Notre Dame Journal of Formal Logic
Volume 29, Number 3, Summer 1988

Semantical Analysis of Superrelevant Predicate Logics with Quantification

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As is well-known, the definition of the Kripke-type semantics for the relevant propositional logics was given by Routley and Meyer in [9], but the definition of the algebraic semantics of these logics was given by Dunn in [2]. The Kripke-type semantics (called the relevant RPg-spaces) and the algebraic semantics (which we call the simple C_R-matrices) for relevant propositional logics were also defined by Maksimova ([5]). It can be easily proved that Routley and Meyer's semantics are equivalent to relevant RPg-spaces, and that Dunn's semantics are equivalent to simple C_R-matrices. In [5], Maksimova showed that there exists a close relationship between relevant RPg-spaces and simple C_R-matrices. An essential aspect of this relationship is that for any relevant RPg-space there exists the respective simple C_R-matrix, the contents of which are identical with the contents of the relevant RPg-space; and with any simple G_R-matrix it is possible to correlate the respective relevant RPg-space. However the contents of that relevant RPg-space need not be identical with the contents of the initial matrix, though in the case of finite matrices the identity of contents holds.

In this paper we pick up the subject of semantics for quantified relevant logics, which is an important and underdeveloped one. Some basic problems and solutions in this field were noted by Routley in [8]. We introduce here two types of semantics which we call respectively general relevant RPg-spaces (g.r. RPg-spaces) and structurally general relevant RPg-spaces (s.g.r. RPg-spaces). In general, by a g.r. RPg-space we mean any triple $\langle \underline{S}, V, \mathfrak{A} \rangle$ such that $\underline{S}$ is a relevant RPg-space, V is a nonempty set, and $\mathfrak{A}$ is a nondegenerate (V, S)-simple C_{RQ}-matrix, and by an s.g.r. RPg-space we mean any triple $\langle \underline{S}_0, V, \underline{S}_1 \rangle$ such that $\underline{S}_0$ and $\underline{S}_1$ are relevant RPg-spaces and V is a nonempty set. We state that though the contents of any g.r. RPg-space as well as the contents of any s.g.r. RPg-space determine some superrelevant predicate logic, i.e. a predicate logic containing the relevant predicate logic RQ; for the superrelevant predicate logics they are incomplete.

Received June 7, 1984; revised August 1, 1986

The g.r. RPg-spaces are a generalization of the Kripke-type structures $\langle \underline{S}, V \rangle$, where $\underline{S}$ is a relevant RPg-space and V is a nonempty set (cf. the relevant quantificational model structures defined in [9]), and of the algebraic structures $\langle \mathfrak{A}, V \rangle$, where $\mathfrak{A}$ is an $\mathfrak{m}$-simple C_{RQ}-matrix and V is a nonempty set (the structures of the type $\langle \mathfrak{A}, V \rangle$ amounting to De Morgan monoid-valued models of RQ and its supersystems, cf. [1]), in the following sense: (1) If $\langle \underline{S}, V, \mathfrak{A} \rangle$ is a g.r. RPg-space such that $\mathfrak{A}$ is a two-element Boole algebra, then $\langle \underline{S}, V, \mathfrak{A} \rangle$ and $\langle \underline{S}, V \rangle$ determine the same superrelevant predicate logic; and (2) if $\langle \underline{S}, V, \mathfrak{A} \rangle$ is a g.r. RPg-space such that $\underline{S}$ is a one-element relevant RPg-space, then $\langle \underline{S}, V, \mathfrak{A} \rangle$ and $\langle \mathfrak{A}, V \rangle$ determine the same superrelevant predicate logic. Having introduced g.r. RPg-spaces and knowing the relationship between relevant RPg-spaces and simple C_R-matrices, it would be natural to introduce s.g.r. RPg-spaces too. However every s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1 \rangle$ could be replaced by the direct product $\langle \underline{S}_0 \times \underline{S}_1, V \times V \rangle$ because contents of these structures are the same; thus it is not essentially a new kind of structures. In the case of g.r. RPg-spaces the similar trick cannot be used. The direct product of $\mathfrak{m}$-simple C_{RQ}-matrices does not have to be an $\mathfrak{m}$-simple C_{RQ}-matrix.

Our reason for introducing the g.r. RPg-spaces is the fact that, thanks to these structures it is possible to characterize a wider class of the superrelevant predicate logics, in comparison to structures of the type $\langle \underline{S}, V \rangle$ and structures of the type $\langle \mathfrak{A}, V \rangle$. However the s.g.r. RPg-spaces are useful to get many solutions characterizing the g.r. RPg-spaces.

The paper is divided into three parts. In the first part we introduce the definitions of the general relevant RPg-space and the structurally general relevant RPg-space, and we also state theorems on the contents of these structures. In the second part we give proofs of the incompleteness of these semantics for superrelevant predicate logics.[1] Finally, in the third part we state some relations between the g.r. RPg-spaces and the s.g.r. RPg-spaces.

1 Given the symbols: $p^{(n)}$, $q^{(n)}$, $r^{(n)}, \ldots$ of n-ary predicate variables, $n \in \omega$, and the countably infinite set $\{x, y, z, \ldots, x_0, y_0, z_0, \ldots\}$ of individual variables we define in the standard way the set AT of atomic formulas. By FOR we denote the set of all formulas built up by means of connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, $\exists$ (conjunction, disjunction, relevant implication, negation, universal quantifier, and existential quantifier, respectively) and atomic formulas. The symbol $Var(\alpha)$ denotes the set of all free variables occurring in α. When $Var(\alpha) \subseteq \{x_1, \ldots, x_n\}$, we shall write $\alpha(x_1, \ldots, x_n)$. The symbol $\alpha(x_i/x_j)$ is used to denote the result of simultaneous substitution of x_j for every free occurrence of x_i in α, with normal restrictions.

Let C_{RQ}: $2^{FOR} \mapsto 2^{FOR}$ be the consequence operation defined as follows: for all $X \subseteq FOR$ and $\alpha \in FOR$, $\alpha \in C_{RQ}(X)$ iff α is provable from X by means of some instances of axiom schemas of system relevant implication with quantification RQ (see [9]) together with the rule of modus ponens, the rule of adjunction, and the rule of generalization. In the case when C_1 and C_2 are consequence operations in FOR and $C_1(X) \subseteq C_2(X)$ for all $X \subseteq FOR$, we shall write $C_1 \leq C_2$.

We say that $\alpha, \beta \in FOR$ are similar (in symbols $\alpha \sim \beta$) if one of them can be obtained by changing some bound variables in the other one (for the formal definition of similarity see [7]). By a substitution in FOR we shall understand any function h: $FOR \mapsto FOR$, which is a homomorphism with respect to $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall x_1$, $\forall x_2, \ldots, \exists x_1$, $\exists x_2, \ldots$ and which satisfies the following conditions (cf. [4]):

(i) $Var(h(\alpha)) \subseteq Var(\alpha)$, for all $\alpha \in FOR$
(ii) for every $\alpha \in AT$ and for every $i,j \in \omega$, there is some $\beta \in FOR$ of a special form (see [6]) such that $h(\alpha) \sim \beta$ and $h(\alpha(x_i/x_j)) \sim \beta(x_i/x_j)$.

For a given $X \subseteq FOR$ by the symbol $Sb(X)$ we shall denote the closure of X under all substitutions in FOR. Each subset $X \subseteq FOR$ such that $RQ \subseteq X$ and $X = C_{RQ}(Sb(X))$ will be called a superrelevant predicate logic.

By a relevant RPg-space we shall mean by Maksimova [5] (cf. also [9]) any quadruple $\underline{S} = \langle S,R,P,g \rangle$, where S is a nonempty set, R is a ternary relation on S, $P \subseteq S$, g: $S \mapsto S$ is a mapping, and which satisfies the following conditions for any $a, a_1, b, b_1, c, c_1 \in S$:

RPg1 There exists $d \in P$ such that $Rdaa$
RPg2 There exists $d \in P$ such that $Rada$
RPg3 If $a_1 \leq_{\underline{S}} a$ and $Rabc$, then Ra_1bc, where "$a \leq_{\underline{S}} b$" means "there exists $d \in P$ such that $Rdab$"
RPg4 If $b_1 \leq_{\underline{S}} b$ and $Rabc$, then Rab_1c
RPg5 If $c \leq_{\underline{S}} c_1$ and $Rabc$, then $Rabc_1$
RPg6 If $Rabc$, then $Rabd$ and $Rdbc$ for some $d \in S$
RPg7 If $Rabc$ and $Rcde$, then $Radd_1$ and Rbd_1e for some $d_1 \in S$
RPg8 If $Rabc$ and $Rcde$, then $Radd_1$ and Rd_1be for some $d_1 \in S$
RPg9 $g(g(a)) = a$
RPg10 If $Rabc$, then $Rag(c)g(b)$
RPg11 $Rag(a)a$.

