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# A STRONG COMPLETENESS THEOREM FOR 3-VALUED LOGIC: PART II

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Proof[1] was given in [1] that $\mathbf{SC}_3$, the 3-valued sentential calculus, has a strongly complete axiomatization. Pushing our investigation one step further,[2] we obtain here a like result about $\mathbf{QC}_3$, the 3-valued quantificational calculus of order one.[3]

**1** The primitive signs of $\mathbf{QC}_3$ are

(a) '$\sim$', '$\supset$', '$\forall$', '(', ')', and ',',
(b) a denumerable infinity of individual variables, to be referred to by means of '$X$',[4]
(c) a denumerable infinity of individual parameters, to be referred to by means of '$\mathsf{X}$',[5] and
(d) for each $d$ from 0 on, a denumerable infinity of predicate parameters of degree $d$, to be referred to by means of '$\mathsf{F}^d$'.[6]

We presume the variables in (b), the parameters in (c), and the parameters in (d) to be alphabetically ordered; and we take the alphabetically first parameter of degree $d$ in (d) to be '$\mathsf{p}$'.

The atomic wffs of $\mathbf{QC}_3$ are all formulas of the sort $\mathsf{F}^d(\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_d)$, where $\mathsf{F}^d$ is a predicate parameter of degree $d$ ($d \geqslant 0$) and $\mathsf{X}_1, \mathsf{X}_2, \ldots,$ and $\mathsf{X}_d$ are individual parameters. The wffs of $\mathbf{QC}_3$ (presumed at one point below to be alphabetically ordered) are the atomic wffs just defined, plus all formulas of the sorts (i) $\sim A$, where $A$ is well-formed, (ii) $(A \supset B)$, where $A$ and $B$ are well-formed, and (iii) $(\forall X)A$, where—for some individual parameter $\mathsf{X}$—the result $A(\mathsf{X}/X)$ of replacing $X$ everywhere in $A$ by $\mathsf{X}$ is well-formed.[7] The length $\mathcal{L}(A)$ of an atomic wff is 1; the length $\mathcal{L}(\sim A)$ of a negation $\sim A$ is $\mathcal{L}(A) + 1$; the length $\mathcal{L}((A \supset B))$ of a conditional $(A \supset B)$ is $\mathcal{L}(A) + \mathcal{L}(B) + 1$; and the length $\mathcal{L}((\forall X)A)$ of a quantification $(\forall X)A$ is $\mathcal{L}(A(\mathsf{X}/X)) + 1$, where $\mathsf{X}$ is the alphabetically earliest individual parameter of $\mathbf{QC}_3$. We avail ourselves of the following ten abbreviations:

$$\text{'}\mathbf{f}\text{'} =_{df} \text{'}\sim(\mathsf{p} \supset \mathsf{p})\text{'}$$
$$(A \vee B) =_{df} ((A \supset B) \supset B)^{8}$$
$$(A \,\&\, B) =_{df} \sim(\sim A \vee \sim B)$$
$$(A \equiv B) =_{df} ((A \supset B) \,\&\, (B \supset A))$$

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$$(A \mathrm{I} B) =_{df} (A \supset (A \supset B))$$
$$-A =_{df} (A \supset \sim A)^9$$
$$\mathrm{J}_1(A) =_{df} \sim(A \supset \sim A)$$
$$\mathrm{J}_3(A) =_{df} \sim(\sim A \supset A)$$
$$\mathrm{J}_2(A) =_{df} \sim(\mathrm{J}_1(A) \vee \mathrm{J}_3(A))$$
$$(\exists X)A =_{df} \sim(\forall X) \sim A;$$

and we omit outer parentheses whenever clarity permits.

Sets of wffs play a major role in the paper. We take an individual parameter to be foreign to a set $S$ of wffs if the parameter does not occur in any member of the set, and we declare $S$ infinitely extendible if $\aleph_0$ individual parameters are foreign to $S$. Given a mapping $M$ of one set of individual parameters into another, we understand by the $M$-rewrite of a wff $A$ the result of simultaneously replacing in $A$ all individual parameters from the first set by their respective values under $M$; and we understand by the $M$-rewrite of a set $S$ of wffs the set $\emptyset$ when $S$ is empty, otherwise the set consisting of the $M$-rewrites of the various members of $S$. Lastly, given two sets $S$ and $S'$ of wffs, we declare $S'$ isomorphic to $S$ if—for some one-to-one mapping $M$ of the individual parameters of $\mathbf{QC}_3$ into all the individual parameters of $\mathbf{QC}_3$—$S'$ is the $M$-rewrite of $S$.

The axioms of $\mathbf{QC}_3$ are all wffs of the sorts *A1*-*A4* on p. 325 of [1], plus all those of the sorts:

*A5.* $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$,
*A6.* $A \supset (\forall X)A$,[10]
*A7.* $(\forall X)A \supset A(\mathrm{X}/X)$,

plus all those of the sort $(\forall X)A$, where—for some individual parameter $\mathrm{X}$ foreign to $(\forall X)A$—$A(\mathrm{X}/X)$ is an axiom of $\mathbf{QC}_3$. The notions of provability, syntactic (in)consistency, and maximal consistency are then defined as on pp. 325-326 of [1], but with '$\mathbf{QC}_3$' substituting throughout for '$\mathbf{SC}_3$'.

