}x)) \vee (w \notin N.\ (w \in A \vee \sim w\mathbf{R}'A))\}$

i.e.

4.8 $w \in *A$ iff $(w \in N.\ (\exists x)(x \in A.\ w\mathbf{R}x))$ or $(w \notin N.\ (w \in A \vee \sim w\mathbf{R}'A))$

Theorem 5. *The algebra on an* S1 *model structure is an* S1-*algebra*.

This is in effect Lemmon's theorem 15 [10, p. 57]. S1-algebras are of course more complicated than modal algebras but it is a simple matter to check that using 4.7 2.1-2.4 can all be verified. And it should be clear that any assignment V within a m.s. gives rise to a modal assignment V' in the corresponding algebra such that $\mathsf{V}'(\alpha) = \{w \in W\colon \mathsf{V}(\alpha, w) = 1\}$ (and conversly). A wff α will therefore be S1-model valid iff for every modal assignment V' in the algebra on any S1 m.s. $\mathsf{V}'(\alpha) \in D$ (for $N = -{*0}$ by 4.8)

Given a finite S1-algebra $\langle K \times - * \rangle$ we may define an S1 m.s. $\langle W\, N\, \mathbf{R}\, \mathbf{R}' \rangle$ such that the algebra on $\langle W\, N\, \mathbf{R}\, \mathbf{R}' \rangle$ is isomorphic to $\langle K \times - * \rangle$. The general pattern follows [10, p. 57f] (Lemmon's theorem 17) with appropriate modifications. Since $\langle K \times - \rangle$ is a finite Boolean algebra then it is isomorphic to the algebra of all subsets of a given finite set W. Let ϕ be the isomorphism. Each atom of the original algebra will thus be $\phi\{w_i\}$ for some $w_i \in W$. We define $\langle W\, N\, \mathbf{R}\, \mathbf{R}' \rangle$ as follows:

4.9 $w_i \in N$ iff $\phi\{w_i\} \not\subset *0$
4.10 $w_i \mathbf{R} w_j$ iff $\phi\{w_i\} \not\subset *0$ and $\phi\{w_i\} \subset *\phi\{w_j\}$
4.11 $w_i \mathbf{R}' A$ $(A \subseteq W)$ iff $\phi\{w_i\} \subset *0$ and $\phi\{w_i\} \not\subset *\phi A$

(Note that since $\phi\{w_i\}$ is an atom if $\phi\{w_i\} \not\subset a$ then $\phi\{w_i\} \times a = 0$)

Theorem 6. $\langle W\, N\, \mathbf{R}\, \mathbf{R}' \rangle$ *is an* S1-*model structure*.

$\mathbf{R}$ is reflexive over N since by 2.1 $\phi\{w_i\} \subset *\phi\{w_i\}$. For $w_i \notin N$ then $\phi\{w_i\} \subset *0$ and so by 4.10 $\sim w_i \mathbf{R} w_j$. Further for any $w_j \notin N$ there is some $w_i \in N$ such that $w_i \mathbf{R} w_j$ for if not then where A is the set of all w_i such that $w_i \mathbf{R} w_j$ then ϕA $(= *\phi\{w_j\}) \subset *0$. Whence by 2.4 $\phi\{w_j\} = 0$ contrary to the fact that $\phi\{w_j\}$ is an atom. For $\mathbf{R}'$ suppose that for some $w_i \in W$, $w_i \mathbf{R}' A$ and $w_i \mathbf{R}' B$ and $A \times B = 0$. Then since ϕ is a Boolean isomorphism $\phi A \times \phi B = 0$ and so by 2.3 $*0 \subset *\phi A \times *\phi B$. Whence by 4.11 $\phi\{w_i\} \subset *\phi A \times *\phi B$. But $\phi\{w_i\}$ is an atom and so either $\phi\{w_i\} \subset *\phi A$ or $\phi\{w_i\} \subset *\phi B$, i.e. (by 4.11) either $\sim w_i \mathbf{R}' A$ or $\sim w_i \mathbf{R}' B$. QED

Theorem 7. *The algebra on* $\langle W\ N\ \mathbf{R}\ \mathbf{R}'\rangle$ *is isomorphic to* $\langle K \times - *\rangle$.

