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On An Implication Connective of RM

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Introduction The Dunn-McCall system *RM* was developed and studied by the "Entailment" school (mainly by Meyer and Dunn), but it can hardly be called "relevance logic" because of theorems like $\sim(A \to A) \to (B \to B)$ and $(A \to B) \vee (B \to A)$ (see [1], 29.5); yet it is a strong and decidable logic which still avoids $A \to (B \to A)$ and $\sim A \to (A \to B)$.

Some new light on *RM* is shed here (so we hope) by investigating an implication connective $\supset$ definable in it by $(A \to B) \vee B$. "$\supset$" has most of the properties one might expect an implication to have in a paraconsistent logic[1]: respecting M.P., the "official" deduction theorem, and a strong version of the Craig interpolation theorem: $RM \vdash A \supset B$ iff either $RM \vdash B$, or there is an interpolant C for A and B. (In classical logic there is also the possibility that $\vdash \sim A$.) These facts are all proved in Section 1.

In Section 2 we investigate *RM* as a system in the $\{\sim, \vee, \wedge\supset\}$ language. We give a simple axiomatization of its $\{\sim\supset\}$ fragment, which suffices for characterizing the Sugihara matrix.[2] In this fragment $\to$ is definable (so the Sobociński logic[3] is a proper subsystem of it), but $\vee$ is not. We get the full system *RM* by adjoining some natural axioms concerning $A \vee B$ and $\sim(A \vee B)$ to its $\{\sim\supset\}$ fragment. In contrast to extending with $\vee$ the $\{\sim\to\}$ fragment, this extension causes no essential changes.

From the simple classical laws concerning combinations of $\sim$ with $\supset$, $\vee$, and $\wedge$, *RM* only lacks $\sim(A \supset B) \supset A$ and $\sim A \supset (A \supset B)$. By adding, in Section 3, the first schema to *RM*, we get a three-valued logic equivalent to what was called RM_3 in [1]. This system might be considered an optimal paraconsistent logic, since its positive fragment (in the $\{\supset, \wedge, \vee\}$ language) is identical with the classical one. It avoids $\sim A \supset (A \supset B)$, but every proper extension of it (closed under substitutions) is equivalent to *PC*.

Preliminaries The system *RM* is obtained from the system *R* by adding to it the mingle axiom $A \to (A \to A)$. We assume the reader is acquainted with this system and its properties, as described in [1], 29.3–4.

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A Sugihara matrix is a structure $\langle S, \leq, \sim, \rightarrow\rangle$ in which $\langle S, \leq\rangle$ is a linearly ordered set, and $\sim$ is a unary operation satisfying the *De-Morgan conditions*: $\sim\sim a = a$; $a \leq b \Rightarrow \sim b \leq \sim a$; and $a \rightarrow b$ is $\sim a \vee b$ if $a \leq b$, $\sim a \wedge b$ otherwise (where $\vee$ and $\wedge$ are the usual lattice operations). We call $a \in S$ *designated* iff $\sim a \leq a$. (This definition is a version of Dunn's concept of Sugihara chain appearing in [1], p. 421.)

Among the Sugihara matrices particularly important are the matrices S_Z, S_Q, and S_i, $S_i(0)$. S_Z, which we shall call here sometimes *the Sugihara matrix*, consists of the integers with their usual order relation and where $\sim a$ is taken to be $-a$ (S_Q is based in the same way on the rational numbers). $S_i(0)$ is the submatrix of S_Z consisting of the integers between (and including) $-i$ and i. S_i is $S_i(0) - \{0\}$. Meyer has proved S_Z to be characteristic for *RM*.

By an *extension* of a system L we mean a set of sentences in the language of L which contains all theorems of L and is closed under the rules of L and under substitutions. Dunn has shown that any proper extension of *RM* has some S_i or $S_i(0)$ $(1 \leq i < \infty)$ as a characteristic matrix. It further follows from his work ([1], 29.4, and [5]) that if T is an *RM*-theory and ϕ a sentence such that $T \nvdash_{RM} \phi$, then there is a valuation v in S_Q such that $v(A) \geq 0$ for any $A \in T$, but $v(\phi) < 0$.

