

## NP-Completeness of a Combinator Optimization Problem

M. S. JOY and V. J. RAYWARD-SMITH

**Abstract** We consider a deterministic rewrite system for combinatory logic over combinators  $S, K, I, B, C, S', B',$  and  $C'$ . Terms will be represented by graphs so that reduction of a duplicator will cause the duplicated expression to be “shared” rather than copied. To each normalizing term we assign a weighting which is the number of reduction steps necessary to reduce the expression to normal form. A lambda-expression may be represented by several distinct expressions in combinatory logic, and two combinatory logic expressions are considered equivalent if they represent the same lambda-expression (up to  $\beta$ - $\eta$ -equivalence). The problem of minimizing the number of reduction steps over equivalent combinator expressions (i.e., the problem of finding the “fastest running” combinator representation for a specific lambda-expression) is proved to be NP-complete by reduction from the “Hitting Set” problem.

**1 Introduction** The uses of the lambda-calculus (see Barendregt [1]) and combinatory logic (see Curry et al. [4],[5]) as notations for defining functions are well known. As branches of mathematical logic they have been explored in great depth. In recent years, however, both disciplines have been used in computer science as models for the evaluation of functional programs. The lambda-calculus has served as a starting point for, for instance, SECD machines (cf. Glaser et al. [7]) and combinatory logic for graph reduction machines (cf. Turner [14],[16]).

There is a “natural” correspondence between a lambda-expression and the function it represents, but to evaluate a function in such a form leads to complications. This is due to the use in the lambda-calculus of variable names, which results in environments needing to be stored when recursively defined functions are called in order to avoid clashes of local variable names. In combinatory logic no such variables are used, so the evaluation of a function is simplified. However such a combinator expression will probably not be easy to read. It is common practice to consider a function as being initially a lambda-expression and then to apply an algorithm to the

*Received December 12, 1988; revised August 3, 1990*

lambda-expression to eliminate all the variables and introduce combinators. We assume the reader is familiar with the fundamentals of the lambda-calculus and combinatory logic. A good introduction can be found in Hindley and Seldin [8]. Having created such a combinator expression, it can be considered in a natural way as being a graph, and to evaluate the function it represents we can apply rewrite rules to the graph until the graph becomes the required form representing “the answer.”

We shall consider the set  $\{S, K, I, B, C, S', B', C'\}$  of combinators, partly because it is a set in common use, partly since it has known abstraction algorithms associated with it. The results we prove will be applicable to many sets of combinators, but the details of the proof are valid only for this set.

A combinatory logic will often be augmented by extra primitives, such as integers, in order to improve its efficiency as a computer code. In order to simplify our analysis we shall assume that *no* such extra primitives are used. If we assume a small finite set of combinators in our combinatory logic, we can think of each as corresponding to a single “machine instruction” and can thus form a measure of time for the function to evaluate as being the “number of instructions (reduction steps) executed.” This metric is naïve, but it will be sufficient for our purposes.

For simplicity in describing the result here, we shall assume that our combinatory logic is augmented by a (countable) set of variables. Variables and combinators will be considered as “atomic” expressions.

Suppose we have a function  $f$  written as a combinator expression. We consider the size  $|f|$  of the combinator expression to be the number of occurrences of atoms (combinators or variables) in it. Suppose  $f$  evaluates, using “normal order” reduction, to “the answer” (that is, an expression in normal form) in  $r$  reduction steps (assuming, of course, that  $f$  is a function which evaluates in finite time). Then the problem of minimizing  $r$  over equivalent combinatory logic expressions of size  $|f|$  is NP-complete. We prove this by reduction from the “Hitting Set” problem.

Investigation into this result was motivated by recent techniques for the implementation of functional programming languages involving the use of combinatory logic not just as a semantic domain, but with combinators implemented as primitive machine instructions (cf. Clarke et al. [3] and Stoye [13]). Given a translation of a functional program to such combinator code, it is often desirable to optimize the code, and our result establishes an upper bound to the possibilities for such code improvement techniques.

This result was proved first in Joy [9] and was published (without proof) in Joy et al. [10].

**2 The optimization problem** The main result of this paper is that the following Optimization Problem (“OP”) is NP-complete.

### **Optimization Problem (OP)**

*Instance:* A combinator expression  $E$  whose only atomic subexpressions are variables  $x_1, \dots, x_m$ , and an integer  $k$ .