We already have the isomorphism for the Boolean operations. All that is necessary is to extend it to *. I.e. where *′ is defined in $\langle W\ N\ \mathbf{R}\ \mathbf{R}'\rangle$ according to 4.7 we must show:

4.12 $\phi *' A = *\phi A$

Suppose $w \epsilon *'A$, i.e. $\phi\{w\} \subset \phi *'A$. Then by 4.8-4.11

4.13 $(\phi\{w\} \not\subset *0.\ (\exists x)(x \text{ is an atom}.\ x \subset \phi A.\ \phi\{w\} \subset *x)) \vee$
$(\phi\{w\} \subset *0.\ (\phi\{w\} \subset \phi A \vee \phi\{w\} \subset *\phi A)$

Now either $\phi\{w\} \subset *0$ or not. If the former then the first disjunct of 4.13 is false and so the second must be true. If $\phi\{w\} \subset \phi A$ then by 2.1 $\phi\{w\} \subset *\phi A$ and so if the second disjunct is true then $\phi\{w\} \subset *\phi A$. Suppose that $\phi\{w\} \not\subset *0$ then the first disjunct of 4.13 must be true. Whence by lemma 1 $(\exists x)(*x \subset (*\phi A + *0)\ .\ \phi\{w\} \subset *x)$, whence $\phi\{w\} \subset *\phi A + *0$. But $\phi\{w\} \not\subset *0$ and $\phi\{w\}$ is an atom so $\phi\{w\} \subset *\phi A$. Thus $*\phi A$ contains as atoms every $\phi\{w\}$ such that $w \epsilon *'A$, i.e. since ϕ is a Boolean isomorphism $\phi *'A \subset *\phi A$.

For the converse suppose $\phi\{w\} \subset *\phi A$. Then either $\phi\{w\} \subset *0$ or not. If so then since $\phi\{w\} \subset *\phi A$ the second disjunct of 4.13 will be true. If $\phi\{w\} \not\subset *0$ then if there is no atom x of ϕA such that $\phi\{w\} \subset *x$ then where $x_1, \ldots, x_n$ are all the atoms of ϕA, $\phi\{w\}$, being an atom, $\not\subset (*x_1 + \ldots + *x_n)$, i.e. since $\phi\{w\} \times *0 = 0$, $\phi\{w\} \not\subset (*x_1 + \ldots *x_n) + *0 = *(x_1 + \ldots + x_n) + *0 = *\phi A + *0$. Whence since $\phi\{w\} \times *0 = 0$ $\phi\{w\} \not\subset *\phi A$ contrary to hypothesis. So if $\phi\{w\} \epsilon *\phi A$ then 4.13 holds. I.e. by 4.7 $*\phi A \subset \phi' A$. Whence 4.12 holds.

With theorems 3 and 4 these last three theorems (4, 5 and 6) show that S1-model-valid formulae are precisely the theorems of S1.

5 *Other systems.* We can if we wish impose the conditions of transitivity and symmetry on the relation **R** in an S1-model. The corresponding conditions for the algebras are;

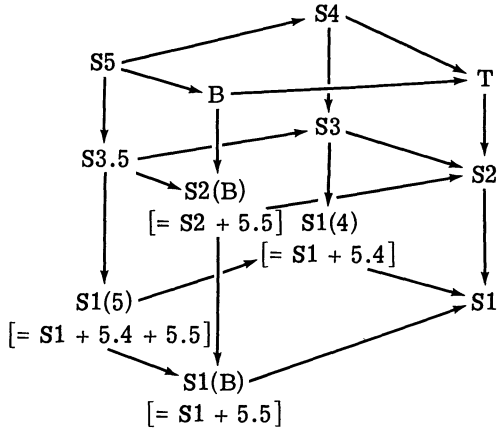
5.1 $*(*a \times -*0) \subset *a + *0$
5.2 $a \subset -*-*a + *0$

With 2.6 (and using 2.1) 5.1 reduces to $*(*a \times -*0) = *a$ which is Lemmon's condition for a transitive epistemic algebra [11, p. 196] and 5.2 is his condition for a symmetric algebra. To 5.1 and 5.2 correspond the axioms:

5.4 $Lp \supset ((p \prec p) \prec Lp)$
5.5 $p \supset LMp$

(1.3 is not to apply to these axioms.)

If we add the strict form of 5.5 to S1 we obtain the Brouwerian system so we may call S1 + 5.5 S1(B)[5]. It is of course strictly weaker than B (the Brouwerian system). We shall call S1 + 5.4, S1(4) and S1 + 5.4 + 5.5, S1(5). In the presence of 5.4 5.5 is interchangeable with $Mp \supset LMp$. S1(5) is contained in Åqvist's S3.5 [1, p. 82], [6, p. 284f]. The following relations hold between these systems and known systems:



S1(4) is a proper subsystem of Canty's R1.[6] S1(4), S1(B) and S1(5) all have an infinite number of distinct modalities[7] and would not appear to be very interesting. A proof that they all have the finite model property can be given by adapting Lemmon's proofs [11, pp. 196f, 210] in the way we have done for S1 in theorem 4.

