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REPAIRING PROOFS OF ARROW'S GENERAL IMPOSSIBILITY THEOREM AND ENLARGING THE SCOPE OF THE THEOREM

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The standard and textbook proofs of Arrow's general impossibility theorem are, like the original proofs, invalid. That is the first main claim argued in what follows. In the course of setting out crucial details of a logically adequate proof, however, the usual conditions on the theorem are liberalised, and the extent to which certain other conditions can be generalised or discarded is investigated (for an exact statement see the concluding summary).

The importance of a logically adequate proof is in no way diminished because, as it fortunately turns out, the theorem is correct under the intended (if often inadequately formulated) conditions. But that makes it easy to say that it is trivial to fuss over quantificational details of the standard proofs (proof failure comes ultimately in every case from quantificational errors, omission of necessary quantifiers or mistaken orderings of quantifiers, both major sources of invalidity in logic and mathematics) for every economist knows what is meant by the theorem, that it is essentially correct and its proof intuitively clear, and that a rigorous proof can be produced. The claim is false, as will emerge, even of the economic textbook writers. The textbooks have failed to produce what it is essential to have, especially in the case of a theorem with such far-reaching consequences (even if it is after all only an exercise in second-order quantificational logic), namely a correct and rigorous proof. The history of mathematics is replete with cases where what everyone was thought to know proved false, or where what was intuitively clear turned out to be mistaken or correct only under restrictive conditions.

The troubles with proofs of Arrow's theorem arise from the misuse of conditions required for the impossibility result. Neither the original proofs of Arrow's theorem as to the impossibility of a reasonable social welfare function (set out in [1], pp. 51-59 and pp. 97-100), nor most of the many copies and variations in the literature clarify the central role of the controversial Principle I (the principle of independence of irrelevant

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alternatives) in the proof of the theorem. Though Arrow does appeal to the condition in claiming ([1], p. 98) that his notions of decisiveness are well-defined, as Sen remarks ([8], p. 42),

> Arrow's proof is somewhat opaque, particularly since the use of the crucial condition I (i.e. his Condition 3) is never clarified; in fact this condition is never even mentioned in the proof.[1]

And Sen's own proof ([8], p. 42-46), while it does appeal to Condition I at one of the requisite points, does so in an informal way only, thereby enabling a step not licensed by the stated formal conditions.

In order to see why proofs of the theorem fail it is instructive to examine a typical case—*Case 1 of the dominance lemma*—where proofs break down. Proof of Arrow's theorem can be conveniently split into two lemmas, a dominance lemma (lemma 3*a of Sen [8], according to which if some individual is almost decisive then he is a dictator) and a voting paradox lemma. (The structure of the argument is elaborated below.) *Both* arguments make repeated use of Condition I, but the main neglected or confused use of I occurs in the dominance lemma.

Consider the argument ([1], pp. 98-99; [8], p. 43) that in a three-alternative case (with alternative set $K = \{x, y, z\}$) where individual J is almost decisive for x against y, then J is decisive for x against z, i.e., in symbols:

$$xD_Jy \supset x\bar{D}_Jz \tag{1}$$

where $x\bar{D}_Jz =_{df} xP_Jz \supset xPz$, $xD_Jy =_{df} xP_Jy \;\&\; (i \in F)\,(i \neq J \supset yP_ix) \supset. \; xPy$, and F is the class of factors or individuals.[2] The argument for (1) should go as follows: Suppose that xD_Jy and further that xP_Jz. It suffices to show that xPz, for then (1) follows by two applications of the deduction theorem (see, e.g., Mendelson [4]). The argument Arrow and Sen in fact use assumes more: it is supposed not just that xD_jy, but that xP_Jy, yP_Jz, and yP_ix and yP_iz for $i \in F - \{J\}$. Given these assumptions it is easily shown, using the Pareto condition and transitivity of P, that xPz. It is then claimed that xPz holds whenever xP_Jz "regardless of the orderings of other individuals" ([1], p. 98), whence it is concluded $xP_Jz \supset xPz$. Sen ([8], p. 43) does point out that the argument requires the use of Condition I, but proceeds to use the surprising inference

$$\frac{xP_Jy \;\&\; yP_Jz \;\&\; (i \in F - \{J\})\,(yP_ix \;\&\; yP_iz) \supset. \; xPz}{xP_Jz \supset xPz}$$

instead of the independence Condition I that is cited as a premiss. Condition I is indeed quite different.