*Question:* Does there exist an expression  $E'$ , without variables, such that the expression  $(E'x_1 \dots x_m)$  reduces, using a normal order reduction strategy, in  $k$  (or less) reduction steps to  $E$ ?

$E'$  is a combinator expression equivalent to the lambda-expression  $(\lambda x_1. \dots \lambda x_m. E)$ .

We will establish  $OP \in NP$ , and then the NP-completeness of OP will be proved by exhibiting a polynomial transformation to OP from a known NP-complete problem. We choose to use the following problem proved to be NP-complete in Garey and Johnson [6]

### Hitting Set (HS)

*Instance:* Collection  $C$  of distinct subsets of a finite set  $S$  such that  $c_i \in C$  satisfies  $|c_i| = 2$  and  $S = \cup C$ , a positive integer  $k \leq |S|$ .

*Question:* Does there exist a subset  $S'$  of  $S$  such that

1.  $|S'| \leq k$ , and
2. for each  $c_i \in C$ ,  $c_i \cap S' \neq \emptyset$ ?

Before we can detail the transformation  $HS \propto OP$ , we need to establish our notation and prove some intermediate results. We do this in Sections 3 and 4. In Section 5 we return to the transformation and give the necessary detail.

## 3 Notations and assumptions

**3.1 Combinator expressions** A combinator expression is

1. a variable  $v$ , or
2. a combinator (an element of  $\{S, K, I, B, C, S', B', C'\}$ ), or
3. an application  $(L M)$  where  $L$  and  $M$  are combinator expressions.

By default, parentheses may be omitted for clarity on the assumption of left-associativity, for example

$$S w (I y) z$$

is equivalent to

$$(((S w) (I y)) z).$$

We adopt the convention that lower-case letters (with or without subscripts) denote variables unless otherwise stated. We introduce *no* extra atoms, such as numbers. The above definition of a combinatory logic is still sufficiently rich to be equivalent to a Turing Machine, that is, for any partial recursive function there exists an expression in the combinatory logic which can be used to compute that function. In order to simplify our calculation later on, we do not formally define the lambda-calculus. Instead we include variables in our definition of combinatory logic. Let  $CL$  denote the set of all such combinator expressions.

The *size* of a combinator expression is given by:

$$|E| = 1, \text{ if } E \text{ is an atom, else } |(F G)| = |F| + |G|.$$

For instance,  $|S w x(I (I y))| = 6$ .

Our plan of attack is to restrict our attention to a subset of lambda-expressions which we know will reduce to normal form in a finite time after they have been given the correct number of arguments. These are “proper combinators” of the shape  $\lambda v_1. \dots \lambda v_m. E$ , where  $E$  contains no lambdas and, as atomic subexpressions, only

elements of  $\{v_1, \dots, v_m\}$ . Thus they can be thought of as simple functions which rearrange, possibly with duplications, their arguments. If  $v_1 \dots v_m$  are provided as arguments such an function with  $m$  lambdas *will* reduce to normal form (viz.  $E$ ).

The conversion of such an expression with  $m$  lambdas to a combinator expression containing no lambdas and no variables is equivalent to a map *abstract* from  $CL$  to  $CL$ , such that, for each  $E$  in  $CL$ ,

1. *abstract*( $E$ ) contains no variables, and
2. (*abstract*( $E$ )  $v_1 \dots v_m$ ) reduces to  $E$ .

We use the symbol “ $\equiv$ ” to mean “lexically equal to,” and the symbol “ $=$ ” (as a relation between combinator expressions) to mean “are equivalent,” that is, represent the same lambda-expression. Thus, if  $E$  and  $F$  are combinator expressions such that ( $E v_1 \dots v_m$ ) reduces to an expression  $G$  containing only variables as atomic subexpressions, and ( $F v_1 \dots v_m$ ) reduces to  $G$  also, then  $E = F$ .