Section 1 The connective $\supset$ of RM

1.1 Definition $\quad A \supset B \stackrel{def}{\equiv} (A \rightarrow B) \vee B$.

1.2 Deduction theorem $\quad \mathfrak{I}, A \vdash_{RM} B$ *iff* $\mathfrak{I} \vdash_{RM} A \supset B$.

Proof: Suppose $\mathfrak{I} \vdash A \supset B$.[4] Since $A \vdash (A \rightarrow B) \rightarrow B$ and $\vdash B \rightarrow B$, we have $A \vdash ((A \rightarrow B) \vee B) \rightarrow B$, i.e., $A \vdash (A \supset B) \rightarrow B$. So $\mathfrak{I}, A \vdash B$ in this case. For the converse, suppose $\mathfrak{I} \nvdash A \supset B$. By Meyer's and Dunn's completeness theorems for *RM*, there exists a valuation v in S_Q such that $v(\phi) \geq 0$ for every $\phi \in \mathfrak{I}$ and $v(A \supset B) < 0$. Hence, $v(B) < 0$ and $v(A \rightarrow B) < 0$, so $v(A) > v(B)$. If $v(A) \geq 0$, then obviously $\mathfrak{I}, A \nvdash B$. Otherwise $v(B) < v(A) < 0$ and $|v(A)| < |v(B)|$. Now define v' by

$$v'(\phi) = \begin{cases} 0 & |v(\phi)| < |v(B)| \\ v(\phi) & \text{otherwise.} \end{cases}$$

It is easy to prove that v' is a well-defined valuation, that $v'(\phi) \geq 0$ for $\phi \in \mathfrak{I} \cup \{A\}$ and $v'(B)$ $(= v(B)) < 0$. Hence, $\mathfrak{I}, A \nvdash B$.

1.3 Remarks

(a) The intuition behind the $\supset$-definition is that $\mathfrak{I}, A \vdash B$ iff either $\mathfrak{I} \vdash B$ or there is a proof of B in $\mathfrak{I}$ from the hypothesis A that actually uses A, in which case $A \rightarrow B$ must be provable in $\mathfrak{I}$. This intuition is not correct in R and *RM* since $A \rightarrow B \vdash A \rightarrow (A \wedge B)$, but neither $A \rightarrow (A \wedge B)$ nor $(A \rightarrow B) \rightarrow A \rightarrow. (A \wedge B)$ are theorems of *RM*. It is strange, therefore, that it leads to correct results.

(b) In a language containing a propositional constant t and in RM^t (the conservative extension of *RM* by the axioms t and $t \rightarrow (B \rightarrow B)$), $A \supset B$ is equivalent to $A \wedge t \rightarrow B$ (we show this immediately below). Now $A \wedge t \rightarrow B$ serves in R and

RM to define the enthymematic implication (see [2] and [7]), for which the deduction theorem is easily proved. This can be used to give another proof of 1.2.

To show that the claimed equivalence holds, we note first that since $\vdash_{\overline{RM^t}} B \to (t \to B)$, we have that $\vdash_{\overline{RM^t}} B \to (A \wedge t \to B)$. Obviously also $\vdash_{\overline{RM^t}} (A \to B) \to (A \wedge t \to B)$, so $\vdash_{\overline{RM^t}} (A \supset B) \to (A \wedge t \to B)$.