The independence condition at issue is effectively as follows ([1], p. 27, [8], p. 41):

Condition I. Let $R_1, \ldots, R_n$ and $R_1^1, \ldots, R_n^1$ be two sets of individual orderings and R and R^1 corresponding social orderings, and let S be any subset of the set K of alternatives. Then,

$$(x, y \in S)\ (i \in F)\ (xR_iy \equiv xR_i^1y) \supset (x \in S) \cdot (y \in S)xRy \equiv (y \in S)xR^1y.$$

Since $C(S, R) =_{df} \{x \in S\colon (y \in S)xRy\}$, the condition given is simply

$$(x, y \in S)\ (i \in F)\ (xR_iy \equiv xR_i^1y) \supset .\ C(S, R) = C(S, R^1).$$

In case environment S is a two-element set, Condition I can be put in the more satisfactory form:

IR. $(x, y \in S)\ (i \in F)\ (xR_iy \equiv xR_i^1y) \supset (x, y \in S)\ (xRy \equiv xR^1y)$

or, equivalently,

IP. $(x, y \in S)\ (i \in F)\ (xP_iy \equiv xP_i^1y) \supset (x, y \in S)\ (xPy \equiv xP^1y)$,

where, for each i, $xP_iy =_{df} \sim yR_ix$, and $xPy =_{df} \sim yRx$ (*cf.*, [1], p. 14). I follows from IR immediately, whatever set S, by quantifier distribution. For the converse it is convenient to apply the following result (lemma 2 of [1]):

Lemma *Where* $S = \{x, y\}$, $xPy \equiv .\ x = \iota z(z \in C(S, R))$, *i.e.*, xPy *iff* x *is the sole element of* $C(S, R)$.

Proof: as in [1], p. 15.

Lemma *Where S is a two-element set, IR is deductively equivalent to I.*

Proof: It remains to show given I that IP is derivable. Suppose $S = \{x, y\}$ and that $(x, y \in S)\ (i \in F)\ (xP_iy \equiv xP_i^1y)$. Then $C(S, R) = C(S, R^1)$. It is enough to prove that $xPy \equiv xP^1y$, for then IP will follow by quantifier generalisation and distribution (or by generalisation and the deduction theorem). But

$$\begin{aligned} xPy &\equiv .\ x = \iota z(z \in C(S, R)) \\ &\equiv .\ x = \iota z(z \in C(S, R^1)) \\ &\equiv\ xP^1y, \text{ upon applying the previous lemma.} \end{aligned}$$

More generally, IR is deductively equivalent to the following strengthened form of I, namely

IS. $(x, y \in S)\ (i \in F)\ (xR_iy \equiv xR_i^1y) \supset .\ (S^1 \subseteq S)\ .C(S^1, R) = C(S^1, R^1)$.

IS follows given IR simply by restriction principles and quantifier distribution. For the converse, the proof of the lemma is copied, and an appropriate two-element set $S^1 = \{x, y\}$, which IS supplies, is selected. In no details which follow, however, will I be required for more than two-element sets. Accordingly, the directly applicable principle IP can be used.

Consider now how Principle IP—or I—might be applied in Case 1 to the two-element set $S = \{x, z\} \subseteq K$. The assumption of further premisses xP_Jy, yP_Jz, and so on, can be avoided simply by defining a new ordering P_i^1 which agrees with P_i on S, and this is plainly what IP invites one to do. To begin again, then, on Case 1, suppose xP_Jy. Let P_i^1, for $i \in F$, be a new ordering on K which agrees with P_i on S and for which xP_J^1y, yP_J^1z and, for every $i \in F - \{J\}$, yP_i^1x and yP_i^1z. That there is such an ordering to be considered is assured by the following condition (Arrow's Condition 1′, [1], p. 96, though his Condition 1 suffices for the case in question):

Condition U. The domain of the social welfare function includes all consistent [admissible] sets of individual orderings ([8], p. 41).

A quantificational model, with a three-element domain $K = \{x, y, z\}$, guarantees the consistency of the orderings P_i^1 for $i \in F$; and then U guarantees a corresponding social ordering, P^1 on K.

Now, it would seem, the original argument with P_i and P can be reapplied with P_i^1 and P^1, whence it follows that xP^1z. However, P_i^1 agrees with P_i for $i \in F$ on S, i.e., $(x_1 z \in S)\ (i \in F)\ (xP_iz \equiv xP_i^1z)$, whence, by IP, $(x, z \in S)\ (xPz \equiv xP^1z)$. Hence as xP^1z, xPz.