Our truth-values are (the designated) 1 and (the undesignated) 2 and 3.[11] Truth-value assignments are functions from the atomic wffs of $\mathbf{QC}_3$ to $\{1, 2, 3\}$, and the truth-values under these of negations and conditionals are reckoned as on p. 326 of [1].[12] As for quantifications, $(\forall X)A$ evaluates to 1 under a truth-value assignment $\alpha$ if $A(\mathrm{X}/X)$ does so for every individual parameter $\mathrm{X}$ of $\mathbf{QC}_3$; $(\forall X)A$ evaluates to 3 under $\alpha$ if $A(\mathrm{X}/X)$ does so for at least one individual parameter $\mathrm{X}$ of $\mathbf{QC}_3$; otherwise, $(\forall X)A$ evaluates to 2 under $\alpha$.[13] We take a set $S$ of wffs to be truth-value verifiable if there is a truth-value assignment under which all members of $S$ evaluate to 1; we take $S$ to be semantically consistent if either $S$ or some set isomorphic to $S$ is truth-value verifiable;[14] we take $S$ to entail a wff $A$ if $S \cup \{-A\}$ is semantically inconsistent; and we take the wff $A$ to be valid if $\emptyset$ entails $A$.

**2** Our completeness proof, an extension of that in [1], uses five fresh results: *L3*(c) and *L4*(a)-(d) below. Proof of *L3*(c) can be recovered from [4], pp. 336-337, and so is omitted here; but proofs of *L4*(a)-(d) are given in full. Our first lemma is *L1* in [1], pp. 326-327, which we shall presume the reader to have on hand. Our second lemma deals with truth-functional matters, and our third with quantificational ones.

*L2*. (a) *If* $S \vdash A \supset B$, *then* $S \vdash (B \supset C) \supset (A \supset C)$.
(b) *If* $S \vdash A \supset B$ *and* $S \vdash B \supset C$, *then* $S \vdash A \supset C$.
(c) *If* $S \vdash {\sim}A \supset {\sim}B$, *then* $S \vdash B \supset A$.
(d) *If* $S \vdash A \supset B$ *and* $S \vdash {\sim}B$, *then* $S \vdash {\sim}A$.
(e) *If* $S \cup \{A\} \vdash B$ *and* $S \vdash A' \supset A$, *then* $S \cup \{A'\} \vdash B$.
(f) *If* $S \vdash A \vee B$, *then* $S \vdash B \vee A$.
(g) *If* $S \vdash A \vee B$ *and* $S \vdash A \supset A'$, *then* $S \vdash A' \vee B$.
(h) *If* $S \vdash A \vee B$ *and* $S \vdash A \supset A'$, *then* $S \vdash (A' \,\&\, A) \vee B$.
(i) *If* $S \vdash A \vee B$ *and* $S \vdash B \supset B'$, *then* $S \vdash A \vee B'$.
(j) *If* $S \vdash A \vee (B \vee C)$ *and* $S \vdash B \supset B'$, *then* $S \vdash A \vee (B' \vee C)$.
(k) *If* $S \vdash A \vee (B \vee C)$ *and* $S \vdash C \supset C'$, *then* $S \vdash A \vee (B \vee C')$.
(l) *If* $S \vdash A \vee (B \,\&\, C)$ *and* $S \vdash C \supset C'$, *then* $S \vdash A \vee (B \,\&\, C')$.
(m) *If* $S \cup \{C\} \vdash A \vee B$, *then* $S \cup \{C\} \vdash A \vee (B \,\&\, C)$.
(n) *If* $S \cup \{C\} \vdash A \vee (B \vee {\sim}C)$, *then* $S \cup \{C\} \vdash A \vee B$.
(o) *If* $S \vdash A \mathbin{\mathrm{I}} -A$, *then* $S \vdash -A$.
(p) *If* $S \vdash \mathsf{J}_1(A) \vee \mathsf{J}_2(A)$, *then* $S \vdash {\sim}\mathsf{J}_3(A)$.
(q) $S \vdash {\sim}\mathsf{J}_3(A) \supset (\mathsf{J}_1(A) \vee \mathsf{J}_2(A))$.
(r) $S \vdash -\mathsf{J}_3(A) \supset {\sim}\mathsf{J}_3(A)$.
(s) *If* $S \cup \{\mathsf{J}_3(A)\} \vdash B$, *then* $S \vdash \mathsf{J}_3(A) \supset B$.

