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REFLECTIONS ON AN EXTENSIONALITY THEOREM

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1 Like any artifact, a formal language may not coincide with the intention with which it was constructed, either by having unanticipated but important properties or by lacking properties it was intended to have. Some theorems about a formal language assert, directly or indirectly, that the language has certain intended properties. Formulation of such a theorem brings it to explicit awareness that the language was to have a certain property. Recognition of the need for a proof of the theorem is the recognition that lacking a proof one simply does not know whether the language actually constructed is the language one intended to construct. The proof itself is a check that we have constructed what we intended to construct.

The theorem this paper investigates functions as a lemma in the soundness proof for the language $\mathcal{L}$ of Mates' rigorous and concise text *Elementary Logic* [1]. Its versions for other systems will be obvious enough not to require separate comment.

2 Of the logic texts with which I am familiar only Mates [1] explicitly notes this theorem. But it does not comment on its general content, significance, or need of proof even apart from its role in proving soundness. This paper aims at filling those gaps.

The proof of the theorem is both lengthy and complex (that is true of the only proof I have been able to devise; Mates has told me that his proof shares those properties) and is omitted here. My sole present concern is to reflect on the significance of the theorem and its need of proof.

I do not know whether these reflections belong only to the technical and not to the philosophical side of logic. My main points are ones of which I was long unaware and my conversations with other philosophers interested in logic lead me to guess that many of them may also be unware of these points. If this is so, there will be some value in making these points obvious.

3 $\mathcal{L}$ is a standard first-order language including predicates, individual constants and variables, connectives, and quantifiers. 'Formula', 'free

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occurrence of a variable', 'sentence', etc., are defined so as to give them their expected extensions in $\mathcal{L}$. A derivation is a finite sequence of lines on each of which occurs a sentence. A model-theoretic semantics is provided which serves to define 'true under an interpretation' for the sentences of $\mathcal{L}$, and therewith also such terms as 'valid' and 'consequence'.

The following clause defines 'true under an interpretation' for universal quantifications:

Truth clause If $\Psi = (\alpha)\Phi$, then Ψ is true under an interpretation I iff $\Phi\alpha/\beta$ is true under every β-variant of I; β being the first individual constant not in Φ.

Here $\Phi\alpha/\beta$ denotes the result of replacing each free occurrence of variable α in formula Φ by constant β. A β-variant of an interpretation I is any interpretation different from I at most in what it assigns to β.

The rule of universal specification (**US**) runs as follows:

US For any individual constant ω, the sentence $\Phi\alpha/\omega$ may be entered on a line in a derivation if the sentence $(\alpha)\Phi$ appears on an earlier line: the premises of the new line are all those of that earlier line.

An informal account of the soundness of **US**: The sentence $(\alpha)\Phi$ is true under I iff Φ is true of every object in the domain D of I. The sentence $\Phi\alpha/\omega$ is true under I iff Φ is true of the element of D which I assigns to ω. Hence, for any constant ω, if $(\alpha)\Phi$ is true under I, so is $\Phi\alpha/\omega$. Indeed, no rule is more directly perceived to be sound then is **US**. If everything is Φ, so is ω. If every number has a unique prime factorization, so does 33.

The proof, however, is not such smooth sailing, for the truth clause speaks of a *single* constant, whereas **US** speaks of *any* constant. By the truth clause the sentence $(x)(y)(Fxy \rightarrow Gxy)$ is true under I iff $(y)(Fay \rightarrow Gay)$ is true under every a-variant of I. By **US**, however, the distinct sentence $(y)(Fcy \rightarrow Gcy)$ may be derived. But what shows that if the former sentence is true under every a-variant of I, then the latter sentence is true under I? This is the difficulty—the solution of which leads us to the theorem with which this paper is concerned.

It might be thought that the difficulty could be eliminated by some change in the truth clause. Suppose we assert that $(\alpha)\Phi$ is true under I iff $\Phi\alpha/\beta$ is true under I for every individual constant β. With this the difficulty vanishes, but now the truth clause is substitutional and thus fails when the domain of I is infinite but not denumerable. Indeed, our difficulty arises precisely in the context of objectual quantification. Its solution is, in effect, a proof that the left to right implication of the substitutional clause is a consequence of the objectual clause.

Suppose we assert that $(\alpha)\Phi$ is true under I iff $\Phi\alpha/\beta$ is true under every β-variant of I for some individual constant β. The difficulty remains, for we now need to prove that if $\Phi\alpha/\beta$ is true under every β-variant of I for *some* constant β, then $\Phi\alpha/\omega$ is true under I for every constant ω.