There is just one major hitch to this argument. The argument is premissed on the assumption that xD_Jy, but for the new argument using P_i^1 to succeed the premiss xD_J^1y is required. Since xP_J^1y & $(i \neq J)yP_i^1x$, this further premiss requires precisely that xP^1y. Unfortunately for standard proofs of Arrow's theorem, nothing guarantees that xP^1y. For the proof to go through it has to be *assumed* that xP^1y (or that xD_J^1y); and this requires an assumption, not covered by condition U which concerns only individual rankings[3], as to the admitted class of *social* rankings. So the argument, as it stands, fails.

It is not difficult to see what has gone wrong. The hypothesis is that xD_Jy has to extend to other rankings on K for the argument to work; but U, as a requirement on individual rankings only, cannot assure us that it does.

It is one thing to show that a given class of arguments breaks down, namely familiar arguments for a lemma in Arrow's theorem; it is quite another, however, to show that no repair can be effected without amending the lemma. The further thesis that the given assumptions are insufficient for the proof of the lemma can however be established by the method of counter-model.

It is easy to show, by a counter-model, that the assumed lemma itself cannot succeed. Observe that the argument for the dominance lemma in no way depends on the—undesirable—requirement of strong transitivity, i.e., for x, y, z in K,

$$xPy \mathbin{\&} {\sim}zPy \supset . \ xPz,$$

but only the more reasonable transitivity condition[4]:

$$xPy \mathbin{\&} yPz \supset . \ xPz, \text{ for } x, y, z \text{ in } K.$$

If the given proofs of the lemma could succeed, they would also succeed for a weakened lemma with a social welfare function differing from Arrow's only in substituting transitivity for strong transitivity. But they cannot, as is now shown.

Consider the pure Pareto social welfare ranking determined for each set of individual rankings P_i as follows: for x, y in K

$$xPy \text{ iff } (i \in F)xP_iy. \qquad \text{(PP)}$$

It has first to be shown that PP does indeed provide a ranking which satisfies principles P, I, U, and D.[5] P is evidently a function defined on $\{P_i \colon i \in F\}$.

ad transitivity. Suppose xPy and yPz. Then $(i \in F)\ (xP_iy)$ & $(i \in F)\ (yP_iz)$, so $(i \in F)\ (xP_iy\ \&\ yP_iz)$, whence by transitivity of P_i, $(i \in F)\ (xP_iz)$, that is, xPz.

ad asymmetry. Suppose xPy and also, for a *reductio*, yPx. Then $(i \in F)$ $(xP_iy\ \&\ yP_ix)$, which is impossible since P_i is asymmetric.

Thus P is indeed a strict partial ordering. It may not however be strongly transitive. For suppose $\sim xP_Jz$ and $\sim xP_Ky$ with $J \neq K$, $F = \{K, J\}$ and xP_Jy and xP_Ky. (Then, by strong transitivity on P_K and P_J, xP_Kz and zP_Jy.) By PP, xPy and $\sim xPy$ but $\sim xPz$, whereas by strong transitivity xPz.[6]

ad P. Immediate.

ad I. Suppose $(x, y \in S \subseteq K)\ (i \in F)\ (xP_iy \equiv xP_i^1y)$ - (i). Suppose further that for some w, z in S, $\sim(wPz \equiv wP^1z)$, say wPz and $\sim wP^1z$. Then, by PP, for some $i \in F$, $\sim wP_i^1z$, but wP_iz, contradicting (i).

ad U. P is always defined for every admissible ordering.

ad D.[7] Consider an arbitrary J and let xP_Jy, and consider a ranking where $\sim xP_Ky$. Then $\sim xPy$, by PP. So J is not a dictator.

Reconsider Case 1 where PP determines the overall ordering, and suppose that $F = \{J, K\}$, that xP_Jz, xI_Jy, and yP_Jz, and that K is indifferent between x, y, z, i.e., xI_Ky, yI_Kz, and xI_Kz, where as usual xI_iy iff $\sim xP_iy$ & $\sim yP_ix$. Then xD_Jy, since $\sim xP_Jy$, xP_Jy, but, by PP, $\sim xPy$. That is, xD_Jy & $x\bar{D}_Jy$, contradicting (1). Hence the weakened "lemma" is invalid.