*Proof:* (a) Since $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$ is an axiom, $S \vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$ by *L1*(a). So (a) by *L1*(d). (b) By (a) and *L1*(d). (c) Proof like that of (a). (d) $S \vdash (A \supset B) \supset ({\sim}B \supset {\sim}A)$ by *L1*(l) and *L1*(a). So (d) by *L1*(d). (e) Suppose $S \cup \{A\} \vdash B$. Then $S \vdash A \mathbin{\mathrm{I}} B$ by *L1*(q), and hence $S \cup \{A'\} \vdash A \mathbin{\mathrm{I}} B$ by *L1*(a). But $(A \mathbin{\mathrm{I}} B) \supset ((A' \supset A) \supset (A' \mathbin{\mathrm{I}} B))$ is valid in the sense of [1]. So $S \cup \{A'\} \vdash (A \mathbin{\mathrm{I}} B) \supset ((A' \supset A) \supset (A' \mathbin{\mathrm{I}} B))$ by the completeness theorem of [1] and *L1*(a), and hence $S \cup \{A'\} \vdash (A' \supset A) \supset (A' \mathbin{\mathrm{I}} B)$ by *L1*(d). So, if $S \vdash A' \supset A$, then $S \cup \{A'\} \vdash A' \supset A$ by *L1*(a), hence $S \cup \{A'\} \vdash A' \mathbin{\mathrm{I}} B$ by *L1*(d), and hence $S \cup \{A'\} \vdash B$ by *L1*(c)-(d). (f) Since $(A \vee B) \supset (B \vee A)$ is valid in the sense of [1], $S \vdash (A \quad B) \supset (B \vee A)$ by the completeness theorem of [1] and *L1*(a). Hence (f) by *L1*(d). (g)-(l) Proofs like that of (f). (m)-(n) Proofs similar to that of (e). (o)-(p) Proofs similar to that of (f). (q)-(r) By the completeness theorem of [1] and *L1*(a). (s) Proof similar to that of (e).

*L3*. (a) *If* $S \vdash (\forall X)(A \supset B)$, *then* $S \vdash (\forall X)A \supset (\forall X)B$.
(b) $S \vdash (\forall X')A(X'/X) \supset (\forall X)A$.
(c) *If* $S \vdash A(\mathsf{X}/X)$, *then* $S \vdash (\forall X)A$, *so long as* $\mathsf{X}$ *is foreign to* $S$ *and* $(\forall X)A$.
(d) $S \vdash (\forall X)(A \supset B) \supset (A \supset (\forall X)B)$.[15]
(e) $S \vdash (\forall X)(A \vee B) \supset (A \vee (\forall X)B)$.[16]
(f) *If* $S \vdash (\forall X)(A \vee B)$, *then* $S \vdash A \vee (\forall X)B$, *so long as* $X$ *is foreign to* $A$.
(g) $S \vdash A(\mathsf{X}/X) \supset (\exists X)A$.
(h) *If* $S \vdash A(\mathsf{X}/X) \vee (B(\mathsf{X}/X) \vee C(\mathsf{X}/X))$, *then* $S \vdash (\forall X)A \vee ((\exists X)B \vee (\exists X)C)$, *so long as* $\mathsf{X}$ *is foreign to* $S$, $(\forall X)A$, $(\exists X)B$, *and* $(\exists X)C$.
(i) $S \vdash (\forall X)A \supset (\exists X)A$.
(j) $S \vdash ((\exists X)\mathsf{J}_k(A) \,\&\, (\forall X) \sum_{i=1}^{k} \mathsf{J}_i(A)) \supset \mathsf{J}_k((\forall X)A)$, *for any* $k$ *from* 1 *through* 3.
(k) $S \vdash (\forall X) {\sim} \mathsf{J}_3(A) \supset (\forall X)(\mathsf{J}_1(A) \vee \mathsf{J}_2(A))$.

(l) $S \vdash (\forall X) - \mathsf{J}_3(A) \supset (\forall X) \sim \mathsf{J}_3(A)$.
(m) $S \vdash -(\forall X) - A \;\mathrm{I}\; (\exists X)A$.
(n) *If* $S \vdash A \supset (\exists X) \sim B$, *then* $S \vdash A \supset \sim(\forall X)B$.
(o) *If* $S \vdash (\forall X)(\sim\sim A \supset B) \supset C$, *then* $S \vdash (\forall X)(A \supset B) \supset C$.
(p) *If* $S \vdash A \supset (B \supset (\forall X) \sim C)$, *then* $S \vdash A \supset (B \supset \sim(\exists X)C)$.
(q) $S \vdash B \supset (\exists X)(A \supset B)$.
(r) $S \vdash (\exists X)(A \supset B) \supset ((\forall X)A \supset B)$.
(s) $S \vdash ((\exists X)(A \supset B) \supset B) \equiv (((\forall X)A \supset B) \supset B)$.