Let $\mathfrak{A} = \langle \underline{A},D \rangle$ be a matrix, such that $\underline{A} = \langle A,\wedge,\vee,\rightarrow,\neg \rangle$ is an algebra similar to $FOR_{\{\wedge,\vee,\rightarrow,\neg\}}$ (= a set of formulas whose all connectives belong to $\{\wedge,\vee,\rightarrow,\neg\}$) and $\langle A,\wedge,\vee \rangle$ is a distributive lattice in which D is a filter having the property: $a \wedge b = a$ iff $a \rightarrow b \in D$ for all $a,b \in A$; and moreover, the following conditions are satisfied for all $a,b,c \in A$:

(1) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$
(2) $a \leq (a \rightarrow b) \rightarrow b$
(3) $a \rightarrow (a \rightarrow b) \leq a \rightarrow b$
(4) $(a \rightarrow b) \wedge (a \rightarrow c) \leq a \rightarrow (b \wedge c)$
(5) $(a \rightarrow c) \wedge (b \rightarrow c) \leq (a \vee b) \rightarrow c$
(6) $a \rightarrow \neg b \leq b \rightarrow \neg a$
(7) $\neg\neg a = a$,

where "$a \leq b$" means "$a = a \wedge b$".

In this case, we say that $\mathfrak{A} = \langle \underline{A},D \rangle$ is a simple C_R-matrix (cf. [5]). A simple C_R-matrix $\mathfrak{A} = \langle \underline{A},D \rangle$ will be called $\mathfrak{m}$-simple C_{RQ}-matrix, if for each set T, $2 \leq \overline{\overline{T}} \leq \mathfrak{m}$, it satisfies:

(8) The algebra $\underline{A}$ is $\mathfrak{m}$-complete; i.e., there exists $\bigwedge_{t \in T} a_t$ and $\bigvee_{t \in T} a_t$ for any subset $\{a_t | t \in T\} \subseteq A$

(9) The filter D is $\mathfrak{m}$-complete; i.e., $\{a_t | t \in T\} \subseteq D$ implies $\bigwedge_{t \in T} a_t$, $\bigvee_{t \in T} a_t \in D$

(10) If $\bigvee_{t \in T} a_t \in D$, then $a_t \in D$ for some $t \in T$

(11) $\bigwedge_{t \in T} (a \to b_t) \leq a \to \bigwedge_{t \in T} b_t$

(12) $\bigwedge_{t \in T} (a_t \to b) \leq \bigvee_{t \in T} a_t \to b$

(13) $\bigwedge_{t \in T} (a \vee b_t) \leq a \vee \bigwedge_{t \in T} b_t$

(14) $\bigwedge_{t \in T} a_t \wedge \bigwedge_{t \in T} b_t \leq \bigwedge_{t \in T} (a_t \wedge b_t)$.

By a general relevant RPg-space we mean a triple $\langle \underline{S}, V, \mathfrak{A} \rangle$ satisfying the following conditions:

(i) $\underline{S}$ is a relevant RPg-space

(ii) V is a nonempty set

(iii) $\mathfrak{A}$ is a nondegenerate $\kappa(V,S)$-simple C_{RQ}-matrix, where $\kappa(V,S)$ is the smallest cardinal number both greater than $\overline{\overline{V}}$ and $\overline{\overline{\{\langle b,c \rangle \in S^2 | Rabc\}}}$ for any $a \in S$.

By an interpretation function in the g.r. RPg-space $\langle \underline{S}, V, \mathfrak{A} \rangle$ we shall understand any function J such that:

(1) For each 0-ary predicate variable $p^{(0)}$ and each $a \in S$, $J(p^{(0)}, a) \in A$; and if $a \leq_{\underline{S}} b$, then $J(p^{(0)}, a) \leq J(p^{(0)}, b)$

(2) For each n-ary predicate variable ($n \geq 1$) and each $a \in S$, $J(p^{(n)}, a)$ is a function from V^n to A; and if $a \leq_{\underline{S}} b$, then $J(p^{(n)}, a)(a_0, \ldots, a_{n-1}) \leq J(p^{(n)}, b)(a_0, \ldots, a_{n-1})$ for any $a_0, \ldots, a_{n-1} \in V$.

A general relevant RPg-model (g.r. RPg-model) is a quadruple $\langle \underline{S}, V, \mathfrak{A}, J \rangle$, where $\langle \underline{S}, V, \mathfrak{A} \rangle$ is a g.r. RPg-space and J is an interpretation function in $\langle \underline{S}, V, \mathfrak{A} \rangle$. By an assignment for the g.r. RPg-space $\langle \underline{S}, V, \mathfrak{A} \rangle$ we shall understand any function $\tilde{a}$ from the set of all individual variables to the set V. The definition of the value-function v for the g.r. RPg-model $\langle \underline{S}, V, \mathfrak{A}, J \rangle$ is inductively given in this way that for any assignment $\tilde{a}$ for $\langle \underline{S}, V, \mathfrak{A} \rangle$ and for each $a \in S$:

(i) $v(p^{(0)}, a) = J(p^{(0)}, a)$

(ii) $v(p^{(n)} x_0 \ldots x_{n-1}, \tilde{a}, a) = J(p^{(n)}, a)(\tilde{a}(x_0) \ldots \tilde{a}(x_{n-1}))$, $n \geq 1$

(iii) $v(\alpha \wedge \beta, \tilde{a}, a) = v(\alpha, \tilde{a}, a) \wedge v(\beta, \tilde{a}, a)$

(iv) $v(\alpha \vee \beta, \tilde{a}, a) = v(\alpha, \tilde{a}, a) \vee v(\beta, \tilde{a}, a)$

(v) $v(\alpha \to \beta, \tilde{a}, a) = \bigwedge (v(\alpha, \tilde{a}, b) \to v(\beta, \tilde{a}, c) | b, c \in S$ and $Rabc)$

(vi) $v(\neg\alpha, \tilde{a}, a) = \neg v(\alpha, \tilde{a}, g(a))$

(vii) $v(\forall x \alpha, \tilde{a}, a) = \bigwedge (v(\alpha, \tilde{a}', a) | \tilde{a}'$ is an assignment that agrees with $\tilde{a}$ except on $x)$

(viii) $v(\exists x \alpha, \tilde{a}, a) = \bigvee (v(\alpha, \tilde{a}', a) | \tilde{a}'$ is an assignment that agrees with $\tilde{a}$ except on $x)$.

We say that $\alpha \in FOR$ is satisfied in the g.r. RPg-model $\langle \underline{S}, V, \mathfrak{A}, J \rangle$, if there exists an assignment $\tilde{a}$ for $\langle \underline{S}, V, \mathfrak{A} \rangle$ such that $v(\alpha, \tilde{a}, a) \in D$ for any $a \in$

P. If for every assignment $\tilde{a}$ for $\langle \underset{\sim}{S}, V, \mathfrak{A}\rangle$ the formula α is satisfied in the $\langle \underset{\sim}{S}, V, \mathfrak{A}, J\rangle$, we say that α is true in $\langle \underset{\sim}{S}, V, \mathfrak{A}, J\rangle$. And the formula α is true in g.r. RPg-space $\langle \underset{\sim}{S}, V, \mathfrak{A}\rangle$, if α is true in every g.r. RPg-model $\langle \underset{\sim}{S}, V, \mathfrak{A}, J\rangle$. The set of all formulas true in the g.r. RPg-model $\langle \underset{\sim}{S}, V, \mathfrak{A}, J\rangle$ (in the g.r. RPg-space $\langle \underset{\sim}{S}, V, \mathfrak{A}\rangle$)—the contents of $\langle \underset{\sim}{S}, V, \mathfrak{A}, J\rangle$ (the contents of $\langle \underset{\sim}{S}, V, \mathfrak{A}\rangle$)—will be denoted by $E(\underset{\sim}{S}, V, \mathfrak{A}, J)(E(S, V, \mathfrak{A}))$.

Theorem 1.1 *For every g.r. RPg-space $\langle \underset{\sim}{S}, V, \mathfrak{A}\rangle$, $E(\underset{\sim}{S}, V, \mathfrak{A})$ is a superrelevant predicate logic.*

Proof: Considering each axiom of RQ separately we get that it is true in any g.r. RPg-model and that the rules of RQ preserve truth.