The key to a repair to the proof can be guessed from what is required to show that J is a dictator, and may be located in the original defective Arrow proof ([1], p. 52 in particular). The trouble lies in the definitions of *decisive* and *almost decisive*, which should have been defined, like dictatorship, not with respect to a single ranking but with respect to the set of all admissible rankings; thus, for $J \in F$ and $x, y \in K$:

$x\bar{D}_Jy$ iff, for every set of admissible rankings such that xP_Jy then xPy, i.e., in the notation of [6], iff $(\mathbf{p})\,(xP_Jy \supset xPy)$. Similarly,

$$xD_Jy \text{ iff } (\mathbf{p})\,(xP_Jy\ \&\ (i \in F - \{J\})\ (yP_ix) \supset .\ xPy).^8$$

Formal precision may be had through the following definitions:

1. A *decision structure* (d.s.) is a system $\mathbf{C} = \langle K, F\rangle$ where K and F are non-null sets (of alternatives, and factors or individuals, respectively). $\mathbf{C}$ is *usual* if K has more than two elements.

2. A *ranking* $\mathbf{p}$ *on* a d.s. $\mathbf{C}$ is a function which assigns for each factor i in F, a two-place relation $P_i = \mathbf{p}(i)$, called a *factor* ranking, on $K \times K$, which is a strict partial ordering, i.e., for i in F, P_i is transitive and asymmetric on K.

3. A *general ranking method* (GRM) on a d.s. $\mathbf{C}$ is a function which assigns for each ranking $\mathbf{p}$ on $\mathbf{C}$ an overall ranking P on $K \times K$ which is

strongly transitive and asymmetric. A GRM would provide a universal social welfare function.

4. An *(Arrow) rational* GRM on a d.s. is a GRM which satisfies the following conditions:

P (Pareto): $(\mathbf{p})(x, y \in K).\ (i \in F)xP_i y \supset .\ xPy.$

IP (Independence): $(\mathbf{p})(\mathbf{p}^1)(S \subseteq K).(x, y \in S)(i \in F)(xP_i y \equiv xP_i^1 y) \supset (x, y \in S)(xPy \equiv xP^1 y).$

M (Multiple-value or non-dictatorship): $(i \in F) \sim (\mathbf{p})(x, y \in K).xP_i y \supset xPy.$

Factor J in F is *dominant* (dictator) *in* $S \subseteq K$ iff $(\mathbf{p})(x, y \in S)xP_J y \supset xPy$; and J is *dominant* iff J is dominant in K. According to M no factor is dominant.

Impossibility Theorem *There is no rational GRM on any usual decision structure.*

The critical corrections to standard "proofs" of the theorems concern the following definitions. Where $V \subseteq F$ and $x, y \in K$,

$$x\bar{D}_v y =_{df} (\mathbf{p}).(i \in V)xP_i y \supset .\ xPy,$$
$$xD_v y =_{df} (\mathbf{p}).((i \in V)xP_i y \ \&\ (i \notin V)yP_i x) \supset xPy.$$

V is *almost decisive* iff (in K) $(Px, y \in K)xD_v y$, i.e., iff for some alternatives x and y, V is almost decisive for x against y. For $J \in F$, $\bar{D}_J =_{df} \bar{D}_{\{J\}}$ and $D_J =_{df} D_{\{J\}}$. Alternatively, these further definitions can be avoided by use of Quine's set theory ML of [5] as logic.

Dominance Lemma *Given that there is a GRM satisfying P and IP on a usual d.s.* $\mathbf{C} = \langle K, F\rangle$, *if some* J *in* F *is almost decisive then* J *is dominant.*

What has to be shown is of the form

$$(\mathbf{p})A(\mathbf{p}) \supset (\mathbf{p})B(\mathbf{p}),$$

not—as the standard "proofs" assume—of the stronger erroneous form

$$A(\mathbf{p}) \supset B(\mathbf{p}).$$

It is enough to show, however, how to repair standard proofs.

Proof: Suppose J is almost decisive. Then for some elements, say x and y in K, $xD_J y$. Since $\mathbf{C}$ is usual it contains at least one other element; let z be an arbitrarily chosen element of K distinct from x and y:

Central case. J is dominant in $K' = \{x, y, z\} \subseteq K$.