Turning to systems weaker than S1 we may obtain algebras and models for Feys' S1° by dropping 2.1. The omission of 2.2 and 2.3 gives the system S1′ of [7]. In this system many of the characteristically S1 theorems are provable only as rules of inference. The omission of 2.3 and the replacement of 2.1 by $a \subset *a + *0$ gives the system obtained by removing the clumsy 'or axiom' clause from 1.3. If we choose to take $D = \{1\}$ in an S1-algebra we get a system which it would be nice to be able to call E1 (after the model of E2, E3 etc.) *viz* 1.1, 1.2, 1.3, 1.4, 1.5 and

5.6 *If* α *is* **PC**-*valid then* $\vdash \alpha$

5.7 *Substitution of proved material equivalents in any theorem.*

The system Lemmon calls E1 is unfortunately a subsystem of his S0.5 [9, p. 183] and therefore does not have 5.7.[8] There would also be E-systems corresponding to S1(4), S1(B) and S1(5).

Of the questions remaining open there is one whose solution might be facilitated by the algebraic and semantic treatment of S1, *viz* whether 1.2 can be replaced by

5.8 $L(p \supset q) \supset (Lp \supset Lq)$

Lemmon calls this system S0.9 and in 1957 [9, p. 180] had no proof that it was weaker than S1 and I have seen none since. Although as Lemmon suggests it would be rather good if 1.2 could be replaced by 5.8 my feeling is that the corresponding algebraic condition *viz*

5.9 $*a \subseteq *(a \times -b) + *b$

is not sufficiently powerful to derive 2.3.

NOTES

1. We are using the notation of [6]. Moreover we give the version of Lemmon's S1 basis found in [6, p. 246f].

2. An introductory account of the relation between Boolean algebra and modal logic will be found in chapter 17 of [6]. Our terminology comes essentially from McKinsey's [13]. The fullest account of the algebraic approach to modal logic is found in Lemmon's two papers [10] and [11].

3. S2-models are due to Kripke [8]. It is possible to define N in terms of $\mathbf{R}$ but the advantage is theoretical only.

4. As far as interpretation of S1-models goes it is difficult to see anything remotely sensible coming out even along the lines of [3] though it would be rash to predict. What can be said is that Lewis' reasons for arriving at S1 have only an accidental connection with its formal structure.

5. [14. p. 59]. Sobociński and Thomas have done a certain amount of work on adding Brouwerian axioms to sub-systems of S1 (for fuller details v. [5, p. 123]). We make one or two remarks *infra* about algebraic and semantic characterizations of subsystems of S1 and it may be that dealing with their Brouwerian extensions is not difficult.

6. Canty's R1 [2, p. 312] is in effect S1$+L(p\supset q)\supset(L(q\supset p)\supset L(Lp\supset Lq))$ though Canty axiomatizes it with a finite number of schemata and modus ponens as the only rule of inference. The derivation of 5.4 in R1 is straightforward. As far as the converse goes it is possible to obtain a transitive S1-model which rejects $L(p\supset q)\supset(L(q\supset p)\supset L(Lp\supset Lq))$.

7. The following S1-model $\langle W N \mathbf{R}\, \mathbf{R}' V\rangle$ will reject $LM_n p\supset LM_{n-1}p$: $W=\{w_0, w_1, \ldots, w_n\}$, $N=\{w_0\}$, $\langle w_i, w_j\rangle\ \varepsilon\ \mathbf{R}$ iff $i=1$, $\langle w_i, A\rangle\ \varepsilon\ \mathbf{R}'$ iff $i\geqslant 2$ and $A=\{w_0, w_1, \ldots, w_j\}$ for some $j<i-2$, $V(p, w_0)=1$, $V(p, w_i)=0$ $(i>0)$. This will show that the series of modalities $LM_n p (n=1,2,\ldots)$ is infinite. Since $\mathbf{R}$ is both symmetrical (over N) and transitive this result holds for S1(5) which is therefore unlike S3.5 in having no reduction to a normal form of specified degree [4].

8. Lemmon seems to think that E1 has replacement for tautologous equivalents but E1 is a subsystem of S0.5 and it is known [6, p. 388] that S0.5 does not admit this rule. Lemmon considers it a consequence in S0.5 of 1.2 and 5.11 but of course this combination is not sufficient to prove e.g. $LLp\supset LL\sim\sim p$.

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