By a structurally general relevant RPg-space we mean any triple $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$, which satisfies the following conditions:

(i) $\underset{\sim}{S}_0 = \langle S_0, R_0, P_0, g_0\rangle$ and $\underset{\sim}{S}_1 = \langle S_1, R_1, P_1, g_1\rangle$ are relevant RPg-spaces
(ii) V is a nonempty set.

A function J is said to be an interpretation function in the s.g.r. RPg-space $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$, if for each $a \in S_0$ and for each $w \in S_1$:

(1) For any 0-ary predicate variable $p^{(0)}$, $J(p^{(0)}, a, w) \in \{\mathbb{1}, \mathbb{0}\}$; and if $a \leq_{\underset{\sim}{S}_0} b$, $w \leq_{\underset{\sim}{S}_1} u$ and $J(p^{(0)}, a, w) = \mathbb{1}$, then $J(p^{(0)}, b, u) = \mathbb{1}$
(2) For any n-ary $(n \geq 1)$ predicate variable $p^{(n)}$, $J(p^{(n)}, a, w) \subseteq V^n$; and if $a \leq_{\underset{\sim}{S}_0} b$ and $w \leq_{\underset{\sim}{S}_1} u$, then $J(p^{(n)}, a, w) \subseteq J(p^{(n)}, b, u)$.

A structurally general relevant RPg-model (s.g.r. RPg-model) is a quadruple $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$, where $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$ is an s.g.r. RPg-space and J is an interpretation function in $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$. Any function $\tilde{a}$ from the set of all individual variables to the set V is called an assignment for the s.g.r. RPg-space $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$. The definition of the value-function v for the s.g.r. RPg-model $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$ is inductively given in this way that for any assignment $\tilde{a}$ for $\langle S_0, V, S_1\rangle$, for each $a \in S_0$ and for each $w \in S_1$:

(i) $v(p^{(0)}, a, w) = \mathbb{1}$ iff $J(p^{(0)}, a, w) = \mathbb{1}$
(ii) $v(p^{(n)}x_0 \ldots x_{n-1}, \tilde{a}, a, w) = \mathbb{1}$ iff $J(p^{(n)}, a, w)\tilde{a}(x_0) \ldots \tilde{a}(x_{n-1})$, $n \geq 1$
(iii) $v(\alpha \wedge \beta, \tilde{a}, a, w) = \mathbb{1}$ iff $v(\alpha, \tilde{a}, a, w) = \mathbb{1}$ and $v(\beta, \tilde{a}, a, w) = \mathbb{1}$
(iv) $v(\alpha \vee \beta, \tilde{a}, a, w) = \mathbb{1}$ iff $v(\alpha, \tilde{a}, a, w) = \mathbb{1}$ or $v(\beta, \tilde{a}, a, w) = \mathbb{1}$
(v) $v(\alpha \rightarrow \beta, \tilde{a}, a, w) = \mathbb{1}$ iff for each $b, c \in S_0$ and for each $u, z \in S_1$: if R_0abc and R_1wuz and $v(\alpha, \tilde{a}, b, u) = \mathbb{1}$, then $v(\beta, \tilde{a}, c, z) = \mathbb{1}$
(vi) $v(\neg\alpha, \tilde{a}, a, w) = \mathbb{1}$ iff $v(\alpha, \tilde{a}, g_0(a), g_1(w)) = \mathbb{0}$
(vii) $v(\forall x\alpha, \tilde{a}, a, w) = \mathbb{1}$ iff for every assignment $\tilde{a}'$ which agrees with $\tilde{a}$ except on x, $v(\alpha, \tilde{a}', a, w) = \mathbb{1}$
(viii) $v(\exists x\alpha, \tilde{a}, a, w) = \mathbb{1}$ iff for some assignment $\tilde{a}'$ which agrees with $\tilde{a}$ except on x, $v(\alpha, \tilde{a}', a, w) = \mathbb{1}$.

We say that $\alpha \in FOR$ is satisfied in the s.g.r. RPg-model $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$, if there exists an assignment ã for $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$ such that $v(\alpha, \tilde{a}, a, w) = \mathbb{1}$ for each $a \in P_0$ and for each $w \in S_1$. If for every assignment $\tilde{a}$ for $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$ the formula α is satisfied in the $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$, we say that α is true in $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$. And the formula α is true in the s.g.r. RPg-space $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1\rangle$, if α is true in every s.g.r. RPg-model $\langle \underset{\sim}{S}_0, V, \underset{\sim}{S}_1, J\rangle$. The set of all formulas true in the s.g.r.

RPg-model $\langle \underline{S}_0, V, \underline{S}_1, J\rangle$ (in the s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$) — the contents of $\langle \underline{S}_0, V, \underline{S}_1, J\rangle$ (the contents of $\langle \underline{S}_0, V, \underline{S}_1\rangle$) — will be denoted by $E(\underline{S}_0, V, \underline{S}_1, J)$ ($E(\underline{S}_0, V, \underline{S}_1)$).

Theorem 1.2 *For every s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$, $E(\underline{S}_0, V, \underline{S}_1)$ is a superrelevant predicate logic.*

Proof: Enlarges with minor modifications the corresponding proof of Routley and Meyer ([9]) showing that the axioms of RQ are true in any s.g.r. RPg-model and that the rules of RQ preserve truth.

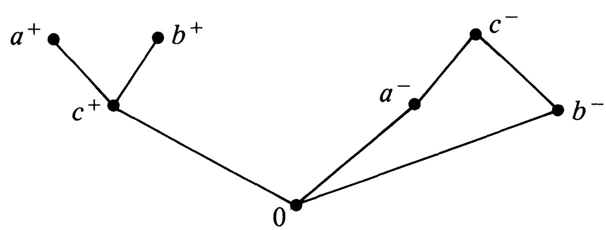
2 If for the superrelevant predicate logic L there exists a set $\{\langle \underline{S}_i, V_i, \mathfrak{A}_i\rangle | i \in I\}$ of g.r. RPg-spaces such that $L = \bigcap_{i\in I} E(\underline{S}_i, V_i, \mathfrak{A}_i)$, then we say that L has characteristic class of g.r. RPg-spaces. Similarly, if L is a superrelevant predicate logic and if there exists a set $\{\langle \underline{S}_0^i, V^i, \underline{S}_1^i\rangle | i \in I\}$ of s.g.r. RPg-spaces such that $L = \bigcap_{i\in E} E(\underline{S}_0^i, V^i, \underline{S}_1^i)$, then we say that L has characteristic class of s.g.r. RPg-spaces.

Let the symbol RQF denote the superrelevant predicate logic $RQ + F (F = \exists x(p(x) \to \forall y p(y)))$ and let the symbol H denote the formula $(q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}) \vee \forall x \forall y(s(x) \to s(y))$.

Lemma 2.1 $H \notin RQF$.

Proof: We define the s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$ as follows:

Let $\mathbb{A} = \langle A, \leq_A\rangle$ be the poset such that $A = \{a,b,c\}$ and $\leq_A = \{\langle x, x\rangle | x \in A\} \cup \{\langle c,a\rangle, \langle c,b\rangle\}$. Let $A^- = \{x^- | x \in A\}$, $A^+ = \{x^+ | x \in A\}$, $\leq^- = \{\langle x^-, y^-\rangle | y \leq_A x\}$ and $\leq^+ = \{\langle x^+, y^+\rangle | x \leq_A y\}$. Let $\langle S_0, \leq\rangle$ be the poset such that $S_0 = A^- \cup A^+ \cup \{0\}$, where $0 \notin A^- \cup A^+$, and $\leq = \leq^- \cup \leq^+ \cup \{\langle 0,0\rangle\}$. Now we define the relevant RPg-space $\underline{S}_0 = \langle S_0, R_0, P_0, g_0\rangle$ in the following way: $P_0 = \{0\}$; g_0 is the function $S_0 \mapsto S_0$ such that $g_0(x^-) = x^+$, $g_0(x^+) = x^-$ for all $x \in A$ and $g_0(0) = 0$; $R_0 = (A^- \cup A^+)^3 \cup \{\langle 0,x,y\rangle | x \leq y\} \cup \{\langle x,0,y\rangle | x \leq y\} \cup \{\langle x,y,0\rangle | y \leq g_0(x)\}$. Because $\leq_{\underline{S}_0} = \leq$, then the order $\leq_{\underline{S}_0}$ may be indicated by the diagram



Let $\langle S_1, \leq\rangle$ be the poset such that $S_1 = \{u^+, u^-, 0\}$ and $\leq = \{\langle x,x\rangle | x \in S_1\}$. We set the mapping g_1: $S_1 \mapsto S_1$, defined as follows: $g_1(u^+) = u^-$, $g_1(u^-) = u^+$ and $g_1(0) = 0$. Moreover we put $P_1 = \{0\}$ and $R_1 = \{u^+, u^-\}^3 \cup \{\langle 0,x,x\rangle | x \in \{u^+, u^-\}\} \cup \{\langle x,0,x\rangle | x \in \{u^+, u^-\}\} \cup \{\langle x,x,0\rangle | x \in \{u^+, u^-\}\} \cup \{\langle 0,0,0\rangle\}$. So we have defined the relevant RPg-space $\underline{S}_1 = \langle S_1, R_1, P_1, g_1\rangle$. The order can be demonstrated by the following diagram:

$$\dot{u}^+ \qquad \dot{0} \qquad \dot{u}^-$$

Let $V = \{a,b\}$. This gives us the s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$.