We use the symbol “ $>$ ” to denote “reduces to,” and “ $>_X$ ” to mean “reduces in one  $X$ -reduction step to,” where  $X$  is a combinator. The combinators used, originally introduced by Turner [15], have definitions as follows ( $a, b, c$ , etc., are used here as meta-variables):

$$\begin{array}{ll}
 S a b c & >_S a c (b c) \\
 K a b & >_K a \\
 I a & >_I a \\
 B a b c & >_B a (b c) \\
 C a b c & >_C a c b \\
 S' a b c d & >_{S'} a (b d) (c d) \\
 B' a b c d & >_{B'} a b (c d) \\
 C' a b c d & >_{C'} a (b d) c
 \end{array}$$

The graph rewrite rules are given in diagrammatic form in Figure 1 below; all lines are directed downwards (the arrows are omitted for clarity). In each rule except those for  $I$  and for  $K$  the root node of the redex is *overwritten*. For the  $I$  and  $K$  rules the pointer to the redex is redirected. An  $I$  or  $K$  reduction where the redex is the root of the whole graph is handled as a special case.

We assume that reduction is normal order, that is, “leftmost-outermost.” This strategy minimizes the number of reduction steps needed to reduce an expression to normal form (as redexes are reduced *only* if they are needed; cf. Klop [12]).

Initially, before any reductions are applied to an expression, that expression is stored either as a tree or as a graph in which the only nodes with in-degree greater than 1 are atoms. This corresponds with the notion of a program being read in from a source in a way which naturally implies a simple storage mechanism (knowledge about code-sharing is itself a difficult problem).

The phrase *code-sharing* will refer to nodes in a graph with in-degree greater than 1, and our result depends on the code-sharing yielded by the  $S$  and  $S'$  combinators (the *duplicators*). Thus

$$S (a b) (c (d e)) (x y z) > (a b (x y z) (c (d e) (x y z)))$$

will cause the subgraph which  $(x y z)$  represents to be shared after the reduction step rather than copied.

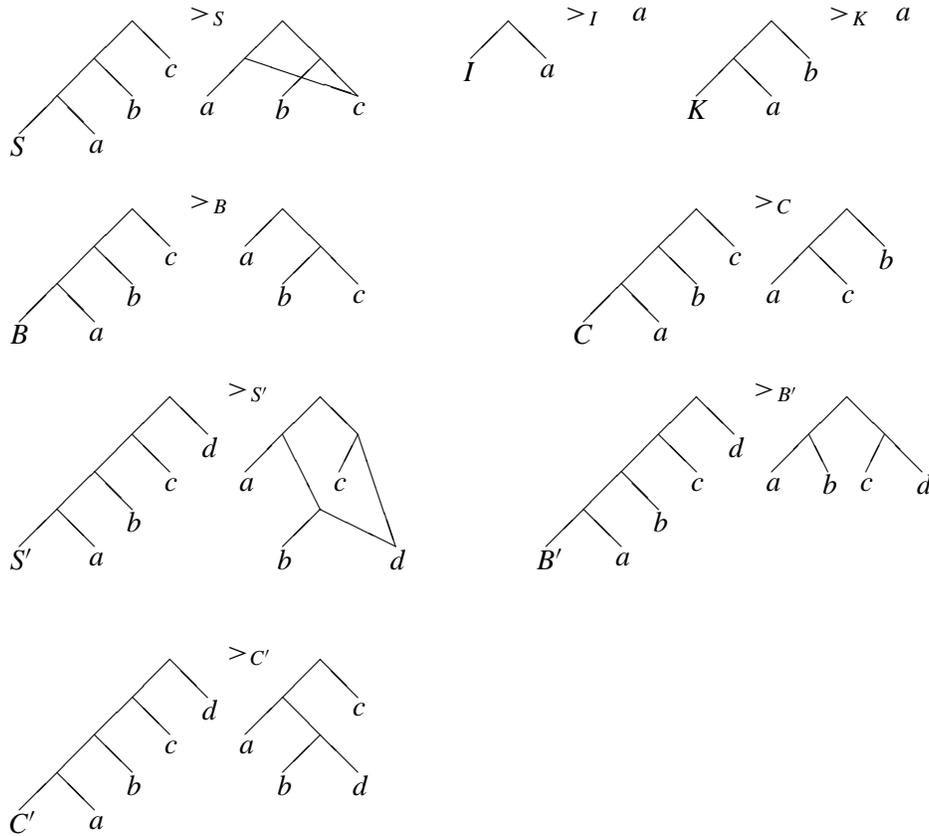


Figure 1: The Graph Rewrite Rules

**3.2 An almost optimal abstraction algorithm** We describe an abstraction algorithm, originally due to Turner [14] (although we phrase it somewhat differently) which produces code that in many cases is optimal. We shall prove the optimality of the algorithm for some of our expressions.