For the converse we observe that from the following three theorems of RM^t: t, $t \to (\sim t \to t)$ and $t \to (A \to A)$, we easily get that $\vdash_{\overline{RM^t}} \sim t \to (A \to A)$, or $\vdash_{\overline{RM^t}} \sim A \to (A \to t)$. Since also $\vdash_{\overline{RM}} \sim A \to (A \to A)$, we have that $\vdash_{\overline{RM^t}} \sim A \to (A \to A \wedge t)$ and so $\vdash_{\overline{RM^t}} A \vee (A \to A \wedge t)$. Using distribution, we have also $\vdash_{\overline{RM^t}} (A \wedge t) \vee (A \to A \wedge t)$. But obviously $\vdash_{\overline{RM^t}} A \wedge t \to [(A \wedge t \to B) \to B]$ and $\vdash_{\overline{RM^t}} (A \to A \wedge t) \to [(A \wedge t \to B) \to (A \to B)]$. Hence $\vdash_{\overline{RM^t}} [(A \wedge t \to B) \to B] \vee [(A \wedge t \to B) \to (A \to B)]$. From this $\vdash_{\overline{RM^t}} (A \wedge t \to B) \to (A \supset B)$ follows at once.

1.4 Craig interpolation theorem *$\vdash A \supset B$ iff $\vdash B$ or there is a sentence C, containing only propositional variables common to A and B, such that $\vdash A \supset C$, $\vdash C \supset B$.*

Proof: We confine ourselves to sentences in the language of $A \supset B$. Suppose $\vdash A \supset B$ and $\nvdash B$. We assume that there is no interpolant C and then get a contradiction. Let $S = \{C | \vdash A \supset C$, and C only contains variables common to A and $B\}$. By 1.2, S is closed under M.P. and adjunction. Hence (and since $\nvdash B$), our no-interpolant assumption and 1.2 again imply that $S \nvdash B$. Let $T_0 \supseteq S$ be a maximal theory in the language of S such that $T_0 \nvdash B$, and let T_1 be a maximal extension of T_0 in the language of B *such that* $T_1 \nvdash B$. Both T_0 and T_1 are easily seen to be prime,[5] and T_1 is a conservative extension of T_0.

Now, $T_0 \cup \{A\}$ is also a conservative extension of T_0, for if $T_0 \cup \{A\} \vdash D$, D in the language of T_0, then there are Theorems $C_1 \ldots C_n$ of T_0 such that $\vdash A \supset (C_1 \supset (\ldots (C_n \supset D) \ldots))$. Therefore $C_1 \supset (\ldots \supset (C_n \supset D) \ldots) \in S \subset T_0$, and $T_0 \vdash D$ as well.

Let T_2 be a maximal conservative extension of T_0 in the language of A, which contains A. T_2 is also prime.

We now define three equivalence relations $\sim_i$ for sentences in the language of T_i ($i = 0, 1, 2$) by:

$$\phi \sim_i \psi \overset{def}{\equiv} T_i \vdash \phi \leftrightarrow \psi.$$

Let $[\phi]^i$ be the equivalence class of ϕ relative to $\sim_i$, with S_i the set of equivalence classes. Let $\leq_i$, $\wedge_i$, $\vee_i$, $\to_i$, $\sim_i$ be defined on S_i in the obvious manner. By the completeness proof of RM and its extensions (see [5]), S_i is a finite Sugihara matrix in which exactly the theorems of T_i are true under the canonical valuation v_i (defined by $v_i(\phi) = [\phi]^i$ for every ϕ). Moreover, since T_0 is prime and the T_i ($i = 1, 2$) are conservative extensions of it, the mappings h_i: $S_0 \to S_i$ ($i = 1, 2$) defined by $h([\psi]^0) = [\psi]^i$ are embeddings of S_0 in S_i for which: (*) $h_i v_0 = v_i$ ($i = 1, 2$).

It is now easy to construct two embeddings g_i ($i = 1, 2$) of S_i in the infinite Sugihara matrix S_Z in such a way that (**) $g_1 h_1 = g_2 h_2$. (For example, S_0

can be mapped on a finite arithmetical sequence with a large enough difference, and then the definition of g_i can be completed.)