Now we define the interpretation function in the s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$ putting:

for $\langle x,y\rangle \in \{\langle 0,0\rangle, \langle b^+,0\rangle, \langle c^+,0\rangle\}$, $J(q^{(0)},x,y) = \mathbb{0}$
and $J(q^{(0)},x,y) = \mathbb{1}$ otherwise
for $\langle x,y\rangle \in \{\langle 0,0\rangle, \langle a^+,0\rangle, \langle c^+,0\rangle\}$, $J(r^{(0)},x,y) = \mathbb{0}$
and $J(r^{(0)},x,y) = \mathbb{1}$ otherwise
$a \in \bigcap (J(s,x,y) | \langle x,y\rangle \in (S_1 \times S_2) - \{\langle 0,0\rangle\})$, and
$b \in \bigcap (J(s,x,y) | \langle x,y\rangle \in (S_1 \times S_2) - \{\langle 0,0\rangle, \langle c^+,0\rangle\})$.

Then we have $H \notin E(\underline{S}_0, V, \underline{S}_1, J)$. It remains to prove that for any formula $\alpha(x)$ constructed by $q^{(0)}, r^{(0)}$, and s, $\exists x(\alpha(x) \to \forall y\alpha(y)) \in E(\underline{S}_0, V, \underline{S}_1, J)$. Suppose to the contrary, i.e. that $\exists x(\alpha(x) \to \forall y\alpha(y)) \notin E(\underline{S}_0, V, \underline{S}_1, J)$. Hence it follows that for every assignment $\tilde{a}$ for $\langle \underline{S}_0, V, \underline{S}_1, J\rangle$, $v(\alpha(x) \to \forall y\alpha(y), \tilde{a}, 0, 0) = \mathbb{0}$. Let $\tilde{a}$ and $\tilde{a}'$ be assignments for $\langle \underline{S}_0, V, \underline{S}_1, J\rangle$ such that $\tilde{a}(x) = a$ and $\tilde{a}'(x) = b$. Therefore there exist $a_0, b_0 \in S_0$ and $w_0, u_0 \in S_1$ and such that $R_0 0 a_0 b_0$, $R_1 0 w_0 u_0$, $v(\alpha(x), \tilde{a}, a_0, w_0) = \mathbb{1}$, and $v(\forall y\alpha(y), \tilde{a}, b_0, u_0) = \mathbb{0}$. Hence we obtain that $v(\alpha(x), \tilde{a}, b_0, u_0) = \mathbb{1}$ and $v(\alpha(x), \tilde{a}', b_0, u_0) = \mathbb{0}$. Analogically, there exist $a_1, b_1 \in S_0$ and $w_1, u_1 \in S_1$ such that $R_0 0 a_1 b_1$, $R_1 0 w_1 u_1$, $v(\alpha(x), \tilde{a}', a_1, w_1) = \mathbb{1}$, and $v(\forall y\alpha(y), \tilde{a}', b_1, u_1) = \mathbb{0}$. Consequently, $v(\alpha(x), \tilde{a}', b_1, u_1) = \mathbb{1}$ and $v(\alpha(x), \tilde{a}, b_1, u_1) = \mathbb{0}$. It can easily be seen that $b_0 \notin A^-$ and $b_1 \notin A^-$, because for each $a \in A^-$ and for each $w \in S_1$, $v(\alpha(x), \tilde{a}, a, w) = v(\alpha(x), \tilde{a}', a, w)$. The elements b_0, b_1 cannot be compared with respect to the relation $\leq_{\underline{S}_0}$, since then $v(\alpha(x), \tilde{a}, b_0, u_0) = v(\alpha(x), \tilde{a}', b_1, u_1) = \mathbb{1}$ and $v(\alpha(x), \tilde{a}, b_1, u_1) = v(\alpha(x), \tilde{a}', b_0, u_0) = \mathbb{0}$ could not occur. Hence either $b_0 = a^+$, $b_1 = b^+$ or conversely $b_1 = a^+$, $b_0 = b^+$. But one can check, that for any $a \in \{a^+, b^+\}$ and for any $w \in S_1$, $v(\alpha(x), \tilde{a}, a, w) = v(\alpha(x), \tilde{a}', a, w)$, which contradicts the assumption that $\exists x(\alpha(x) \to \forall y\alpha(y)) \notin E(\underline{S}_0, V, \underline{S}_1, J)$.

Lemma 2.2 *For any g.r. RPg-space $\langle \underline{S}, V, \mathfrak{A}\rangle$, $F \in E(\underline{S}, V, \mathfrak{A})$ implies $H \in E(\underline{S}, V, \mathfrak{A})$.*

Proof: If $\overline{\overline{V}} = 1$, then for any g.r. RPg-space $\langle \underline{S}, V, \mathfrak{A}\rangle$, $H \in E(\underline{S}, V, \mathfrak{A})$. Suppose that $\overline{\overline{V}} \geq 2$ and $H \notin E(\underline{S}, V, \mathfrak{A})$. Then there exists an interpretation function J in the g.r. RPg-space $\langle \underline{S}, V, \mathfrak{A}\rangle$ such that $H \notin E(\underline{S}, V, \mathfrak{A}, J)$. Hence for some element $a_0 \in P$, $v((q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}), a_0) \notin D$. Let u_0 be an element of V. Consider now the interpretation function J_1 in $\langle \underline{S}, V, \mathfrak{A}\rangle$ such that for any $a \in S$: $J_1(q^{(0)}, a) = J(q^{(0)}, a)$; $J_1(r^{(0)}, a) = J(r^{(0)}, a)$; $J_1(p,a)(u) = J(q^{(0)}, a)$ if $u = u_0$; $J_1(p,a)(u) = J(r^{(0)}, a)$ if $u \neq u_0$. Let v_1 be the value-function for the g.r. RPg-model $\langle \underline{S}, V, \mathfrak{A}, J_1\rangle$. Then it is obvious that for any assignment $\tilde{a}$ for $\langle \underline{S}, V, \mathfrak{A}\rangle$, $v_1(\exists x(p(x) \to \forall y p(y)), \tilde{a}, a) = v_1((q^{(0)} \to \forall y p(y)) \vee (r^{(0)} \to \forall y p(y)), \tilde{a}, a) = v_1((q^{(0)} \to q^{(0)} \wedge r^{(0)}) \vee (r^{(0)} \to q^{(0)} \wedge r^{(0)}), a)$. So, if $v_1((q^{(0)} \to q^{(0)} \wedge r^{(0)}) \vee (r^{(0)} \to q^{(0)} \wedge r^{(0)}), a) \in D$, then after an easy calculation we obtain that also $v_1((q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}), a) \in D$, and $v((q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}), a) \in D$. Therefore, on the strength of the assumptions, $v_1(\exists x(p(x) \to$

$\forall yp(y)), \tilde{a}, a_0) \notin D$ for any assignment $\tilde{a}$ for $\langle \underline{S}, V, \mathfrak{A}\rangle$, and consequently $F \notin E(\underline{S}, V, \mathfrak{A})$.

Theorem 2.1 *The superrelevant predicate logic RQF does not possess characteristic classes of the g.r. RPg-spaces.*

Proof: On the strength of Lemmas 2.1 and 2.2.