*Proof:* (a) Since $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$ is an axiom, $S \vdash (\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$ by $L1$(a). Hence (a) by $L1$(d). (b) In case $X'$ and $X$ are the same, (b) by $L1$(g) and $L1$(a). So suppose $X'$ and $X$ are distinct, and let $\mathsf{X}$ be foreign to $(\forall X)A$. $(\forall X')A(X'/X) \supset A(\mathsf{X}/X) (= (\forall X')A(X'/X) \supset (A(X'/X))(\mathsf{X}/X'))$ is an axiom. Hence, by the hypothesis on $\mathsf{X}$, so is $(\forall X)((\forall X')A(X'/X) \supset A)$. Hence, by $L1$(a), $S \vdash (\forall X)((\forall X')A(X'/X) \supset A)$. Hence, by (a), $S \vdash (\forall X)(\forall X')A(X'/X) \supset (\forall X)A$. But $(\forall X')A(X'/X) \supset (\forall X)(\forall X')A(X'/X)$ is an axiom. Hence, by $L1$(a), $S \vdash (\forall X')A(X'/X) \supset (\forall X)(\forall X')A(X'/X)$. Hence (b) by $L2$(b). (c) See proof of (3.7.12) in [4]. (d) Since $A \supset (\forall X)A$ is an axiom, $S \vdash A \supset (\forall X)A$ by $L1$(a). Hence $S \vdash ((\forall X)A \supset (\forall X)B) \supset (A \supset (\forall X)B)$ by $L2$(a). But $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$ is an axiom. So $S \vdash (\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$ by $L1$(a). So (d) by $L2$(b). (e) See proof of Lemma 6.7.2 in [5]. (f) Suppose $X$ is foreign to $A$, in which case $(\forall X)(A \vee B) \supset (A \vee (\forall X)B)$ is well-formed. Then (f) by (e) and $L1$(d). (g) See proof of Lemma 6.8.5 in [5]. (h) Suppose $S \vdash A(\mathsf{X}/X) \vee (B(\mathsf{X}/X) \vee C(\mathsf{X}/X))$, suppose $\mathsf{X}$ is foreign to $S$, $(\forall X)A$, $(\exists X)B$, and $(\exists X)C$, and let $X'$ be new. Then $S \vdash A(\mathsf{X}/X) \vee ((\exists X)B \vee C(\mathsf{X}/X))$ by (g) and $L2$(j), hence $S \vdash A(\mathsf{X}/X) \vee ((\exists X)B \vee (\exists X)C)$ by (g) and $L2$(k), hence $S \vdash ((\exists X)B \vee (\exists X)C) \vee A(\mathsf{X}/X)$ by $L2$(f), hence $S \vdash (\forall X')(((\exists X)B \vee (\exists X)C) \vee A(X'/X))$ by (c), hence $S \vdash ((\exists X(B \vee (\exists X)C) \vee (\forall X')A(X'/X)$ by (f) and the hypothesis on $X'$, hence $S \vdash ((\exists X)B \vee (\exists X)C) \vee (\forall X)A$ by (b) and $L2$(k), and hence $S \vdash (\forall X)A \vee ((\exists X)B \vee (\exists X)C)$ by $L2$(f). (i) Let $\mathsf{X}$ be an arbitrary individual parameter. Since $(\forall X)A \supset A(\mathsf{X}/X)$ is an axiom, $S \vdash (\forall X)A \supset A(\mathsf{X}/X)$ by $L1$(a). Hence (i) by (g) and $L2$(b). (j) See proof of Lemma 6.8.24 in [5]. (k) Let $\mathsf{X}$ be an individual parameter foreign to $(\forall X)(\sim \mathsf{J}_3(A) \supset (\mathsf{J}_1(A) \vee \mathsf{J}_2(A)))$. By $L2$(q) $\vdash \sim \mathsf{J}_3(A(\mathsf{X}/X)) \supset (\mathsf{J}_1(A(\mathsf{X}/X)) \vee \mathsf{J}_2(A(\mathsf{X}/X)))$. So, by the hypothesis on $\mathsf{X}$, $\vdash (\forall X)(\sim \mathsf{J}_3(A) \supset (\mathsf{J}_1(A) \vee \mathsf{J}_2(A)))$. So, by $L1$(a), $S \vdash (\forall X)(\sim \mathsf{J}_3(A) \supset (\mathsf{J}_1(A) \vee \mathsf{J}_2(A)))$. So (k) by (a). (l) Proof like that of (k), but using $L2$(r) in place of $L2$(q). (m) See proof of Lemma 6.8.29 in [5]. (n) Let $\mathsf{X}$ be an individual parameter foreign to $(\forall X)(B \supset \sim\sim B)$. By $L1$(k) $\vdash B(\mathsf{X}/X) \supset \sim\sim B(\mathsf{X}/X)$; hence, by (c), $\vdash (\forall X)(B \supset \sim\sim B)$; hence, by (a), $\vdash (\forall X)B \supset (\forall X)\sim\sim B$; hence, by $L1$(l) and $L1$(d), $\vdash (\exists X) \sim B \supset \sim(\forall X)B$; hence, by $L1$(a), $S \vdash (\exists X) \sim B \supset \sim(\forall X)B$; and hence (n) by $L2$(b). (o)-(p) Proofs similar to that of (n). (q) See proof of Lemma 6.8.10 in [5]. (r) See proof of Lemma 6.8.11 in [5]. (s) See proof of Lemma 6.8.8 in [5].