The algorithm takes the form of a map  $abs$  from  $\{\text{variables of } CL\} \times CL \rightarrow CL$ . For notational convenience we write  $abs_x(E)$  in preference to  $abs(\langle x, E \rangle)$ , and  $abs_{x,y}(E)$  as shorthand for  $abs_x(abs_y(E))$ .  $E$  and  $F$  are here arbitrary combinator expressions, and  $k$  is an arbitrary combinator expression which contains no variables. The first possible of the following rules should be applied.

$$\begin{aligned}
 abs_x(x) &\equiv I, \\
 abs_x(E x) &\equiv E, \text{ if } x \text{ does not occur in } E, \\
 abs_x(E) &\equiv K E, \text{ if } x \text{ does not occur in } E, \\
 abs_x(k x F) &\equiv (S k abs_x(F)), \text{ if } x \text{ occurs in } F, \\
 abs_x(k x F) &\equiv (C k F), \text{ if } x \text{ does not occur in } F, \\
 abs_x(k E F) &\equiv (S' k abs_x(E) abs_x(F)), \text{ if } x \text{ occurs in both } E \text{ and } F, \\
 abs_x(k E F) &\equiv (C' k abs_x(E) F), \text{ if } x \text{ occurs in } E \text{ but not in } F, \\
 abs_x(k E F) &\equiv (B' k E abs_x(F)), \text{ if } x \text{ occurs in } F \text{ but not in } E, \\
 abs_x(E F) &\equiv (S abs_x(E) abs_x(F)), \text{ if } x \text{ occurs in both } E \text{ and } F,
 \end{aligned}$$

$abs_x(E F) \equiv (C abs_x(E) F)$ , if  $x$  occurs in  $E$  but not in  $F$ ,  
 $abs_x(E F) \equiv (B E abs_x(F))$ , if  $x$  occurs in  $F$  but not in  $E$ .

To illustrate this algorithm, consider  $abs_{x,y}(y x x)$ . The successive stages are as follows:

$$\begin{aligned}
& abs_{x,y}(y x x) \\
&= abs_x(abs_y(y x x)) \\
&= abs_x(C (abs_y(y x)) x) \\
&= abs_x(C (C (abs_y(y)) x) x) \\
&= abs_x(C (C I x) x) \\
&= S' C (abs_x(C I x)) (abs_x(x)) \\
&= S' C (C I) I.
\end{aligned}$$

**4 Intermediate definitions and results** The construction of the transformation  $HS \propto OP$  relies on the use of combinator expressions of the form  $W_{x,y}^n$ , which we now define. The functions  $\psi$  and  $\bar{V}$  will also be used later on.

Let  $x, y$  and  $v$  be variables,  $n$  a positive integer, and  $f$  and  $g$  combinator expressions, then we define

$$\begin{aligned}
\psi_{0,f,g} &\equiv g, \text{ and} \\
\psi_{r,f,g} &\equiv (f \psi_{r-1,f,g}) \text{ if } r > 0.
\end{aligned}$$

Thus  $\psi_{r,f,g} \equiv f^r g$ , and  $f^r \equiv \psi_{r-1,f,f}$ .

Let  $\bar{n} = 16n$ , then we define, as illustrated in Figure 2 below,

$$\begin{aligned}
V_{x,y}^n &\equiv \bar{V}_{n,1,v,x,y}, \text{ where} \\
\bar{V}_{n,m,v,x,y} &\equiv \bar{V}_{n,m+1,v,x,y}(v^{m\bar{n}} x)(v^{m\bar{n}} y), \text{ if } n > m \geq 1, \text{ otherwise} \\
\bar{V}_{n,n,v,x,y} &\equiv (v^{n\bar{n}} x)(v^{n\bar{n}} y).
\end{aligned}$$

This is illustrated in Figure 2. Finally we define

$$W_{x,y}^n \equiv (V_{x,y}^n V_{y,x}^n).$$

We note that  $|V_{y,x}^n| = n\bar{n}(n+1) + 2n$ , which is polynomial in  $n$ .