Finally, we define $v(P)$ as $g_i v_i(P)$ for P atomic in the language of T_i. By (*) and (**) v is well defined, and $v(\phi) = g_i h_i(\phi)$ for every ϕ in the language of T_i. In particular, since $T_2 \vdash A$ and $T_1 \nvdash B$, $v(A)$ is designated and $v(B)$ is not. This contradicts the validity of $A \supset B$.

1.5 Remark In [1], pp. 416–417, it is shown that the Craig theorem fails for $A \to B$ in *RM*, and it was conjectured that, "There is an appropriate version of that theorem, perhaps involving sentential constants, which does hold for RM". Theorem 1.4 gives an affirmative answer to this conjecture: using 1.3(b), 1.4 entails that if $RM^t \vdash A \wedge t \to B$, then there is an interpolant C such that $RM^t \vdash A \wedge t \to C$, $RM^t \vdash C \wedge t \to B$. (Note that C may contain t.[6])

1.6 Theorem *The* $\{\supset, \wedge, \vee\}$ *fragment of RM is identical to the corresponding fragment of the system LC*[7] *of Dummett.*

Proof: This is essentially proved in [6], taking 1.3(b) into account.

1.7 Theorem on definability
(a) $\to$ *is definable in RM using* $\sim$ *and* $\supset$.
(b) $\supset$ *is undefinable in RM using* $\sim$ *and* $\to$.
(c) $\vee$ *is undefinable in RM using* $\sim$ *and* $\supset$.
(d) $\to$ *is undefinable in RM using* $\sim$ *and* $\vee$.
(e) $\to$ *is undefinable in RM using* $\supset$, $\vee$ *and* $\wedge$.

Proof: (a) We leave it to the reader to check that $A \to B$ is equivalent in the Sugihara matrix to $\sim(A \supset B) \supset \sim (B \supset A)$. We note, however, that $\to$ is most naturally defined in the $\{\sim, \supset, \wedge\}$ language by $(A \supset B) \wedge (\sim B \supset \sim A)$.

(b) For any sentence A in the $\{\sim \to\}$ language and a valuation v in the Sugihara matrix $|v(A)| = max\ (|v(P_1)| \dots |v(P_n)|)$, where $P_1 \dots P_n$ are the atomic variables of A. $P \supset Q$, on the other hand, lacks this property. (If $v(P) = 1$, $v(Q) = 0$, then $v(P \supset Q) = 0$.)

(c) Call an atomic variable P a 0-atom of ϕ if for any valuation v in S_Z $v(\phi) = 0 \Rightarrow v(P) = 0$. Now $p \vee q$ has no 0-atom, but any sentence in the $\{\sim, \supset\}$ language has. This is easily shown by induction on the length of ϕ: if ϕ is atomic the claim is trivial. Also, any 0-atom of B is also a 0-atom of $\sim B$ and $A \supset B$.

(d)–(e) We leave the proofs to the reader.

Section 2 Axiomatizing RM and $RM_{\supset}$

2.1 The system $RM_{\supset}$

A1 $A \supset (B \supset A)$
A2 $A \supset (B \supset C) \supset. (A \supset B) \supset (A \supset C)$
A3 $A \supset \sim\sim A$
A4 $\sim\sim A \supset A$
A5 $(\sim A. \supset B) \supset. (A \supset B) \supset B$
A6 $A \supset. \sim B \supset \sim(A \supset B)$
A7 $\sim(A \supset B) \supset \sim B$
A8 $(A \supset B) \supset. \sim(A \supset B) \supset A.$
Inference rule. $A, A \supset B/B$ (M.P.)

2.2 Theorem *All theorems of $RM_{\supset}$ are valid in the Sugihara matrix and so are provable in RM.*

Proof: By 1.2, in order to prove the validity of a sentence of the form $A_1 \supset ((A_2 \supset \ldots \supset (A_n \supset B)) \ldots)$ in the Sugihara matrix, it is enough to consider valuations in which $A_1 \ldots A_n$ all get designated values and show that B also gets a designated value. We leave details to the reader.