*L4.* (a) $S \vdash (\forall X) - A \supset -(\exists X)A$.
(b) $S \vdash (\exists X')(A(X'/X) \supset (\forall X)A)$.
(c) *If* $S \vdash (\forall X)A$, *then* $S \vdash A(\mathsf{X}/X)$ *for every individual parameter* $\mathsf{X}$ *of* $\mathbf{QC}_3$.
(d) *If* $S \vdash \sim A(\mathsf{X}/X)$ *for any individual parameter* $\mathsf{X}$ *of* $\mathbf{QC}_3$, *then* $S \vdash \sim(\forall X)A$.

*Proof:*
(a) Let $\mathsf{X}$ be foreign to $(\forall X)A$, $(\exists X)B$, and $(\exists X)C$. $\mathsf{J}_1(A(\mathsf{X}/X)) \vee (\mathsf{J}_2(A(\mathsf{X}/X)) \vee \mathsf{J}_3(A(\mathsf{X}/X)))$ is valid in the sense of [1]. So by the completeness theorem of [1]

$$\vdash \mathsf{J}_1(A(\mathsf{X}/X)) \vee (\mathsf{J}_2(A(\mathsf{X}/X)) \vee \mathsf{J}_3(A(\mathsf{X}/X))),$$

so by *L3*(h) and the hypothesis on $\mathsf{X}$

$$\vdash (\forall X)\mathsf{J}_1(A) \vee ((\exists X)\mathsf{J}_2(A) \vee (\exists X)\mathsf{J}_3(A)),$$

so by *L1*(a)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash (\forall X)\mathsf{J}_1(A) \vee ((\exists X)\mathsf{J}_2(A) \vee (\exists X)\mathsf{J}_3(A)),$$

so by *L2*(n)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash (\forall X)\mathsf{J}_1(A) \vee (\exists X)\mathsf{J}_2(A),$$

so by *L3*(i) and *L2*(h)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash ((\exists X)\mathsf{J}_1(A)\ \&\ (\forall X)\mathsf{J}_1(A)) \vee (\exists X)\mathsf{J}_2(A),$$

so by *L3*(j) and *L2*(g)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash \mathsf{J}_1((\forall X)A) \vee (\exists X)\mathsf{J}_2(A),$$

so by *L2*(m)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash \mathsf{J}_1((\forall X)A) \vee ((\exists X)\mathsf{J}_2(A)\ \&\ (\forall X) \sim \mathsf{J}_3(A)),$$ [17]

so by *L3*(k) and *L2*(l)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash \mathsf{J}_1((\forall X)A) \vee ((\exists X)\mathsf{J}_2(A)\ \&\ (\forall X)(\mathsf{J}_1(A) \vee \mathsf{J}_2(A))),$$

so by *L3*(j) and *L2*(i)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash \mathsf{J}_1((\forall X)A) \vee \mathsf{J}_2((\forall X)A),$$

so by *L2*(p)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash \sim \mathsf{J}_3((\forall X)A),$$

so by *L1*(c) and *L1*(r)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) \sim \mathsf{J}_3(A)\} \vdash -(\forall X) - \mathsf{J}_3(A),$$

so by *L3*(l) and *L2*(e)

$$\{\mathsf{J}_3((\forall X)A),\ (\forall X) - \mathsf{J}_3(A)\} \vdash -(\forall X) - \mathsf{J}_3(A),$$

so by *L1*(q)

$$\{\mathsf{J}_3((\forall X)A)\} \vdash (\forall X) - \mathsf{J}_3(A)\ \mathrm{I} - (\forall X) - \mathsf{J}_3(A),$$

so by *L2*(o)

$$\{\mathsf{J}_3((\forall X)A)\} \vdash -(\forall X) - \mathsf{J}_3(A),$$

so by *L3*(m) and *L1*(d)

$$\{\mathsf{J}_3((\forall X)A)\} \vdash (\exists X)\mathsf{J}_3(A),$$

so by $L2$(s)

$$\vdash \mathsf{J}_3((\forall X)A) \supset (\exists X)\mathsf{J}_3(A),$$

so by $L3$(n)

$$\vdash \mathsf{J}_3((\forall X)A) \supset \sim(\forall X)(\sim A \supset A),$$

so by $L2$(c)

$$\vdash (\forall X)(\sim A \supset A) \supset (\sim(\forall X)A \supset (\forall X)A),$$

so in particular

$$\vdash (\forall X)(\sim\sim A \supset \sim A) \supset ((\exists X)A \supset (\forall X)\sim A),$$

so by $L3$(o)

$$\vdash (\forall X) - A \supset ((\exists X)A \supset (\forall X)\sim A),$$

so by $L3$(p)