This is proved by exhaustion of cases, i.e., for every pair $\langle u, v\rangle \subseteq K'$, $u\bar{D}_J v$. Since $u\bar{D}_J u$, it can be assumed that $u \neq v$; so there are six cases:

Case 1: *ad* $xD_J z$. Suppose yet again, but now for an arbitrarily chosen ranking $\mathbf{p}$ in $\mathbf{C}$, that $xP_J y$. Let $S = \{x, z\}$, and let $\mathbf{p}'$ be another ranking which agrees with $\mathbf{p}$ on S, i.e., $(u, v \in S)(i \in F)(uP_i v \equiv uP_i' v)$, and such that $xP_J^1 y$, $yP_J' z$ and, for every $i \in F - \{J\}$, $yP_i' x$ and $yP_i' z$. By P, $yP'z$, and now since xD_Jy instantiating at $\mathbf{p}'$, $xP'y$. Hence, by transitivity of P', $xP'z$. Then as before by IP, xPz. That is, for arbitrarily chosen $\mathbf{p}, xP_Jz \supset xPz$, so $x\bar{D}_Jz$, as required. Hence too, eliminating the hypothesis xD_Jy, $xD_Jy \supset x\bar{D}_Jz$.

Case 2: *ad* $z\bar{D}_Jy$. Suppose for an arbitrary ranking $\mathbf{p}$ in $\mathbf{C}$, that zP_Jy: to show zPy. Let $\mathbf{p}'$ be another ranking which agrees with $\mathbf{p}$ in $S = \{z, y\}$, for which $zP_J'x$ and $xP_J'y$ and, for every $i \in F - \{J\}$, $zP_i'x$ and $yP_i'x$. A three-element relational model shows that $\mathbf{p}'$ is admissible. Since $zP_J'x$, by transitivity, $zP'x$ by $\mathbf{P}$. Since xD_Jy and also $xP_J'y$ and $yP_i'x$ for $i \in F - \{J\}$, $xP'y$. Hence, by transitivity, $zP'x$. Thus by IP, zPx, as in Case 1. Hence too, eliminating the main hypothesis, $xD_Jy \supset z\bar{D}_Jy$.

The remaining cases 3-6 now follow by symmetry or permutations on the first two cases (see, e.g., Sen [8]).

General case. $u\bar{D}_Jv$ for arbitrary u and v in K. If $u = v$ the result is immediate, and if $\{u, v\} = \{x, y\}$ the central case establishes the result. If one of u and v coincides with one of x and y, say $x = u$ but $v \neq y$, then the central case with $K' = \{x, y, v\}$ establishes the result. Otherwise where neither u nor v coincides with x or y, apply the central case to $K' = \{x, y, u\}$, whence $x\bar{D}_Ju$. But then xD_Ju, so the central case can be reapplied with $K'' = \{x, u, v\}$ to yield $u\bar{D}_Jv$.

Corollary A *Given that there is a rational GRM on a usual d.s.* $\langle K, F\rangle$, *no factor in F is almost decisive.*

The independence condition IP also plays a critical role in the next lemma. A *minimal* [*almost*] *decisive* set (M[A]D set) is a non-null [almost] decisive set with no non-null proper subset which is [almost] decisive.

Voting Paradox Lemma *Given that there is a GRM satisfying IP in a usual d.s.* $\mathbf{C} = \langle K, F\rangle$, *then there are rankings on* $\mathbf{C}$ *for which any MAD set is a singleton.*

Proof: Let V be any MAD set, let x and y be elements of K such that V is almost decisive for x against y, and let z be an arbitrarily chosen element of K, ensured by usualness, distinct from x and y.

If V contains only one factor the lemma is proved, so suppose V contains more than one factor. Define the following sets:

$V_1 =_{df} \xi i (i \in U)$, i.e., V_1 is an arbitrarily selected factor in V. In fact, V_1 can be defined as any proper subset of V; it does not have to be a singleton.

$V_2 =_{df} V - V_1$

$V_3 =_{df} F - V$. V_3 may be null.

Consider next—one ranking suffices for the lemma—the following (voting paradox) factor rankings on $\mathbf{C}$ for the given distinct elements x, y, z of K:

for every $j \in V_1$, xP_jy and yP_jz
for every $j \in V_2$, zP_jx and xP_jy
for every $j \in V_3$, yP_jz and zP_jx.