2.3 Completeness Theorem *Let L be an extension of $RM_{\supset}$. Let ϕ be a sentence in this language such that $L \nvdash \phi$. Then there is a finite Sugihara matrix in which all theorems of L are valid but ϕ is not.*

Proof: Let S be a Sugihara matrix.

The operation $\supset$ on S (corresponding to the connective $\supset$) is defined by:

$$(*) \quad a \supset b = \begin{cases} \sim a & a \leq b \leq \sim a \\ b & \text{otherwise.} \end{cases}$$

We may suppose that L is an extension by schemata of $RM_{\supset}$. Suppose $L \nvdash \phi$ and let $P_1 \ldots P_n$ be the sentenial variables of ϕ. We deal from now on only with sentences in the $\{P_1 \ldots P_n\}$ language.

As usual, the presence of A_1 and A_2 provides a deduction theorem for $RM_{\supset}$, and using A5 we can find a complete L-theory $\mathfrak{I}$ such that $\mathfrak{I} \nvdash \phi$. Define $A \sim_{\mathfrak{I}} B \overset{def}{\equiv} \mathfrak{I} \vdash A \supset B$, $\mathfrak{I} \vdash B \supset A$, $\mathfrak{I} \vdash \sim A \supset \sim B$ and $\mathfrak{I} \vdash \sim B \supset \sim A$. $\sim$ is an equivalence relation. Let $[A]$ denote the equivalence class of A and let S be the set of equivalence classes. Further, define $\sim[A] = [\sim A]$ and $[A] \leq [B]$ iff $\mathfrak{I} \vdash A \supset B$ and $\mathfrak{I} \vdash \sim B \supset \sim A$. By definition of $\sim_{\mathfrak{I}}$ and A1–A4, $\sim$ and $\leq$ are well defined. $\leq$ partially ordered S and the De Morgan conditions are satisfied.

We now show that $\leq$ is linear. First, we note that by A8 (using A1):

$$(**) \quad RM_{\supset} \vdash \sim(A \supset B) \supset (B \supset A).$$

Now, if $\mathfrak{I} \vdash A \supset B$ and $\mathfrak{I} \vdash \sim B \supset \sim A$, then $[A] \leq [B]$. Otherwise, by completeness of $\mathfrak{I}$, $\mathfrak{I} \vdash \sim(A \supset B)$ or $\mathfrak{I} \vdash \sim(\sim B \supset \sim A)$.

If $\mathfrak{I} \vdash \sim(A \supset B)$, then by (**) $\mathfrak{I} \vdash B \supset A$. Also, by A7, $\mathfrak{I} \vdash \sim B$ and so $\mathfrak{I} \vdash \sim A \supset \sim B$. Hence, $[B] \leq [A]$.

If $\mathfrak{I} \vdash \sim(\sim B \supset \sim A)$, then by (**) $\mathfrak{I} \vdash \sim A \supset \sim B$ and by A7, A4, and A1, $\mathfrak{I} \vdash B \supset A$. So again $[B] \leq [A]$.

$(S, \leq, \sim)$ is, therefore, a Sugihara matrix. We now show that if $\supset$ is defined on S according to (*), then $[A] \supset [B] = [A \supset B]$ for all A, B. We argue by cases:

First, suppose $[A] \leq [B] \leq \sim[A]$. Then: (i) $\mathfrak{I} \vdash A \supset B$, (ii) $\mathfrak{I} \vdash B \supset \sim A$, (iii) $\mathfrak{I} \vdash \sim B \supset \sim A$, (iv) $\mathfrak{I} \vdash A \supset \sim B$. Now, by (ii), (iii) and A5, $\mathfrak{I} \vdash \sim A$, and so: (1) $\mathfrak{I} \vdash (A \supset B) \supset \sim A$. From (iv) and A6, $\mathfrak{I} \vdash A \supset \sim(A \supset B)$ and so: (2) $\mathfrak{I} \vdash \sim\sim A \supset \sim(A \supset B)$. (i) gives (3) $\mathfrak{I} \vdash \sim A \supset (A \supset B)$. Finally, by (i), A8, and A3: (4) $\mathfrak{I} \vdash \sim(A \supset B) \supset \sim\sim A$. (1)–(4) show, by definition, that $(A \supset B) \sim_{\mathfrak{I}} \sim A$, as desired.

