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# An Axiomatization of the Equivalential Fragment of the Three-Valued Logic of Łukasiewicz

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The problem of axiomatizing the purely equivalential fragment of the infinite-valued Łukasiewicz logic ($L_\infty$) and the corresponding variety of algebras remains open. Moreover for every $n = 3, 4, \ldots$ one may ask about axiomatization of the purely equivalential fragment of $n$-valued Łukasiewicz logic ($L_n$). In this paper we give an axiomatization of the purely equivalential fragment of $L_3$ and an appropriate set of identities determining the corresponding variety of algebras (see [3]).

Let us recall that the three-valued logic of Łukasiewicz $L_3$ is determined by the following matrix: $L_3 = (\{0,1,2\}, \{0\}, \rightarrow_L, \wedge_L, \vee_L, \sim_L)$ where $x \rightarrow_L y = max(0, y-x)$, $x \wedge_L y = max(x,y)$, $x \vee_L y = min(x,y)$, and $\sim_L x = x \rightarrow_L 2$ (see [5]).

The other well-known three-valued logic is the logic $H_3$ considered by Heyting in [1]. It is determined by the matrix $H_3 = (\{0,1,2\}, \{0\}, \rightarrow_H, \wedge_H, \vee_H, \sim_H)$, where $x \rightarrow_H y = y$ whenever $x < y$ and $x \rightarrow_H y = 0$ otherwise, $x \wedge_H y = max(x,y)$, $x \vee_H y = min(x,y)$, and $\sim_H x = x \rightarrow_H 2$.

Let the symbols $L_3^{\equiv}$ and $H_3^{\equiv}$ denote the purely equivalential fragments in question. Since $x \equiv y =_{df} (x \rightarrow y) \wedge (y \rightarrow x)$ then $L_3^{\equiv}$ and $H_3^{\equiv}$ are determined by the following matrices $\mathbf{L}_3^{\equiv}$ and $\mathbf{H}_3^{\equiv}$ respectively: $\mathbf{L}_3^{\equiv} = (\{0,1,2\}, \{0\}, \equiv_L)$ where $x \equiv_L y = max(x-y, y-x)$ and $\mathbf{H}_3^{\equiv} = (\{0,1,2\}, \{0\}, \equiv_H)$ where $x \equiv_H y = max(x,y)$ whenever $x \neq y$ and $x \equiv_H y = 0$ otherwise.

It is known that neither $L_3 \not\subseteq H_3$ nor $H_3 \not\subseteq L_3$; for example $(\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta) \in H_3 - L_3$ whereas $((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha) \in L_3 - H_3$. Nevertheless we shall prove that the purely equivalential fragments of $L_2$ and $H_3$ are identical.

The equality $L_3^{\equiv} = H_3^{\equiv}$ is an immediate consequence of the fact that the matrices $\mathbf{L}_3^{\equiv}$ and $\mathbf{H}_3^{\equiv}$ are isomorphic. The reader will have no difficulty in verifying that the required isomorphism is the mapping $i: \{0,1,2\} \rightarrow \{0,1,2\}$, such that $i(0) = 0$, $i(1) = 2$, $i(2) = 1$.

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The notion of intuitionistic equivalential algebra introduced in [2] (see also [4]) is an algebraic counterpart of the equivalential fragment of the intuitionistic propositional logic. This fragment was axiomatized by Tax in [6] by means of the single axiom

**(TA)** $$((\beta \equiv (\beta \equiv \alpha)) \equiv ((\beta \equiv (\beta \equiv \alpha)) \equiv (\alpha \equiv (\alpha \equiv (\gamma \equiv \delta))))) \equiv ((\alpha \equiv \delta) \equiv (\gamma \equiv \alpha))$$

and the following rules of inference:

**(DR)** $$\frac{\alpha \equiv \beta,\ \alpha}{\beta}$$ (the detachment rule for the equivalence)

**(TR)** $$\frac{\alpha}{\beta \equiv (\beta \equiv \alpha)}$$ (the Tax rule).

The class of intuitionistic equivalential algebras was defined in [2] as the variety of all algebras of type $\langle 2 \rangle$ satisfying the following identities:

**(i1)** $(a \equiv a) \equiv b = b$
**(i2)** $((a \equiv b) \equiv c) \equiv c = (a \equiv c) \equiv (b \equiv c)$
**(i3)** $((a \equiv b) \equiv ((a \equiv c) \equiv c)) \equiv ((a \equiv c) \equiv c) = a \equiv b$.

The variety of algebras corresponding to the equivalential fragment of the logic $H_3$ has already been axiomatized in [4] by the identities (i1), (i2), (i3), and

**(h1)** $(((a \equiv ((b \equiv c) \equiv c)) \equiv ((b \equiv c) \equiv c)) \equiv ((a \equiv ((c \equiv b) \equiv b)) \equiv ((c \equiv b) \equiv b))) \equiv ((a \equiv (b \equiv c)) \equiv (b \equiv c)) = a$
**(h2)** $(a \equiv (((b \equiv c) \equiv c) \equiv b)) \equiv (((b \equiv c) \equiv c) \equiv b) = a$.

On the basis of axioms of the variety of intuitionistic equivalential algebras the identities (h1), (h2) are equivalent to the one identity

**(h)** $(a \equiv (b \equiv c)) \equiv (b \equiv c) = (a \equiv ((a \equiv b) \equiv b)) \equiv ((a \equiv c) \equiv c)$.

A routine proof will be omitted.

Since just the same variety corresponds to the equivalential fragment of the logic $L_3$, one gets the following:

**Corollary** *The identities* (i1), (i2), (i3), (h) *form an axiomatization of the variety determined by* $L_3^{\equiv}$.

The results of Tax [6] combined with our corollary yield the following:

**Theorem** $L_3^{\equiv}$ *can be axiomatized by adopting* (TA), (DR), (TR), *and the following axiom:* $((\alpha \equiv (\beta \equiv \gamma)) \equiv (\beta \equiv \gamma)) \equiv ((\alpha \equiv ((\alpha \equiv \beta) \equiv \beta)) \equiv ((\alpha \equiv \gamma) \equiv \gamma))$.

## REFERENCES

[1] Heyting, A., "Die Formalen Regeln der intuitionistischen Logik," *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse* (1930), pp. 42–56.

[2] Kabziński, J. K., *Algebry Równoważnościowe*, Ph.D. dissertation, Jagiellonian University, Krakow, 1974.

[3] Kabziński, J. K., "On equivalential fragment of the three-valued logic of Łukasiewicz" (abstract), *Bulletin of the Section of Logic, Polish Academy of Sciences*, vol. 8 (1979), pp. 182–187.

[4] Kabziński, J. K. and A. Wroński, "On equivalential algebras," *Proceedings of the 1975 International Symposium of Multiple-Valued Logics*, Indiana University, Bloomington, May 13–16 (1975), pp. 419–428.

[5] Łukasiewicz, J., "O logice trójwartościowej," *Ruch Filozoficzny*, vol. 5 (1920), pp. 170–171.

[6] Tax, R. E., "On the intuitionistic equivalential calculus," *Notre Dame Journal of Formal Logic*, vol. 14 (1973), pp. 448–456.

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