Lemma 2.3 *For any s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_0\rangle$, $F \in E(\underline{S}_0, V, \underline{S}_1)$ implies $H \in E(\underline{S}_0, V, \underline{S}_1)$.*

Proof: If $\overline{\overline{V}} = 1$, then for any s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$, $H \in E(\underline{S}_0, V, \underline{S}_1)$. Suppose that $\overline{\overline{V}} \geq 2$ and $H \notin E(\underline{S}_0, V, \underline{S}_1)$. Therefore there exists an interpretation function J in the s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$ such that $H \notin E(\underline{S}_0, V, \underline{S}_1, J)$. Hence for some elements $a_0 \in P_0$ and $w_0 \in P_1$, $v((q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}), a_0, w_0) = \mathbb{0}$. Let u_0 be an element of V. Consider now the interpretation function J_1 in $\langle \underline{S}_0, V, \underline{S}_1\rangle$ such that for any $a \in S_0$ and for any $w \in S_1$: $J_1(q^{(0)}, a, w) = J(q^{(0)}, a, w)$; $J_1(r^{(0)}, a, w) = J(r^{(0)}, a, w)$; $J_1(p, a, w)u_0$, if $J(q^{(0)}, a, w) = 1$; and $J_1(p, a, w)u$ iff $J(r^{(0)}, a, w) = \mathbb{1}$, for every $u \in V$ and $u \neq u_0$. If v_1 is the value-function for the s.g.r. RPg-model $\langle \underline{S}_0, V, \underline{S}_1, J_1\rangle$, then we obtain that for any $a \in S_0$, for any $w \in S_1$, and for any assignment $\tilde{a}$ for $\langle \underline{S}_0, V, \underline{S}_1\rangle$ the following equivalences hold: $v_1(\exists x(p(x) \to \forall yp(y)), \tilde{a}, a, w) = \mathbb{1}$ iff $v_1((q^{(0)} \to \forall yp(y)) \vee (r^{(0)} \to \forall yp(y)), \tilde{a}, a, w) = \mathbb{1}$ iff $v_1((q^{(0)} \to q^{(0)} \wedge r^{(0)}) \vee (r^{(0)} \to q^{(0)} \wedge r^{(0)}), a, w) = \mathbb{1}$. The last identity implies the identity $v_1((q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}), a, w) = \mathbb{1}$, which is equivalent to $v((q^{(0)} \to r^{(0)}) \vee (r^{(0)} \to q^{(0)}), a, w) = \mathbb{1}$. Therefore for any assignment $\tilde{a}$ for $\langle \underline{S}_0, V, \underline{S}_1\rangle$, $v_1(\exists x(p(x) \to \forall yp(y)), \tilde{a}, a_0, w_0) = \mathbb{0}$, and consequently $F \notin E(\underline{S}_0, V, \underline{S}_1)$.

Theorem 2.2 *The superrelevant predicate logic RQF does not possess characteristic classes of the s.g.r. RPg-spaces.*

Proof: On the strength of Lemmas 2.1 and 2.3.

By Lemmas 2.1 and 2.2 it is easily seen that the incompleteness result holds for the semantics of the type $\langle \mathfrak{A}, V\rangle$. Also, by Lemmas 2.1 and 2.3 we get that the incompleteness result holds for the semantics of the type $\langle \underline{S}, V\rangle$.

3 Let $\underline{S} = \langle S, R, P, g\rangle$ be a relevant RPg-space. A subset $H \subseteq S$ is called a $\leq_{\underline{S}}$-hereditary if for any $a, b \in S$ it follows from $a \in H$ and $a \leq_{\underline{S}} b$ that $b \in H$.

Theorem 3.1 For any s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1\rangle$ there exists s.g.r. RPg-space $\underline{S}_0^*, V, \underline{S}_1^*\rangle$ such that

(i) $\underline{S}_0^* = \langle S_0^*, R_0^*, P_0^*, g_0^*\rangle$ and $\underline{S}_1^* = \langle S_1^*, R_1^*, P_1^*, g_1^*\rangle$
(ii) $\leq_{\underline{S}_0^*}$ and $\leq_{\underline{S}_1^*}$ are partial orderings on S_0^* and S_1^*, respectively
(iii) P_0^* is a $\leq_{\underline{S}_0^*}$-hereditary subset of S_0^* and P_1^* is a $\leq_{\underline{S}_1^*}$-hereditary subset of S_1^*
(iv) $E(\underline{S}_0, V, \underline{S}_1) = E(\underline{S}_0^*, V, \underline{S}_1^*)$.

Proof: Suppose that $\langle \underline{S}_0, V, \underline{S}_1\rangle$ is an s.g.r. RPg-space. Let $P_0^0 = \{a \in S_0 |$ there exists $b \in P_0$ such that $b \leq_{\underline{S}_0} a\}$ and $P_1^0 = \{a \in S_1 |$ there exists $b \in P_1$ such that $b \leq_{\underline{S}_1} a\}$. By an easy verification we obtain that $\underline{S}_0^0 = \langle S_0, R_0, P_0^0, g_0\rangle$ and $\underline{S}_1^0 = \langle S_1, R_1, P_1^0, g_1\rangle$ are relevant RPg-spaces, and that $\leq_{\underline{S}_0} = \leq_{\underline{S}_0^0}$ and $\leq_{\underline{S}_1} =$

$\leq_{\underline{S}_1^0}$. Next, given the equivalence relations $\equiv_{\underline{S}_0}$ (for any $a, b \in S_0$, $a \equiv_{\underline{S}_0} b$ iff $a \leq_{\underline{S}_0^0} b$ and $b \leq_{\underline{S}_0^0} a$) and $\equiv_{\underline{S}_1}$ (for any $a, b \in S_1$, $a \equiv_{\underline{S}_1} b$ iff $a \leq_{\underline{S}_1^0} b$ and $b \leq_{\underline{S}_1^0} a$), we construct in the standard way the quotient relevant RPg-spaces $\underline{S}_0^* = \langle S_0^*, R_0^*, P_0^*, g_0^* \rangle$ and $\underline{S}_1^* = \langle S_1^*, R_1^*, P_1^*, g_1^* \rangle$ corresponding to $\underline{S}_0^0$ and $\underline{S}_1^0$, respectively. The final and crucial step of the proof consists in showing that the following condition holds:

(*) Let $\langle \underline{S}_0, V, \underline{S}_1, J \rangle$ be an s.g.r. RPg-model and let v be the value-function for the $\langle \underline{S}_0, V, \underline{S}_1, J \rangle$. Then for any $\alpha \in FOR$, if $v(\alpha, \tilde{a}, a, w) = \mathbb{1}$ and $b \geq_{\underline{S}_0} a$, $u \geq_{\underline{S}_1} w$, then $v(\alpha, \tilde{a}, b, u) = \mathbb{1}$.

But then the condition (*) does not call for a new proof, since it is a well-known lemma of Routley and Meyer ([9]). So by the condition (*) it can easily be seen that (iv) holds.

In the remainder of this paper we will discuss only s.g.r. RPg-spaces $\langle \underline{S}_0, V, \underline{S}_1 \rangle$, where P_0, P_1 are hereditary subsets of S_0 and S_1 with respect to the partial orderings $\leq_{\underline{S}_0}$ and $\leq_{\underline{S}_1}$, respectively. G.r. RPg-spaces will be also discussed as g.r. RPg-spaces $\langle \underline{S}, V, \mathfrak{A} \rangle$, in which the relevant RPg-space $\underline{S} = \langle S, R, P, g \rangle$ is such that $P \subseteq S$ is a hereditary subset with respect to the partial ordering $\leq_{\underline{S}}$.

The following construction allows the correlation with any relevant RPg-space $\underline{S}$ of the respective matrix. Let $\underline{S} = \langle S, R, P, g \rangle$ be a relevant RPg-space. The symbol $A(\underline{S})$ denotes the class of all $\leq_{\underline{S}}$-hereditary subsets of $\underline{S}$, whereas the symbol $Alg(\underline{S})$ denotes the algebra with universum $(\underline{S})$ and operations defined as follows: for any $H_1, H_2 \in A(\underline{S})$, $H_1 \wedge H_2 = H_1 \cap H_2$, $H_1 \vee H_2 = H_1 \cup H_2$, $H_1 \to H_2 = \{a \in S \mid$ for every $b, c \in S$: if $Rabc$ and $b \in H_1$, then $c \in H_2\}$, $\neg H_1 = \{a \mid g(a) \notin H_1\}$. It is obvious that in $Alg(\underline{S})$ there exist $\bigwedge_{t \in T} H_t$ and $\bigvee_{t \in T} H_t$, where T is a set of any power, Let $D(\underline{S}) = \{H \in A(\underline{S}) \mid P \subseteq H\}$.

Lemma 3.1 *For any relevant RPg-space $\underline{S} = \langle S, R, P, g \rangle$ such that the set P has the least element, $\mathfrak{A}(\underline{S}) = \langle Alg(\underline{S}), D(\underline{S}) \rangle$ is an $\mathfrak{m}$-simple C_{RQ}-matrix, where $\mathfrak{m}$ is any cardinal.*

Proof: It suffices to verify that: (i) the class $A(\underline{S})$ is closed with respect on the operations $\wedge$, $\vee$, $\to$, $\neg$; (ii) $D(\underline{S})$ is a filter; and (iii) conditions 1, ..., 14 defining an $\mathfrak{m}$-simple C_{RQ}-matrix hold (cf. also [5]).