Suppose we assert that $(\alpha)\Phi$ is true under I iff $\Phi\alpha/\beta$ is true under every β-variant of I for every constant β. The difficulty vanishes, for among the constants β is the constant introduced by **US** and I is among its β-variants. But this new clause gives wrong results.

Proof: Let D of $I = \{1, 2\}$, $I(F) = \{\langle 1, 1\rangle, \langle 1, 2\rangle\}$, and $I(a) = 1$. Then the sentence $(x)Fax$ should be true under I. The amended truth-clause asserts that $(x)Fax$ is true under I iff the result of replacing x in Fax by any constant β is true under every β-variant of I. Let $\beta = a$ and let I' be the a-variant of I assigning 2 to a. Now, Faa is true under I' iff $\langle I'(a), I'(a)\rangle \in I'(F)$. But $I'(F) = I(F)$. Thus, Faa is not true under I' and thus the amended truth-clause declares that $(x)Fax$ is not true under I.

It thus appears that there is no acceptable amendment of the truth-clause which eliminates the difficulty. Further, it is clear that **US** is, as it stands, precisely the rule of inference we desire. The difficulty cannot be avoided but must be directly dealt with.

4 We need to prove: if $\Phi\alpha/\beta$ is true under every β-variant of I and β is the first constant not in Φ, then $\Phi\alpha/\omega$ is true under I for any constant ω. To prove this it is enough to prove: If $\Phi\alpha/\beta$ is true under that β-variant of I which assigns to β what I assigns to ω, then $\Phi\alpha/\omega$ is true under I. Now, that β-variant differs at most from I in what it assigns to β and assigns to β what I assigns to ω; hence, all we need show, in effect, is that the shift from one constant to another unaccompanied by any shift in denotation does not affect truth under an interpretation. That is, we need merely show that if I and I' are interpretations differing at most in what they assign to β and $I(\omega) = I'(\beta)$, then for any sentences Φ' and Φ such that Φ is just like Φ' except for having occurrences of ω wherever Φ' has occurrences of β, Φ is true under I iff Φ' is true under I'. As it turns out one must prove this theorem as an instance of its generalization to the interchange of any finite number of constants.

The basic theorem, then, is this (Proposition 8, p. 66, of [1] *EL*):

Theorem *If a sentence Φ is like Φ' except for having the constants $\omega_1, \omega_2, \ldots, \omega_n$ wherever Φ' has, respectively, the distinct constants $\beta_1, \beta_2, \ldots, \beta_n$, and if an interpretation I' is like an interpretation I except that it assigns to $\beta_1, \beta_2, \ldots, \beta_n$, respectively, what I assigns to $\omega_1, \omega_2, \ldots, \omega_n$, then Φ is true under I iff Φ' is true under I'.*

(It is worth noting in passing that the generalization of this theorem to cover substitutions in the sentences in infinite sets of sentences plays a key role in the *EL* version of the Henkin completeness proof; see the second full paragraph, p. 144 of *EL*.)

Solution for a single case: Fa and Fb are sentences differing only in the interchange of constants. Let I and I' be interpretations which differ at most in what they assign to b and let $I'(b) = I(a)$. Since I and I' differ only in what they assign to b, $I(F) = I'(F)$. By the truth clause for atomic sentences Fa and Fb are respectively true under I and I' iff $I(a) \in I(F)$ and $I'(b) \in I'(F)$. Thus, Fa is true under I iff Fb is true under I'.

In effect, Fa and Fb have the same predicate and a and b denote the same object; hence, *Fa* is true iff *Fb* is true. That is, these sentences differ only in the orthography of their constants, by containing different marks for the same object; and the proof shows us that *this* difference makes no difference at the level of truth and falsity.

5 Let us take it for granted that distinct tokens of the same type have the same *significance*. Thus, each token of the type *a* has the same significance and thus also the same denotation, that being an element in the significance of a constant. Tokens of distinct types may also agree in denotation (a possibility marked by the possibility of interpretations assigning the same object to different constants). But tokens of distinct types which agree in denotation may yet differ in significance, e.g., the orthographical difference of constants the same in denotation may mark a further difference in sense, meaning, connotation, etc. Let us here speak of non-denotative differences in significance. A language for which non-denotative differences in significance have no effect upon truth-value is extensional. Let us call this property in respect of individual constants c-extensionality.

The basic content of the key theorem is now clear: it asserts the c-extensionality of $\mathcal{L}$. That this is a theorem for which we need a proof is plain from the point of view of a proof of soundness, so we can determine whether or not the system we have constructed is sound.