A quantificational model establishes the admissibility of this ranking $\mathbf{p}$ on C. The ranking $\mathbf{p}$ determines several overall rankings:

ad xPy. For every i in V, i.e., in $V_1 \cup V_2$, xP_iy, and for every j not in V, i.e., in V_3, yP_jx, by factor transitivity. Hence since V is almost decisive for x against y, xPy, upon using the definition of almost decisive and instantiating to $\mathbf{p}$.

ad $\sim zPy$. Suppose on the contrary zPy. Then V_2 is an almost decisive set. For suppose for arbitrary ranking $\mathbf{p}'$, $(i \in V_2)\, zP_i'y$ & $(i \notin V_2)\, yP_i'z$. It has to be shown that $zP'y$. Firstly

$$zP_iy \text{ iff } zP_i'y. \tag{a}$$

Suppose zP_iy. Then $i \in V_2$. For suppose $i \notin V_2$. Then since $(i \in V_2)\, zP_iy$ & $(i \notin V_2) yP_iz$, yP_iz whence by asymmetry $\sim zP_iy$. Since $i \in V_2$, $zP_i'y$. Suppose conversely $zP_i'y$. Then $i \in V_2$. For suppose otherwise $i \notin V_2$. Then $yP_i'z$, whence $\sim zP_i'y$. Since $i \in V_2$, zP_iy. Similarly,

$$yP_iz \text{ iff } yP_i'z. \tag{b}$$

Apply principle IP to set $S = \{y, z\}$. Then, since (a) and (b) hold,

$$zPy \text{ iff } zP'y.$$

Hence $zP'y$ is required, and V_2 is almost decisive. But since V_2 is not null and $V_2 \subset V$, V_2 is smaller than the minimal almost decisive set, which is impossible.

ad xPz. From xPy and $\sim zPy$ by strong transitivity. This is the only point in the proof of Arrow's theorem where strong transitivity is required; and its use, as shown below, is essential.

ad V_1 is almost decisive for x against z. The argument is similar to that showing that V_2 is almost decisive. Suppose for arbitrary ranking p', $(i \in V_1)\, xP_i'z$ & $(i \notin V_1)\, zP_i'x$; to show $xP'z$. Since xPz, the desired result will follow by IP if it can be shown for every u, $v \in S = \{x, z\}$—and every $i \in F$,

$$uP_iv \text{ iff } uP_i'v.$$

Since P_i and P_i' are irreflexive, only two cases require further argument, and the following case is typical:

$$zP_ix \text{ iff } zP_i'x. \tag{c}$$

But by hypothesis for $i \in V_1$, $xP_i'z$ and xP_iz and, for $i \notin V_1$, $xP_i'x$ and zP_iz, whence (c) follows.

Since V_1 is non-null and $V_1 \subset V$, the assumption that V is minimal is contradicted. Hence V cannot contain more than one factor.

Note that the argument showing that V_1 and V_2 are almost decisive breaks down if one attempts to substitute decisive for almost decisive sets. With decisive sets, equivalences, such as (c) required for the application of IP, are no longer ensured. A slightly simpler proof of the Lemma, and of the Impossibility Theorem, does however result if almost decisive sets are replaced by exactly decisive sets, where V is *exactly decisive* for x against y, in symbols xE_Vy, iff (i) $(i \in V \equiv. xP_iy)$.

Corollary B *Given that there is a GRM satisfying IP and P on a usual d.s. $\langle K, F\rangle$, then there is a factor in F which is almost decisive.*

Proof: Consider set F. It is non-null, and by Condition P it is decisive, and so almost decisive. Where F is finite, by exhaustive elimination some non-null subset M of F is a MAD set. (For, in brief, if $F_j \subseteq F$ is not minimal then some non-null subset F_{j+1} of F_j is also almost decisive. Since F is finite the sequence of non-null almost decisive sets $F_1(=F)$, $F_2, \ldots, F_k$ must terminate in some set $M = F_k$.)

Where F is not finite a more complicated argument using the axiom of choice can be appealed to. Zorn's lemma, which follows from the axiom of choice, is applied in the following form:

(d) Any non-null partially ordered set in which every chain (i.e., every totally-ordered subset) has a lower bound has a minimal element (*cf.*, Mendelson [4], p. 198), to show, firstly,

(e) For any ordered pair $\langle x, y\rangle$ of distinct alternatives $x, y \in K$, there is a non-null MAD set $V_{\langle x,y\rangle}$ for x as against y (i.e., $V_{\langle x,y\rangle}$ is an almost decisive set for x as against y but no non-null subset $V_{\langle x,y\rangle}$ is almost decisive for x as against y).