Theorem 3.2 *For any s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1 \rangle$ such that the set $P_1 \subseteq S_1$ has a least element, $E(\underline{S}_0, V, \underline{S}_1) = E(\underline{S}_0, V, \mathfrak{A}(\underline{S}_1))$.*

Proof: Suppose that $\langle \underline{S}_0, V, \underline{S}_1, J \rangle$ is an s.g.r. RPg-model and v is the value-function for the $\langle \underline{S}_0, V, \underline{S}_1, J \rangle$. Let $\tilde{a}$ be an assignment for the s.g.r. RPg-space $\langle \underline{S}_0, V, \underline{S}_1 \rangle$. We define the interpretation function J_1 in the g.r. RPg-space $\langle \underline{S}_0, V, \mathfrak{A}(\underline{S}_1) \rangle$ as follows: $J_1(p^{(0)}, a) = \{w \in S_1 \mid J(p^{(0)}, a, w) = \mathbb{1}$ and $J_1(p^{(n)}, a)(\tilde{a}(x_0), \ldots, \tilde{a}(x_{n-1})) = \{w \in S_1 \mid J(p^{(n)}, a, w)\tilde{a}(x_0) \ldots \tilde{a}(x_{n-1})\}$, for $n \geq 1$. Let v_1 denote the value-function for the $\langle \underline{S}_0, V, \mathfrak{A}(\underline{S}_1) \rangle$. It is easy to verify that, for any $\alpha \in FOR$, for any assignment $\tilde{a}$ for the $\langle \underline{S}_0, V, \underline{S}_1 \rangle$ and for each $a \in S_0$, $v_1(\alpha, \tilde{a}, a) = \{w \in S_1 \mid v(\alpha, \tilde{a}, a, w) = \mathbb{1}\}$. The proofs of the remaining steps are easy and will be omitted.

Let $\mathfrak{A} = \langle \underline{A}, D \rangle$ be an $\mathfrak{m}$-simple C_{RQ}-matrix. We say that a proper $\mathfrak{m}$-complete filter F of the algebra $\underline{A}$ is an $\mathfrak{m}$-prime filter if it has the following property: for any $\overline{\overline{T}} \leq \mathfrak{m}$, if $\bigvee_{t \in T} a_t \in F$ then $\{a_t | t \in T\} \cap F \neq \emptyset$. If $\mathfrak{m} < \aleph_0$ then we identify $\mathfrak{m}$-prime filters with prime filters.

Lemma 3.2 *Let $\mathfrak{A} = \langle \underline{A}, D \rangle$ be an $\mathfrak{m}$-simple C_{RQ}-matrix. Then*
(i) *If $\mathfrak{m} \geq \aleph_0$, then the $\mathfrak{m}$-complete filter generated by a nonempty subset A_0 of the universum of the algebra $\underline{A}$ is the set of all elements $a \in A$ such that $a \geq \bigwedge_{t \in T} a_t$ for some elements $a_t \in A_0$, $t \in T$, and for some $\overline{\overline{T}} \leq \mathfrak{m}$*
(ii) *If $\mathfrak{m} < \aleph_0$, then the $\mathfrak{m}$-complete filter generated by a nonempty subset A_0 of the universum of the algebra $\underline{A}$ is the set of all elements $a \in A$ such that $a \geq a_0 \wedge \ldots \wedge a_{n-1}$ for some elements $a_0, \ldots, a_{n-1} \in A_0$.*

Proof: By an easy verification.

It follows easily from Lemma 3.3 that

Lemma 3.3 *Let a_0 be an element of the universum of the algebra $\underline{A}$ and let F be an $\mathfrak{m}$-complete filter of the algebra $\underline{A}$. Then $[F, a_0) = \{a \in A | a \geq a_0 \wedge c$ and $c \in F\}$ is the $\mathfrak{m}$-complete filter generated by the set $F \cup \{a_0\}$.*

Let the symbol $\mathbb{G}_0$ denote the set of all proper $\mathfrak{m}$-complete filters of the $\mathfrak{m}$-simple C_{RQ}-matrix $\mathfrak{A} = \langle \underline{A}, D \rangle$. Let $\mathbb{G} = \mathbb{G}_0 \cup \{\emptyset, A\}$. For any $F_1, F_2 \in \mathbb{G}$ let $F_1 \cdot F_2 = \{z |$ there exist $x \in F_1$ and $y \in F_2$ such that $x \leq y \to z\}$. By the symbol $\mathbb{F}_p$ we denote the set of all $\mathfrak{m}$-prime filters of the algebra $\underline{A}$. Let $\mathbb{F} = \mathbb{F}_p \cup \{\emptyset, A\}$. We now introduce certain specific definitions (cf. [5]), namely: for any $F, F_1, F_2 \in \mathbb{F}$, $R(F, F_1, F_2)$ holds iff $F \cdot F_1 \subseteq F_2$, $P = \{F | F \in \mathbb{F}$ and $F \supseteq D\}$, and $g(F) = \{x \in A | g(x) \notin F\}$.

Lemma 3.4 *Let $\mathfrak{A} = \langle \underline{A}, D \rangle$ be an $\mathfrak{m}$-simple C_{RQ}-matrix such that the set-theoretic union of any chain of $\mathfrak{m}$-complete filters of the algebra $\underline{A}$ is an $\mathfrak{m}$-complete filter. Let $F_1, F_2 \in \mathbb{G}$ and $F \in \mathbb{F}$. Then*
(i) *If $F_1 \cdot F_2 \subseteq F$, then there exists $F_1^* \in \mathbb{F}$ such that $F_1 \subseteq F_1^*$ and $F_1^* \cdot F_2 \subseteq F$*
(ii) *If $F_1 \cdot F_2 \subseteq F$, then there exists $F_2^* \in \mathbb{F}$ such that $F_2 \subseteq F_2^*$ and $F_1 \cdot F_2^* \subseteq F$.*

Proof: Let us note that if $\mathfrak{m} < \aleph_0$ then we consider filters and thus each algebra $\underline{A}$ of the $\mathfrak{m}$-simple C_{RQ}-matrix $\mathfrak{A} = \langle \underline{A}, D \rangle$ satisfies the condition: the set-theoretic union of any chain of filters is a filter. We only prove the lemma for the case when $\mathfrak{m} \geq \aleph_0$. To prove (i) let us assume that $K = \{F' | F \in \mathbb{G}, F_1 \subseteq F'$ and $F' \cdot F_2 \subseteq F\}$. The collection K is nonempty, since $F_1 \in K$. Let $\mathfrak{C}$ be a chain in $\langle K, \subseteq \rangle$ and let $P = \bigcup (X | X \in \mathfrak{C})$. Since $F_1 \subseteq F'$ for each $F' \in \mathfrak{C}$, then $F_1 \subseteq P$. Since $P \cdot F_2 = \bigcup (F' \cdot F_2 | F' \in \mathfrak{C})$, then $P \cdot F_2 \subseteq F$. In the case when $P \in \{\emptyset, A\}$, then we conclude the proof. Let us assume that $P \notin \{\emptyset, A\}$. Now, on the strength of the assumptions of the lemma, P is an $\mathfrak{m}$-complete filter. Hence, by the Kuratowski–Zorn lemma, K has a maximal element F^*. Let us assume that $F^* \notin \mathbb{F}_p \cup \{A\}$. Then there exist elements $a_t \in A$, $t \in T$, and $\overline{\overline{T}} \leq \mathfrak{m}$, such that $\bigvee_{t \in T} a_t \in F^*$ and $a_t | t \in T\} \cap F^* = \emptyset$. For each $t \in T$, let $F_t^* = [F^*, a_t)$ be an $\mathfrak{m}$-complete filter generated by the set $F^* \cup \{a_t\}$. Hence $F_t^* \supset F^*$ for each $t \in T$. Therefore for all $t \in T$ there exist $b_t, c_t \in A$ such that $b_t \to c_t \in F_t^*$, $b_t \in F_2$, and $c_t \notin F$. By Lemma 3.3 there exist $x_t \in F^*$, $t \in T$, so that $x_t \wedge$

$a_t \leq b_t \rightarrow c_t$. Let $x = \bigwedge_{t \in T} x_t$. Now, by an easy verification we get that $x \wedge a_t \leq \bigwedge_{t \in T} b_t \rightarrow \bigvee_{t \in T} c_t$ for all $t \in T$, and, consequently, $\bigvee_{t \in T} (x \wedge a_t) \leq \bigwedge_{t \in T} b_t \rightarrow \bigvee_{t \in T} c_t$. Since $x \wedge \bigvee_{t \in T} a_t \leq \bigvee_{t \in T} (x \wedge a_t)$ then $x \wedge \bigvee_{t \in T} a_t \leq \bigwedge_{t \in T} b_t \rightarrow \bigvee_{t \in T} c_t$. Hence it follows that $\bigwedge_{t \in T} b_t \rightarrow \bigvee_{t \in T} c_t \in F^*$, but because $\bigwedge_{t \in T} b_t \in F_2$, then $\bigvee_{t \in T} c_t \in F$—a contradiction, since F is an $\mathfrak{m}$-prime filter.