Now suppose that one of (i)–(iv) is not true. We show that $A \supset B \sim_{\mathfrak{I}} B$ in this case. Since $B \supset (A \supset B)$ and $\sim(A \supset B) \supset \sim B$ are axioms, we must only show: (a) $\mathfrak{I} \vdash \sim B \supset \sim(A \supset B)$, (b) $\mathfrak{I} \vdash (A \supset B) \supset B$.

Subcase (i). $\mathfrak{I} \nvdash A \supset B$. Then $\mathfrak{I} \vdash \sim(A \supset B)$ and (a) is true. By A8, we also have $\mathfrak{I} \vdash (A \supset B) \supset A$ and so, by A2 and $(A \supset B) \supset (A \supset B)$, we get (b) as well.
Subcase (ii). $\mathfrak{I} \nvdash (B \supset \sim A)$. Then $\mathfrak{I} \vdash \sim(B \supset \sim A)$. By A7 and A4, $\mathfrak{I} \vdash A$. (a) then follows from A6 and (b) from $A \supset (A \supset B) \supset B$.
Subcase (iii). $\mathfrak{I} \nvdash \sim B \supset \sim A$. Similar to case (ii).
Subcase (iv). $\mathfrak{I} \nvdash A \supset \sim B$. Then $\mathfrak{I} \vdash \sim(A \supset \sim B)$ and by A7, A4 $\mathfrak{I} \vdash B$, and (b) follows. Also, by A8 $\mathfrak{I} \vdash (A \supset \sim B) \supset A$ and so $\mathfrak{I} \vdash \sim B \supset A$. Using A6, (a) is true as well.

Using $[A] \supset [B] = [A \supset B]$, it is easy to prove, for any A, that $[A] \in \{[P_1], [\sim P_1] \ldots [P_n], [\sim P_n]\}$, and that $v_0(A) = [A]$, where v_0 is the canonical valuation (defined by $v_0(P) = [P]$ for P atomic). As a consequence, S is indeed finite.

We finally show that $[A]$ is designated in S (i.e., $\sim[A] \leq [A]$) iff $\mathfrak{I} \vdash A$. Since every substitution instance of L-theorems is provable in $\mathfrak{I}$, this suffices by now to prove that S is an L-matrix. Since $v_0(\phi) = [\phi]$ and $\mathfrak{I} \nvdash \phi$, that ϕ is not valid in S follows as well.

Suppose then that $\mathfrak{I} \vdash A$. By A1 then $\mathfrak{I} \vdash \sim A \supset A$, $\mathfrak{I} \vdash \sim A \supset \sim\sim A$, so $\sim[A] = [\sim A] \leq [A]$. Conversely, if $\sim[A] \leq [A]$, then $\mathfrak{I} \vdash \sim A \supset A$. Finally, by A5 and $A \supset A$, $\mathfrak{I} \vdash A$.

2.4 Theorem *Any proper extension of $RM_{\supset}$ has a finite characteristic matrix which belongs to the sequence: S_1, $S_1(0)$, S_2, $S_2(0)$, S_3, $S_3(0)$.... Moreover, the logics corresponding to this sequence are all distinct and form a decreasing sequence.*