Consider the class D of all sets of factors which are almost decisive for x against y. D is non-null. For, by Condition P, the class F of all factors is almost decisive for x against y. Further, D is partially-ordered by set inclusion. Let C be an arbitrary chain in D. Then C is bounded below by its intersection $\cap_c$. It is immediate, from the definition of intersection that for every element $C_i \in C$, $\cap_c \subseteq C_i$. It remains to show that $\cap_c \in D$. Suppose then for every $i \in \cap_c$, xP_iy, and every $j \notin \cap_c$, yP_jx. It suffices to show xPy. Since $i \in \cap_c$ implies $i \in C_k$ for each $C_k \in C$, for every element C_k of the chain xP_iy for every $i \in C_k$. (In short, the conditions for the application of Zorn's lemma (c) are met in the case of sets decisive for x against y.) The axiom of choice ensures the well-ordering of the elements of F not in $\cap_c$, i.e., in $X = F - \cap_c$, under some relation R. Let j be the first element in the well-ordering not *dealt with*, i.e., lacking the property P of being dealt with. Since $j \notin \cap_c$ there is some $J \in C$, with J included in every $H \in C$ induced by elements F_{j-} dealt with, such that $j \notin J$. Then for every i not in $\cap_c$ but in $F_{j-} \cup \{j\}$, yP_ix, since for every $i \notin J, yP_ix$. Let Z be the element of C which results when every j in X is dealt with. The complete induction principle (see, e.g., [4], p. 10) is applied to show that every member of X is eventually dealt with, and hence that there is such a set Z in C. For consider an arbitrary $j \in X$ and suppose that every $z \in X$ which R-precedes j is dealt with, i.e., has property P. Then, by the preceding argument, j is dealt with. Hence, by the induction principle, every u in X is dealt with, guaranteeing Z. Hence, since $i \notin Z$ iff $i \in \cap_c$, for $i \in Z$, xP_iy, and for $i \notin Z$, yP_ix, and so, as $Z \in C$, xPy. Thus $\cap_c \in D$.

Accordingly the conditions for Zorn's lemma are satisfied. Hence, by (d), D has a minimal element $V_{\langle x,y\rangle}$; and so (e) is established. Now (e) is used, in turn, to prove

(f) Where $K' = \{x, y, z\}$ is an arbitrary triple in K, there is a non-null minimal almost decisive set M on K' (i.e., there is no non-null proper subset M' of M which is almost decisive with respect to distinct elements of K').

Consider the class E consisting of every MAD set $V_{\langle u,v\rangle}$ for u against v for each pair of distinct elements u and v of K'. By (e), E is non-null and contains only non-null members. Consider next the subset E' of E consisting of those elements of E which contain no other element of E as a proper subset. Since E is non-null, E' is non-null. Let M be an arbitrarily chosen element of E', i.e., $M = \xi x(x \in E')$. $M \neq \Lambda$, since $\Lambda \notin E$.

It suffices to show that M is a MAD set on K'. To show this, it has to be shown that no non-null subset N of M is almost decisive on K'. Suppose, on the contrary, that for u, $v \in K'$, a set $N \subset M$ is almost decisive with respect to $\langle u, v\rangle$. Then some subset M' of M is a MAD set with respect to $\langle u, v\rangle$, i.e., $M' \in E$ and $M' \subset M$, which is impossible by choice of M.

Finally apply the voting paradox lemma with M the chosen MAD set on $K' = \{x, y, z\} \subseteq K$. Then, by the lemma, M is a singleton, and so there is a factor k, where $M = \{k\}$, which is almost decisive.

Proof of the generalised impossibility theorem: Suppose there were a rational GRM on some usual d.s. $\langle K,F\rangle$. By corollary B there would be a factor in F which is almost decisive, but by corollary A no factor is almost decisive, which is impossible.

The main argument of course merely rectifies Arrow's important argument. The generalised theorem does however establish the following points beyond those claimed originally by Arrow: namely, that the impossibility of a general ranking or assessment method is not escaped by admitting an infinite population or infinite set of factors (a matter which becomes more important when factors rather than individuals are considered, but could happen if all future voters over an infinite time horizon were counted in as individuals); nor can it be escaped by allowing infinitely many alternatives, nor by admitting individual or factor rankings that are not strongly transitive, nor by abandoning the functionality requirement, that the overall or social ranking is a function of the factor or individual ranking, in favour of the assumption that the overall ranking is only more loosely related to factor rankings.[9]

NOTES

1. Similarly alternative proofs; compare [3], p. 339 commenting on [9]:

   > Care must be taken with Welden's paper for it appears as if Arrow's theorem is proved without using the condition of the independence of irrelevant alternatives, when in fact it is used in the proof.