The proof of case (ii) is entirely analogous to the previous proof, and therefore is omitted.

Lemma 3.5 *For any $F, F_1 \in \mathbb{G}$:*
(i) $F \cdot F_1 \in \mathbb{G}$
(ii) $D \cdot F = F$
(iii) $F \cdot D \subseteq F$.

Proof: To prove (i) observe that if $F \cdot F_1 \in \{\emptyset, A\}$ then $F \cdot F_1 \in \mathbb{G}$. Now let us assume that $\{a_t | t \in T\} \subseteq F \cdot F_1 \notin \{\emptyset, A\}$. Hence we get that there exist $b_t \in F_1$ for each $t \in T$ such that $b_t \rightarrow a_t \in F$. Then $\bigwedge_{t \in T} b_t \in F_1$ and $\bigwedge_{t \in T} (b_t \rightarrow a_t) \in F$. Since $\bigwedge_{t \in T} (b_t \rightarrow a_t) \leq \bigwedge_{t \in T} b_t \rightarrow \bigwedge_{t \in T} a_t$ then $\bigwedge_{t \in T} b_t \rightarrow \bigwedge_{t \in T} a_t \in F$ and, consequently, $\bigwedge_{t \in T} a_t \in F \cdot F_1$. Now, assuming that $a \in F \cdot F_1$ and $a \leq b$ we get immediately that there exists $c \in F_1$ such that $c \rightarrow a \in F$ and that $c \rightarrow a \leq c \rightarrow b$. Then $c \rightarrow b \in F$ yields $b \in F \cdot F_1$.

To prove (ii) observe that if $x \in D \cdot F$ then there exists $y \in F$ such that $y \rightarrow x \in D$. Hence $y \in x$ and so $x \in F$. Next, assuming that $x \in F$ we get that also $x \in D \cdot F$ by virtue of the fact that $x \rightarrow x \in D$.

To prove (iii) suppose that $x \in F \cdot D$. Thus for some $y \in D$, $y \rightarrow x \in F$. Therefore $(y \rightarrow x) \rightarrow x \in D$ and so $y \rightarrow x \leq x$. Hence $x \in F$.

Lemma 3.6 *Let $\mathfrak{A} = \langle \underline{A}, D \rangle$ be an $\mathfrak{m}$-simple C_{RQ}-matrix such that the set-theoretic union of any chain of $\mathfrak{m}$-complete filters of the algebra $\underline{A}$ is an $\mathfrak{m}$-complete filter. Then $\mathbb{F}(\mathfrak{A}) = \langle \mathbb{F}, R, P, g \rangle$ is a relevant RPg-space.*

Proof: In the first place we shall show that the following condition holds:

(*) For any $F_1, F_2 \in \mathbb{F}$, $F_1 \subseteq F_2$ iff there exists $F \in P$ such that $R(F, F_1, F_2)$.

Let us assume that $F_1 \subseteq F_2$. Hence by Lemma 3.5(ii) $D \cdot F_1 \subseteq F_2$. If $D \in \mathbb{F}_p \cup \{A\}$ then on the strength of Lemma 3.4(i) there exists $F \in \mathbb{F}$ such that $D \subseteq F$ and $F \cdot F_1 \subseteq F_2$. Therefore, $F \in P$ and $R(F, F_1, F_2)$. Conversely, let $R(F, F_1, F_2)$ for some $F \in P$. Then $F \cdot F_1 \subseteq F_2$. Since $F \in P$ then $D \subseteq F$. Hence $D \cdot F_1 \subseteq F_2$, and consequently by Lemma 3.5(ii) $F_1 \subseteq F_2$.

Remembering about the possibility of applying Lemmas 3.4, 3.5, and Condition (*) the reader should not have any difficulties in supplying all the proofs that are omitted here.

To verify RPg 9 observe that if $F \in \mathbb{F}$ then $g(F) \in \mathbb{F}$. Applying the definition of the set $g(F)$ one gets that $x \in g(g(F))$ iff $\neg x \in g(F)$ iff $\neg\neg x \in F$ iff $x \in F$.

To prove RPg 10 suppose to the contrary that $F \cdot F_1 \subseteq F_2$ and $F \cdot g(F_2) \nsubseteq g(F_1)$. Hence there exists $y, z \in A$ such that $y \rightarrow z \in F$, $y \in g(F_2)$ and $z \notin$

$g(F_1)$. Since $y \to z \leq \neg z \to \neg y$, therefore $\neg z \to \neg y \in F$ and $\neg z \in F_1$, and consequently $\neg y \in F_2$. Hence we conclude that $y \notin g(F_2)$—a contradiction.

To prove RPg 11 let us note only that for any $x, y \in A$, $x \wedge \neg y \leq \neg(x \to y)$, since the proofs of the remaining steps are easy.

Lemma 3.7 (cf. [3]) *Let $\mathfrak{A} = \langle \underline{A}, D\rangle$ be a finite simple C_{RQ}-matrix. Then $\mathfrak{A}$ and $\mathfrak{A}(\mathbb{F}_p(\mathfrak{A}))$ are isomorphic.*

Proof: The fact that $\mathbb{F}_p(\mathfrak{A}) = \langle \mathbb{F}_p, R, P, g\rangle$ is a relevant RPg-space is proved in an analogous manner to the proof of Lemma 3.6. We add, however, that in every finite simple C_{RQ}-matrix the set-theoretic union of any chain of filters is a filter. Now, we define the function $h\colon A \mapsto A(\mathbb{F}_p(\mathfrak{A}))$ as follows: $h(x) = \{F \in \mathbb{F}_p | x \in F\}$ for every $x \in A$. We omit the proof that the thus-defined function h is a well-defined function and that h is a bijection. And, moreover, the reader can easily verify that the function h preserves operations $\wedge$, $\vee$, $\neg$; and that for each $x \in A$, $x \in D$ iff $h(x) \in D(\mathbb{F}_p(\mathfrak{A}))$. To prove the inclusion $h(x \to y) \subseteq h(x) \to h(y)$ assume that $F \in h(x \to y)$ and $F \notin h(x) \to h(y)$. Then we get that $x \to y \in F$ and there exist $F_1, F_2 \in \mathbb{F}_p$ such that $R(F, F_1, F_2)$, $F_1 \in h(x)$, and $F_2 \notin h(y)$. Hence we have $F \cdot F_1 \subseteq F_2$, $x \in F_1$ and $y \notin F_2$. Thus $y \in F \cdot F_1$, and consequently $y \in F_2$—a contradiction. Now, we shall prove that for all $y \in A$ and all $x \in A$, $x \neq 0_A$, $h(x) \to h(y) \subseteq h(x \to y)$. Let us assume that for some $F \in \mathbb{F}_p$, $F \notin h(x \to y)$. Hence $x \to y \notin F$. Let us consider the sets $F_1 = \{z | x \leq z\}$ and $F \cdot F_1$. Thus $y \notin F \cdot F_1$, because if $y \in F \cdot F_1$ then $z \to y \in F$ for some $z \in F_1$ and so $x \to y \in F$ on the strength of $z \to y \leq x \to y$. Let F_2 be an element in the set $\mathbb{F}_p$ such that $F \cdot F_1 \subseteq F_2$ and $y \notin F_2$. On the basis of Lemma 3.4(ii) there exists $F^* \in \mathbb{F}_p$ such that $F_1 \subseteq F^*$ and $F \cdot F^* \subseteq F_2$. Hence we obtain that $R(F, F^*, F_2)$, $F^* \in h(x)$, and $F_2 \in h(y)$, therefore $F \notin h(x) \to h(y)$. In this way, it remains to prove that $h(0_A) \to h(y) \subseteq h(0_A \to y)$ for all $y \in A$. But since $\mathfrak{A}$ is a finite simple C_{RQ}-matrix, then in the algebra $\underline{A}$ there exists a greater element 1_A and $0_A \to y = 1_A$. Hence $h(0_A \to y) = \mathbb{F}_p$.