Proof: Using 2.3, the proof proceeds exactly like that of the analogous theorem for *RM*. The only difficulty is to show that the logics corresponding to the various S_i, $S_i(0)$ are all distinct. Dunn's proof for the *RM* case uses the "Dugundgi sentences" which are disjunctions of sentences of the form $p \Leftrightarrow q$ (p, q atomic). Now $p \Leftrightarrow q$ is equivalent in *RM* to $(p \rightarrow q) \circ (q \rightarrow p)$, i.e., to $[(p \rightarrow q) \rightarrow \sim(q \rightarrow p)]$, and so can be expressed, by 1.7(a), in the language of $RM_{\supset}$. However, $\vee$ is not available in this language, so we cannot directly use Dugundgi sentences. Nevertheless, we can replace any schema B of the form $A_1 \vee A_2 \vee \ldots \vee A_n$ by the following schema B^*, in which q can be any atomic variable not occurring in B:

$$B^* = (A_1 \supset q) \supset ((A_2 \supset q) \supset \ldots \supset ((A_n \supset q) \supset q) \ldots).$$

We show that B is valid in a Sugihara matrix S iff B^* is. Since $\vdash_{\overline{RM}} B \supset B^*$, one direction is trivial. For the other direction, suppose B is not valid in S. Then there is a valuation v in S which simultaneously falsifies $A_1, A_2, \ldots, A_n$. We may assume that v is not defined for q and extend its definition by letting $v(q) = max(v(A_1), v(A_2), \ldots, v(A_n))$. Then we get $v(B^*) = v(q)$, which is not designated. Hence B^* is not valid in S too.

Using the above observation, it is clear how to transfer any Dugundgi sentence B to a sentence in $RM_{\supset}$ language which has the same relevant properties (to the proof of the theorem) that B has.

As is clear from 1.7, $RM_{\tilde{\supset}}$ is stronger in its expressive power than $RM_{\tilde{\rightarrow}}$ (Sobociński's three-valued logic), but weaker than the full system RM.[8] To get a system equivalent to RM we must add to $RM_{\tilde{\supset}}$ language either $\vee$ or $\wedge$ with appropriate axioms. We choose to add $\vee$.

2.5 Definition The system $RM^{\supset}$: This is $RM_{\tilde{\supset}}$ augumented by the following:

A9 $A \supset (A \vee B)$
A10 $B \supset (A \vee B)$
A11 $(A \supset C) \supset (B \supset C) \supset ((A \vee B) \supset C)$
A12 $\sim(A \vee B) \supset \sim A$
A13 $\sim(A \vee B) \supset \sim B$
A14 $\sim A \supset \sim B \supset \sim(A \vee B)$.

2.6 Theorem *$RM^{\supset}$ is equivalent to RM and appropriate versions of* 2.2–2.4 (*in the language of* $\sim$, $\supset$, $\vee$) *hold for it. Moreover, all extensions of RM result by adding* A9–A14 *to $RM_{\tilde{\supset}}$'s extensions.*

Proof: Like in 2.2–2.4, we only note that according to the definition of $\leq$ (in the proof of 2.3), A9–A14 is just what is needed for proving the identity $[A \vee B] = max\ ([A], [B])$.

2.7 Remark A9–A14 are the most obvious introduction and elimination laws concerning $A \vee B$ and $\sim(A \vee B)$. An analogous set of axioms can characterize $\wedge$ independently. ($\wedge$ is definable in $RM^{\supset}$ using De Morgan laws.) We could, of course, take both $\vee$ and $\wedge$ as primitive and as axioms—the usual positive axioms concerning them and all forms of De Morgan laws.

2.8 Corollary *If we add $\sim A \supset (A \supset B)$ to either RM or $RM_{\tilde{\supset}}$, we get classical logic (in the corresponding languages).*

Among RM axioms, there is only one that may seem unnatural: A8. If we strengthen it in order to make it an analogue of A7, we get a very interesting system:

2.9 Definition The system $RM_3^{\supset}$ ($RM_{\tilde{\supset}3}$) is the system resulting from the replacement of A8 in $RM^{\supset}$ ($RM_{\tilde{\supset}}$) by

A8′: $\sim(A \supset B). \supset A$.