   Note that Sen's final claim, while true of the second Arrow proof, does not hold of the original argument (see, e.g., [1], p. 53).

2. The definitions are those of [1] and [8], and other notions and assumptions are as explained in [1] or [8]. The logical notion is standard and may be found, e.g., in [4].

   It is important for my case that the standard definitions are those formalised above: that they are is a matter of inspection; see the italicised definitions in Arrow [1], p. 98, and the copies, e.g., in [8], p. 42, [3], p. 339, [2], p. 62, and [6], p. 24.

3. This is explicit in Sen ([8], p. 41) and in the original text of Arrow ([1], p. 24). It is not quite so evident in the amended condition 1^1 ([1], p. 96). To see that the same restriction is imposed, however, it suffices to combine the statements of 1^1 with the definition of *admissible* (given on p. 24), which applies to individual orderings.

4. Contrary to popular assumption, strong transitivity is only required in Arrow's theorem (in the voting paradox lemma) for social preference rankings, *not* for individual rankings.

5. The labelling of these principles is that of Sen [8].

6. Hence too, PP rankings furnish a simple countermodel to proposed versions of Arrow's theorem which do not require strong transitivity of the overall ranking. Strong transitivity is required only once in the proof of the impossibility theorem (in showing that xPz in the voting paradox lemma), but without it the theorem fails. This is significant, since strong transitivity is a suspect condition; it is a condition which should be practically as controversial as the critical Condition I—though it has gained little discussion in the literature. It is commonly assumed that abandoning strong transitivity means abandoning individual transitivity of indifference and therewith the economic theory built upon indifference curves. This is not so. Transitivity of indifference can be maintained while the unconvincing connectivity principle $xP_iy \;\&\; yI_iz \supset. xP_iz$ and the implausible strong transitivity principle $xP_iy \;\&\; {\sim}zP_iy \supset. xP_iz$ are abandoned.

   An extension of PP, designed to recover strong transitivity, provides in turn a simple countermodel to the theorem without Condition I (alternatively it shows the independence of I from the other requirements, and that I is essential). Let P^c be the strongly transitive and asymmetric closure of P where P is determined by principle PP. Then P^c, which furnishes a GRM, meets all requirements for the theorem except I, and it is plain why P^c does not satisfy I—because alternatives outside a given environment S may determine overall rankings in S through the closure conditions. Of course the independence of I is well-known (*cf.* [1], pp. 109ff.), not to say notorious.

7. For condition D it is not enough to require simply that there is no i in F such that $(x, y \in K)xP_iy \supset xPy$. For then non-dictatorial recipes such as the method of majority decision would, in some cases, be dictatorial. Consider, e.g., the individual who always votes with the majority. As is common a problem generated by the use material implication can be avoided by judicious use of quantifiers, as follows: There is no $i \in F$, *for every set* of admissible individual rankings, such that if $(x, y \in K)xP_iy$ then xPy.

8. The mistake made in the standard arguments is rather like that of logic students who state Leibnitz's identity principle as

$$x = y \equiv . f(x) \equiv f(x),$$

without the appropriate universal quantifier, for every f on the right-hand side.

9. For elaboration of this point, and also an examination of ways of enforcing Arrow rationality requirements, see [7].

REFERENCES

[1] Arrow, K. J., *Social Choice and Individual Values*, Second Edition, Yale University Press, New Haven, 1963.

[2] Blau, J. H., "A direct proof of Arrow's theorem," *Econometrica*, vol. 40 (1972), pp. 61-67.

[3] Luce, R. D. and H. Raiffa, *Games and Decisions*, John Wiley, New York, 1957.

[4] Mendelson, E., *Introduction to Mathematical Logic*, Van Nostrand, New York, 1964.

[5] Quine, W. V. O., *Mathematical Logic*, Revised Edition, Cambridge, Massachusetts, 1951.

[6] Rothenberg, J., *The Measurement of Social Welfare*, Greenwood, Connecticut, 1972.

[7] Routley, R., "On the impossibility of an orthodox social theory and of an orthodox solution to environmental problems," *Logique et Analyse*, to appear.

[8] Sen, A. K., *Collective Choice and Social Welfare*, Holden-Day, San Francisco, 1970.

[9] Welden, J. C., "On the problem of social welfare functions," *Canadian Journal of Economics and Political Science*, vol. 18 (1952), pp. 452-463.

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