Theorem 3.3 *Let $\mathfrak{A} = \langle \underline{A}, D\rangle$ be a $\kappa(V,S)$-simple C_{RQ}-matrix such that the set-theoretic union of any chain of $\kappa(V,S)$-complete filters of the algebra $\underline{A}$ is a $\kappa(V,S)$-complete filter. Then $E(\underline{A}, V, \mathfrak{A}(\mathbb{F}(\mathfrak{A}))) \subseteq E(\underline{S}, V, \mathfrak{A})$.*

Proof: By the construction of the relevant RPg-space $\mathbb{F}(\mathfrak{A}) = \langle \mathbb{F}, R, P, g\rangle$ the set P has a least element. So by Lemma 3.1 $\mathfrak{A}(\mathbb{F}(\mathfrak{A}))$ is a $\kappa(V,S)$-simple C_{RQ}-matrix. Let h be the function from A into $A(\mathbb{F})$ given by: $h(x) = \{F \in \mathbb{F} | x \in F\}$ for every $x \in A$. Let us assume that $\alpha \notin E(\underline{S}, V, \mathfrak{A})$. Hence there must exist a g.r. RPg-model $\langle \underline{S}, V, \mathfrak{A}, J\rangle$ such that $\alpha \notin E(\underline{S}, V, \mathfrak{A}, J)$. Now, we define the interpretation function J_1 in the g.r. RPg-space $\langle \underline{S}, V, \mathfrak{A}(\mathbb{F}(\mathfrak{A}))\rangle$ as follows: $J_1(p^{(0)}, a) = h(u)$, if $J(p^{(0)}, a) = u$; and $J_1(p^{(n)}, a)(\tilde{a}(x_0), \ldots, \tilde{a}(x_{n-1})) = h(u)$, if $J(p^{(n)}, a)(\tilde{a}(x_0), \ldots, \tilde{a}(x_{n-1})) = u$, $n \geq 1$.

Let v_1 denote the value-function for $\langle \underline{S}, V, \mathfrak{A}(\mathbb{F}(\mathfrak{A}))\rangle$. We shall prove that

(*) For any formula α, for any assignment $\tilde{a}$ for $\langle \underline{S}, V, \mathfrak{A}\rangle$ and for each $a \in S$: $v_1(\alpha, \tilde{a}, a) = h(u)$, if $v(\alpha, \tilde{a}, a) = u$.

The proof of (*) goes by induction with respect to the length of the formula α. We verify only the cases when $\alpha = \beta \to \gamma$ and $\alpha = \forall x\beta$, because the verification of the remaining cases is easy. By the proof of Lemma 3.7, $h(x \to y) \subseteq h(x) \to$

$h(y)$ for any $x, y \in A$. To prove the inclusion $h(x) \to h(y) \subseteq h(x \to y)$ assume that for some $F \in \mathbb{F}$, $F \notin h(x \to y)$. Hence $x \to y \notin F$. Let $F_1 = \{z | x \leq z\}$. Therefore $y \notin F \cdot F_1$, because if $y \in F \cdot F_1$ then $z \to y \in F$ for some $z \in F_1$ and so $x \to y \in F$ on the strength of the $z \to y \leq x \to y$. Let us consider $F_2 \in \mathbb{F}$ such that $F \cdot F_1 \subseteq F_2$ and $y \notin F_2$. By Lemma 3.4(ii) there exists $F^* \in \mathbb{F}$ such that $F_1 \subseteq F^*$ and $F \cdot F^* \subseteq F_2$. Hence it follows that $R(F, F^*, F_2)$, $F^* \in h(x)$, and $F_2 \notin h(y)$, so $F \notin h(x) \to h(y)$. In order to show that $h\left(\bigwedge_{t \in T} a_t\right) = \bigwedge_{t \in T} h(a_t)$ for each T such that $\bar{\bar{T}} \leq \kappa(V, S)$ let us note that $F \in h\left(\bigwedge_{t \in T} a_t\right)$ iff $\bigwedge_{t \in T} a_t \in F$ iff for each $t \in T$, $a_t \in F$ iff for each $t \in T$, $F \in h(a_t)$ iff $F \in \bigwedge_{t \in T} h(a_t)$. Let $\alpha = \beta \to \gamma$. In this case we have: $v_1(\beta \to \gamma, \tilde{a}, a) = \bigwedge (v_1(\beta, \tilde{a}, b) \to v_1(\gamma, \tilde{a}, c) | b, c \in S \text{ and } Rabc) = \bigwedge (h(v(\beta, \tilde{a}, b)) \to h(v(\gamma, \tilde{a}, c)) | b, c \in S \text{ and } Rabc) = \bigwedge (h(v(\beta, \tilde{a}, b) \to v(\gamma, \tilde{a}, c)) | b, c \in S \text{ and } Rabc) = h\big(\bigwedge (v(\beta, \tilde{a}, b) \to v(\gamma, \tilde{a}, c) | b, c \in S \text{ and } Rabc)\big) = h(v(\beta \to \gamma, \tilde{a}, a))$. If $\alpha = \forall x\beta$, then we obtain that $v_1(\alpha, \tilde{a}, a) = \bigwedge (v_1(\beta, \tilde{a}', a) | \tilde{a}'$ is an assignment that agrees with $\tilde{a}$ except on $x) = \bigwedge (h(v(\beta, \tilde{a}', a)) | \tilde{a}'$ is an assignment that agrees with $\tilde{a}$ except on $x) = h\big(\bigwedge (v(\beta, \tilde{a}', a)) \big| \tilde{a}'$ is an assignment that agrees with $\tilde{a}$ except on $x\big) = h(v(\alpha, \tilde{a}, a))$.

Since $\alpha \notin E(\underline{S}, V, \mathfrak{A}, J)$, then there exists an assignment $\tilde{a}$ for $\langle \underline{S}, V, \mathfrak{A} \rangle$ and $a \in P$ such that $v(\alpha, \tilde{a}, a) \notin D$. Thus, on the strength of the (*), $v_1(\alpha, \tilde{a}, a) = h(v(\alpha, \tilde{a}, a)) \notin h(D) = D(\mathbb{F}(\mathfrak{A}))$, and consequently $\alpha \notin E(\underline{S}, V, \mathfrak{A}(\mathbb{F}(\mathfrak{A})))$.

Corollary 3.1 *Let $\mathfrak{A} = \langle \underline{A}, D \rangle$ be a $\kappa(V, S)$-simple C_{RQ}-matrix such that the union set-theoretical of any chain of $\kappa(V, S)$-complete filters of the algebra $\underline{A}$ is a $\kappa(V, S)$-complete filter. Then $E(\underline{S}, V, \mathbb{F}(\mathfrak{A})) \subseteq E(\underline{S}, V, \mathfrak{A})$.*

Proof: If follows from Theorems 3.2 and 3.3.

Theorem 3.4 *Let $\mathfrak{A} = \langle \underline{A}, D \rangle$ be a finite C_{RQ}-matrix. Then $E(\underline{S}, V, \mathfrak{A}) = E(\underline{S}, V, \mathfrak{A}(\mathbb{F}_p(\mathfrak{A})))$.*

Proof: It follows from Lemma 3.7.

NOTE

1. It should also be mentioned that *RQ* is incomplete for the standard ternary relation semantics and yet complete for a nonstandard ternary relation semantics, distinguished by a special clause for the quantifier. These results were given by Kit Fine in his unpublished work. The author is greatly indebted to the editor for bringing this fact to his attention.

REFERENCES

[1] Anderson, A. R. and N. D. Belnap, Jr., *Entailment*, Vol. I, Princeton University Press, Princeton, 1975.

[2] Dunn, J. M., *The Algebra of Intensional Logics*, Ph.D. Dissertation, University of Pittsburgh, 1966.

[3] Dziobiak, W., and W. J. Blok, "On the lattice of quasivarieties of Sugihara algebras," *Studia Logica*, vol. 45 (1986), pp. 277–280.

[4] Hawranek, J. and M. Tokarz, "Matrices for predicate logics," *Reports on Mathematical Logic*, vol. 9 (1977), pp. 27–30.

[5] Maksimova, L. L., "Structures with implication," *Algebra i Logika*, vol. 12 (1973), pp. 445–467.

[6] Pogorzelski, W. A. and T. Prucnal, "Structural completeness of the first-order predicate calculus," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 21 (1975), pp. 315–320.

[7] Pogorzelski, W. A. and T. Prucnal, "The substitution rule for predicate letters in the first-order predicate calculus," *Reports on Mathematical Logic*, vol. 5 (1975), pp. 77–90.

[8] Routley, R., "Problems and solutions in the semantics of quantified relevant logics I," pp. 305–340 in *Mathematical Logic in Latin America*, ed., A. I. Arruda, R. Chuaqui and N. C. A. da Costa, North Holland, Amsterdam, 1980.

[9] Routley, R. and R. K. Meyer, "The semantics of entailment," pp. 199–243 in *Truth, Syntax and Modality*, ed., H. Leblanc, North Holland, Amsterdam, 1973.

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