$$\vdash (\forall X) - A \supset - (\exists X)A,$$

so by $L1$(a)

$$S \vdash (\forall X) - A \supset - (\exists X)A.$$

(b) $(A \supset B) \supset ((B \supset C) \supset (((B \supset A) \equiv (C \supset A)) \supset (C \supset B)))$ is valid in the sense of [1]. So by the completeness theorem of [1]

$$\vdash (A \supset B) \supset ((B \supset C) \supset (((B \supset A) \equiv (C \supset A)) \supset (C \supset B))),$$

so in particular

$$\vdash (B \supset (\exists X)(A \supset B)) \supset (((\exists X)(A \supset B) \supset ((\forall X)A \supset B)) \supset ((((\exists X)(A \supset B) \supset B) \equiv (((\forall X)A \supset B) \supset B)) \supset (((\forall X)A \supset B) \supset (\exists X)(A \supset B)))).$$

But

$$\vdash B \supset (\exists X)(A \supset B),$$
$$\vdash (\exists X)(A \supset B) \supset ((\forall X)A \supset B),$$

and

$$\vdash ((\exists X)(A \supset B) \supset B) \equiv (((\forall X)A \supset B) \supset B)$$

by $L3$(q), $L3$(r), and $L3$(s), respectively. So by $L1$(d)

$$\vdash ((\forall X)A \supset B) \supset (\exists X)(A \supset B),$$

so in particular

$$\vdash ((\forall X')A(X'/X) \supset (\forall X)A) \supset (\exists X')(A(X'/X) \supset (\forall X)A),$$

so by $L3$(b) and $L1$(d)

$$\vdash (\exists X')(A(X'/X) \supset (\forall X)A),$$

so by $L1$(a)

$$S \vdash (\exists X')(A(X'/X) \supset (\forall X)A).$$ [18]

(c) $(\forall X)A \supset A(\mathsf{X}/X)$ is an axiom of $\mathbf{QC}_3$. So $S \vdash (\forall X)A \supset A(\mathsf{X}/X)$ by $L1$(a). So (c) by $L1$(d).
(d) $S \vdash (\forall X)A \supset A(\mathsf{X}/X)$ by the same steps as in (c). So (d) by $L2$(d).

**3** Let $S$ be a set of wffs that is syntactically consistent *and* infinitely extendible. We extend $S$ into another set $S^\infty$, then extend $S^\infty$ into yet another set $S_\infty$, and proceed to show all members of $S_\infty$ (hence, all members of $S$) true on a certain truth-value assignment $\alpha$.

Towards defining $S^\infty$, let $S^0$ be $S$; and, $(\forall X_n)A_n$ being the alphabetically $n$-th quantification of $\mathbf{QC}_3$, let $S^n$ be for each $n$ from 1 on $S^{n-1} \cup \{A_n(\mathsf{X}_n/X_n) \supset (\forall X_n)A_n\}$, where $\mathsf{X}_n$ is the alphabetically earliest individual parameter of $\mathbf{QC}_3$ foreign to $S^{n-1}$ and $(\forall X_n)A_n$. $S^\infty$ will then be the union of $S^0, S^1, S^2, \ldots$.

Towards defining $S_\infty$, let $S^0$ be $S^\infty$; and, $A_n$ being the alphabetically $n$-th wff of $\mathbf{QC}_3$, let $S_n$ be for each $n$ from 1 on $S_{n-1} \cup \{A_n\}$ or $S_{n-1}$ according as $S_{n-1} \cup \{A_n\}$ is syntactically consistent or not. $S_\infty$ will then be the union of $S_0, S_1, S_2, \ldots$. It is easily verified that:

(0) $S^\infty$ *is syntactically consistent,*
(1) $S_\infty$ *is syntactically consistent,*

and

(2) $S_\infty$ *is maximally consistent.*

Proof of (1) is as on p. 328 of [1] (but using the syntactic consistency of $S^\infty$ rather than that of $S$); and so is proof of (2). As for (0), suppose $S^n$ to be syntactically inconsistent, and hence by $L1$(t) $-(A_n(\mathsf{X}_n/X_n) \supset (\forall X_n)A_n)$ to be provable from $S^{n-1}$, and let $X_n'$ be the alphabetically earliest individual variable of $\mathbf{QC}_3$ foreign to $(\forall X_n)A_n$. Then by $L3$(c) $S^{n-1} \vdash (\forall X_n') - (A_n(X_n'/X_n) \supset (\forall X_n)A_n)$. But by $L4$(a)

$$S^{n-1} \vdash (\forall X_n') - (A_n(X_n'/X_n) \supset (\forall X_n)A_n) \supset -(\exists X_n')(A_n(X_n'/X_n) \supset (\forall X_n)A_n).$$

So by $L1$(d)

$$S^{n-1} \vdash -(\exists X_n')(A_n(X_n'/X_n) \supset (\forall X_n)A_n),$$

i.e.,

$$S^{n-1} \vdash (\exists X_n')(A_n(X_n'/X_n) \supset (\forall X_n)A_n) \supset {\sim}(\exists X_n')(A_n(X_n'/X_n) \supset (\forall X_n)A_n).$$

But by $L4$(b)

$$S^{n-1} \vdash (\exists X_n')(A_n(X_n'/X_n) \supset (\forall X_n)A_n).$$

So by $L1$(d) $S^{n-1}$ is syntactically inconsistent. So $S^n$ is syntactically consistent if $S^{n-1}$ is. But by assumption $S^0$ is syntactically consistent. So each one of $S^0, S^1, S^2, \ldots$, is syntactically consistent. So, by a familiar argument using $L1$(a) and $L1$(b), $S^\infty$ is syntactically consistent.