But quite apart from its role in a soundness proof, it should be plain that we construct $\mathcal{L}$ with the intention that it should be c-extensional, so that we need a proof of the theorem in any case. Indeed, the c-extensionality of $\mathcal{L}$ is one of its fundamental properties. To lack a proof of its c-extensionality would mean that we do not know whether $\mathcal{L}$ is even the kind of language we intended it to be.

6 That $\mathcal{L}$ was *to be* c-extensional was a point to which I was oblivious for a long time. I simply believed *that it was*. That—antecedent to proof—this was at best a hope was not clear to me. That the point might be crystallized into a theorem and then proved never entered my mind. Even after sharply recognizing the need to prove the key theorem in order to prove soundness, for a while it appeared to me only as a technical problem, which I thought was probably due to some complexity forced by the formalism. It was only after a certain amount of reflection on the theorem that I came to see that it asserted a fundamental property of $\mathcal{L}$ and would stand in need of proof even apart from its role in the soundness proof.

I thus was, for a long while, blind to the obvious. My conversations with other philosophers interested in logic seem to show that they too tended to this blindness. Why is this so? I think the blindness has its source in a certain way of thinking about predicates. It is natural to first think of predicates against the background of sentences and names. A predicate is what results from deleting one or more occurrences of one or more names from a sentence. The sentence 'Theaetetus sits' yields the predicate 'sits' by deletion of the name 'Theaetetus'. The resulting predicate is then thought of as true or false of the reference of the name. Thus, abstracting from the name, we think of predicates as expressions true or false of objects whether named or however named. Returning to sentences, we then think of sentences formed from the same predicate by completing it with names of the same objects as the same in truth-value, even if the names differ. But to this there are the common counterexamples. We thus declare that those sentences do not yield predicates by the deletion of names and talk about non-extensional contexts. By 'predicate', then, we come to mean an extensional structure resulting from the right kind of sentence by the deletion of names.

Turning to our formal construction, we carry over this idea and just "see" that the sentence $(y)(Fby \rightarrow Gby)$ yields the predicate $(y)(F - y \rightarrow G - y)$ which is either true or false of a given object 0, however named, so that the sentences $(y)(Fay \rightarrow Gay)$ and $(y)(Fcy \rightarrow Gcy)$ agree in truth-value if a and c each name 0.

What helps to foster this illusion of insight is the fact that the truth-clause for atomic sentences directly yields the right result, for that clause is designed to assure the extensionality of simple predicates and hence that $\mathcal{L}$ is c-extensional at the atomic level. Thus, when one points out the need for the theorem of c-extensionality one is likely to be met with the rejoinder that the problem doesn't amount to much, for, it will be suggested, $(\alpha)\Phi$ contains the predicate Φ and a predicate is simply true or false of objects, however named. Thus, since the truth of $(\alpha)\Phi$ assures the truth of $\Phi\alpha/\beta$ for any denotation of β it also assures the truth of $\Phi\alpha/\omega$ since the denotation of ω is among those of β.

This line of thought is correct when Φ is a formula formed from a single predicate letter with the use of the single variable α; for in that case the only truth clause to which we need appeal is the truth clause for atomic sentences, which directly assures the right result. But none of the other clauses in the recursion on 'true under an interpretation' thus directly assures the right result. We may view the matter this way: c-extensionality is directly provided for by the truth-clause for atomic sentences. The further clauses are *intended* to preserve this c-extensionality (in effect, they are *intended* to ensure that the infinitely many complex predicates formulable from the simple predicates by use of connectives and quantifiers share the extensionality of those simple predicates). The question is this: have we so constructed our recursion on 'true under an interpretation' to realize this intention?

And, in fact, the proof of the theorem is just a check on each further truth clause as to whether it does its intended job of preserving c-extensionality. For the proof naturally takes the form of an induction starting with atomic sentences and then working through the remaining seven clauses for the connectives and quantifiers.

7 The c-extensionality of $\mathcal{L}$ is a necessary condition of the soundness of $\mathcal{L}$. But quite apart from this, it is a fundamental property of $\mathcal{L}$. That our theorem asserts this property of $\mathcal{L}$ is its content and marks its significance. To recognize the need for a proof of the theorem is to realize that apart from a proof we just do not know whether in constructing $\mathcal{L}$ we have constructed what we intended to construct.

REFERENCE

[1] Mates, Benson, *Elementary Logic,* Oxford University Press, Second Edition, 1972.

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