The following Corollaries of 2.4–2.6 are what makes $RM_3^{\supset}$ interesting:

2.10 Theorem
(a) *$RM_3^{\supset}$ ($RM_{\tilde{\supset}3}$) axiomatizes the Sugihara matrix $S_1(0)$ and is therefore equivalent to RM_3* (see [1], 29.4).
(b) *$\sim A \supset (A \supset B)$ is not provable in $RM_3^{\supset}$ and $RM_3^{\supset}$ is a maximal logic having this property. In fact, classical PC is its only extension. The same holds for the $\{\sim\supset\}$ fragment.*
(c) *The positive fragment of $RM_3^{\supset}$ (in the $\{\supset, \vee, \wedge\}$ language) is identical to that of classical PC, and it is the only extension of $RM^{\supset}$ (besides PC itself) having this property.*
(d) *$RM_3^{\supset}$ and $RM_{\tilde{\supset}3}$ can be also axiomatized by adding the Peirce's law to $RM^{\supset}$ and $RM_{\tilde{\supset}}$, respectively.*

$RM_3^{\supset}$ is, by 3.3, a three-valued logic. For the reader's convenience, we display here the corresponding matrices with $\{T, F, I\}$:

	$\sim$	$\supset$	T	I	F	$\vee$	T	I	F	$\wedge$	T	I	F
$*T$	F		T	I	F		T	T	T		T	I	F
$*I$	I		T	I	F		T	I	I		I	I	F
F	T		T	T	T		T	I	F		F	F	F

NOTES

1. See [3] for the meaning of this.
2. Which is, by a theorem of Meyer, characteristic for *RM*. See [1], 29.3, and preliminaries.
3. Developed in [9] and proved by Parks to be identical with the $\{\sim\to\}$ fragment of *RM*. (See [1], pp. 148–149.)
4. We omit, henceforth, subscripts under $\vdash$ whenever no danger of confusion arises.
5. $\mathfrak{T}$ is prime if $\mathfrak{T} \vdash A \vee B \Rightarrow \mathfrak{T} \vdash A$ or $\mathfrak{T} \vdash B$.
6. This is an essentially known result; see Corollary 1 on p. 52 of [8].
7. I want to thank Professor D. Gabbay for first suggesting to me this connection to *LC*.
8. It is surprising therefore that although the language of $RM_{\underset{\supset}{\sim}}$ is weaker than that of *RM*, it has all of *RM* important properties, as 2.3–2.4 show. By this it differs in an essential way from $RM_{\underset{\to}{\sim}}$, which cannot distinguish between the various finite Sugihara matrices and between them and the infinite one.

REFERENCES

[1] Anderson, A. R. and N. D. Belnap, *Entailment*. Vol. 1, Princeton University Press, Princeton, New Jersey, 1975.

[2] Anderson, A. R. and N. D. Belnap, "Enthymemes," *The Journal of Philosophy*, vol. 58 (1961), pp. 713–723.

[3] da-Costa, N. C. A., "Theory of inconsistent formal systems," *Notre Dame Journal of Formal Logic*, vol. 15 (1974), pp. 497–510.

[4] Dummett, M., "A propositional calculus with denumerable matrix," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 96–107.

[5] Dunn, J. M., "Algebraic completeness results for R-mingle and its extensions," *The Journal of Symbolic Logic*, vol. 35 (1970), pp. 1–13.

[6] Dunn, J. M. and R. K. Meyer, "Algebraic completeness results for Dummett's LC and its extensions," *Zeitschrift für mathematische Logic und Grundlagen der Mathematics*, vol. 17 (1971), pp. 225–230.

[7] Meyer, R. K., "Intuitionism, entailment, negation," in *Truth, Syntax and Modality*, ed. H. Leblanc, North-Holland, Amsterdam, 1973, pp. 168–198.

[8] Meyer, R. K., "Relevantly interpolating in RM," ANU Logic Group Research Paper No. 9, 1980.

[9] Sobociński, B., "Axiomatization of a partial system of three-valued calculus of propositions," *The Journal of Computing Systems*, vol. 1 (1952), pp. 23–55.

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