Now let $\alpha$ be the result of assigning to each atomic wff $A$ of $\mathbf{QC}_3$ the truth-value 1 if $S_\infty \vdash A$, the truth-value 3 if $S_\infty \vdash {\sim}A$, otherwise the truth-value 2. Mathematical induction on the length $\mathcal{L}(A)$ of an arbitrary wff $A$ of $\mathbf{QC}_3$ will show that:

(i) *If* $S_\infty \vdash A$, $\alpha(A) = 1$,
(ii) *If* $S_\infty \vdash \sim A$, $\alpha(A) = 3$,

and

(iii) *If neither* $S_\infty \vdash A$ *nor* $S_\infty \vdash \sim A$, $\alpha(A) = 2$.

Basis: $\mathcal{L}(A) = 1$. Proof by the construction of $\alpha$.
Inductive Step: $\mathcal{L}(A) > 1$.
Case 1: $A$ is a negation $\sim B$. See Case 1 on p. 328 of [1].
Case 2: $A$ is a conditional $B \supset C$. See Case 2 on p. 328 of [1].
Case 3: $A$ is a quantification $(\forall X)B$. (i) Suppose $S_\infty \vdash (\forall X)B$. Then by $L4$(c) $S_\infty \vdash B(\mathsf{X}/X)$ for every individual parameter $\mathsf{X}$ of $\mathbf{QC}_3$, hence by the hypothesis of the induction $\alpha(B(\mathsf{X}/X)) = 1$ for every such $\mathsf{X}$, and hence $\alpha((\forall X)B) = 1$. (ii) Suppose $S_\infty \vdash \sim(\forall X)B$, and let $\mathsf{X}$ be the alphabetically earliest individual parameter of $\mathbf{QC}_3$ such that $B(\mathsf{X}/X) \supset (\forall X)B$ belongs to $S_\infty$. Then by $L1$(c) $S_\infty \vdash B(\mathsf{X}/X) \supset (\forall X)B$, hence by $L1$(l) and $L1$(d) $S_\infty \vdash \sim(\forall X)B \supset \sim B(\mathsf{X}/X)$, hence by $L1$(d) $S_\infty \vdash \sim B(\mathsf{X}/X)$, hence by the hypothesis of the induction $\alpha(B(\mathsf{X}/X)) = 3$, and hence $\alpha((\forall X)B) = 3$. (iii) Suppose neither $S_\infty \vdash (\forall X)B$ nor $S_\infty \vdash \sim(\forall X)B$. If $\alpha(B(\mathsf{X}/X))$ equaled 3 for any individual parameter $\mathsf{X}$ of $\mathbf{QC}_3$, then by the hypothesis of the induction $\sim B(\mathsf{X}/X)$ would be provable from $S_\infty$ for that $\mathsf{X}$, and hence by $L4$(d) $\sim(\forall X)B$ would be provable from $S_\infty$, against the hypothesis on $\sim(\forall X)B$. If, on the other hand, $\alpha(B(\mathsf{X}/X))$ equaled 1 for every individual parameter $\mathsf{X}$ of $\mathbf{QC}_3$, then by the hypothesis of the induction $B(\mathsf{X}/X)$ would be provable from $S_\infty$ for every such $\mathsf{X}$. But $B(\mathsf{X}/X) \supset (\forall X)B$ is sure to belong to $S_\infty$, and hence by $L1$(c) to be provable from $S_\infty$, for at least one individual parameter $\mathsf{X}$ of $\mathbf{QC}_3$. So, if $\alpha(B(\mathsf{X}/X))$ equaled 1 for every individual parameter $\mathsf{X}$ of $\mathbf{QC}_3$, then by $L1$(d) $(\forall X)B$ would be provable from $S_\infty$, against the hypothesis on $(\forall X)B$. So $\alpha((\forall X)B) = 2$.

Since every member of $S$ belongs to $S_\infty$ and hence by $L1$(c) is provable from $S_\infty$, every member of $S$ is thus sure to evaluate to 1 under $\alpha$. Hence:

*L5. If S is syntactically consistent and infinitely extendible, then S is truth-value verifiable and hence semantically consistent.*

Suppose next that $S$ is syntactically consistent but *not* infinitely extendible; $\mathsf{X}_i$ being for each $i$ from 1 on the alphabetically $i$-th individual parameter of $\mathbf{QC}_3$, let $M$ be the mapping on the individual parameters of $\mathbf{QC}_3$ such that $M(\mathsf{X}_i) = \mathsf{X}_{2i}$; let $M'$ be the restriction of $M$ to the individual parameters of $\mathbf{QC}_3$ occurring in $S$; and let $S'$ be the $M'$-rewrite of $S$. $S'$ is infinitely extendible, and is easily verified to be syntactically consistent if—as presumed here—$S$ is. So by $L5$ $S'$ is truth-value verifiable. But $S'$ is isomorphic to $S$. So $S$ is semantically consistent.

So, whether or not $S$ is infinitely extendible,

*L6. If S is syntactically consistent, then S is semantically consistent.*

So, by the same argument as on p. 329 in [1]:

*T1. If $S$ entails $A$, then $S \vdash A$.* (Strong Completeness Theorem for $\mathbf{QC}_3$)

So, taking $S$ to be $\emptyset$:

*T2. If $A$ is valid, then $\vdash A$.* (Weak Completeness Theorem for $\mathbf{QC}_3$)

## NOTES

1. Part I of the paper appeared in this Journal (see vol. XV (1974), pp. 325-330) under the title "A strong completeness theorem for 3-valued logic"; it was co-authored by Harold Goldberg, Hugues Leblanc, and George Weaver. The present results were announced at the 1975 International Symposium on Multiple-Valued Logic, Indiana University, Bloomington, and appear on pp. 388-398 of the Symposium's *Proceedings* (under the title "A Henkin-type completeness proof for 3-valued logic with quantifiers"). The Bloomington text unfortunately is marred by misprints, for which the editors of the *Proceedings* are in no way to be blamed. So publication of a corrected text seemed imperative, and I am grateful to Professor Sobociński for making it possible.

2. I owe thanks to Professor A. R. Turquette, who suggested the proof of *L4*(a) below and that of *L4*(b). I also owe thanks to George Weaver for his counsel and advice throughout the writing of the paper.

3. The result is a generalization (for $\mathbf{QC}_3$) of a result in [5].

4. Our individual variables are in effect what the literature understands by bound individual variables.

5. Our individual parameters are in effect what the literature understands by free individual variables.

6. Our predicate parameters are in effect what the literature understands by free predicate variables, and our predicate parameters of degree 0 are what it understands by free sentence variables.

7. Because of (iii) formulas in which identical quantifiers overlap are not counted well-formed.

8. Following customary practice we shall also write '$\sum_{i=1}^{n} A_i$' for '$((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_n)$'.

9. In [1] we wrote '$\bar{A}$' where we now write '$-A$'.

10. With $A \supset (\forall X)A$ presumed to be well-formed, $X$ here is sure to be foreign to $A$.

11. In [1] we used 1, 1/2, and 0 as our truth-values, but 1, 2, and 3 prove handier here.

12. Given the matrices in [1] for $\sim A$ and $(A \supset B)$, those for $-A$, $\mathrm{J}_1(A)$, $\mathrm{J}_2(A)$, and $\mathrm{J}_3(A)$ respectively run:

| $A$ | $-A$ | $\mathrm{J}_1(A)$ | $\mathrm{J}_2(A)$ | $\mathrm{J}_3(A)$ |
|---|---|---|---|---|
| 1 | 3 | 1 | 3 | 3 |
| 2 | 1 | 3 | 1 | 3 |
| 3 | 1 | 3 | 3 | 1 |

13. Our interpretation of $(\forall X)$—like that in [5]—is thus of the substitutional sort, and our semantics for $\mathbf{QC}_3$ is of the truth-value sort. For a brief introduction to truth-value semantics, see [3].

14. Here, as in two-valued logic, *some* syntactically consistent sets of wffs are *not* truth-value verifiable: a case in point is $\{f(x_1), f(x_2), f(x_3), \ldots, \sim(\forall x) f(x)\}$, where '$f$' is a predicate parameter of degree 1, '$x_1$', '$x_2$', '$x_3$', etc. are all the individual parameters of $\mathbf{QC}_3$, and '$x$' is an individual variable. But, as we shall establish below, *all* syntactically consistent sets of wffs *are* semantically consistent in the sense just defined. For alternative accounts of semantic consistency in truth-value semantics, see [2].

15. *L3*(c)-(d) guarantee that any wff of $\mathbf{QC}_3$ provable by the "axiomatic stipulation" on p. 88 of [5] is provable here, and vice-versa. With $(\forall X)(A \supset B) \supset (A \supset (\forall X)B)$ presumed to be well-formed, $X$ here is sure to be foreign to $A$.

16. With $(\forall X)(A \vee B) \supset (A \vee (\forall X)B)$ presumed to be well-formed, $X$ here is sure to be foreign to $A$.

17. From this point on the proof of *L4*(a) is due to Professor Turquette.

18. The entire proof of *L4*(b) is due to Professor Turquette.

## REFERENCES

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