The *left-depth* of a combinator expression is given by

$$\begin{aligned}
\text{left-depth}(E, E) &= 0; \\
\text{left-depth}(E, (F G)) &= 0 \text{ if } E \text{ does not occur in } F, \\
&\quad \text{otherwise } 1 + \text{left-depth}(E, F).
\end{aligned}$$

For example,  $\text{left-depth}(x, (a b (c x d) e f)) = 2$ . We use the phrase “the left-depth of  $E$  in  $F$ ” as shorthand for  $\text{left-depth}(E, F)$ . *Right-depth* is defined similarly, with  $(G F)$  replacing  $(F G)$  in the second clause.

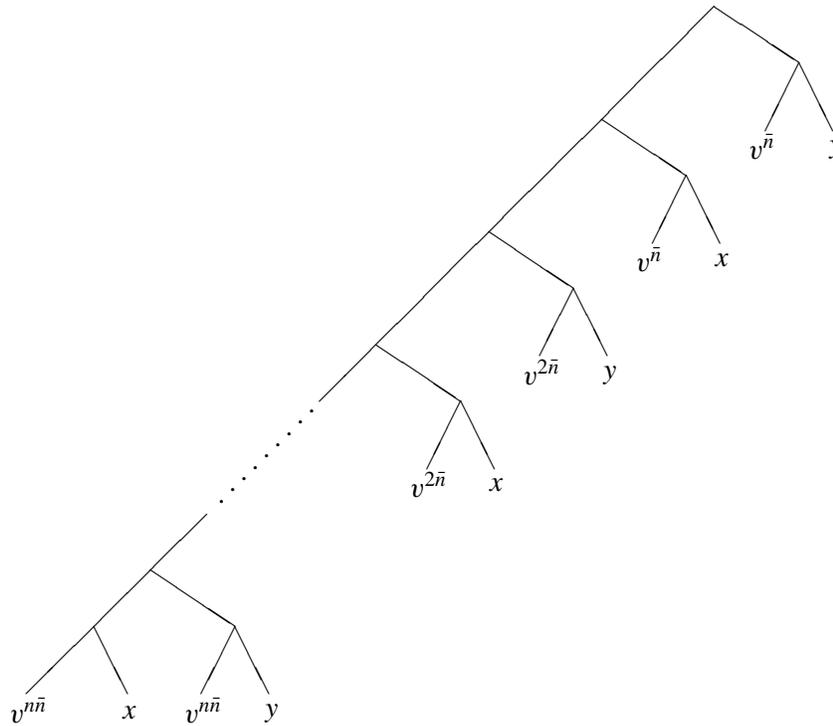


Figure 2:  $V_{x,y}^n$

The *depth* of a combinator expression is given by

$$\begin{aligned} \text{depth}(E, E) &= 0; \\ \text{depth}(E, (F G)) &= 0 \text{ if } E \text{ does not occur in } (F G), \\ &\text{else } 1 + \max(\text{depth}(E, F), \text{depth}(E, G)). \end{aligned}$$

For example,  $\text{depth}(x, (a b (c x d) e f)) = 5$ .

The *spine* of an expression  $E$  is the set of subexpressions of  $E$  whose right-depth in  $E$  is 0. For example,

$$\text{spine}(a b(c x d) e f) = \{(a b (c x d) e f), (a b (c x d) e), (a b (c x d)), (a b), (a)\}.$$

The notation  $[E/F]G$  is used to mean “the combinator expression produced when all occurrences of the expression  $F$  in  $G$  are replaced by the expression  $E$ .” Let  $F$  be a combinator expression in normal form containing  $x_1, \dots, x_m$  as its only atomic subexpressions. Then  $\text{opt}_{x_1, \dots, x_m}(F)$  will be any combinator expression, not containing any element of  $\{x_1, \dots, x_m\}$  such that  $(\text{opt}_{x_1, \dots, x_m}(F) x_1 \dots x_m)$  reduces to  $F$  in the minimum number of reduction steps using normal order reduction, denoted by  $\text{red}_{x_1, \dots, x_m}(F)$ .

We also need to introduce  $Z_1$  and  $Z_2$ :

$$Z_1 = [(C' S (S' C (K (KI))v^{n\bar{n}})v^{n\bar{n}})/(C' Bv^{n\bar{n}}v^{n\bar{n}})]\text{abs}_{x,y}(V_{x,y}^n),$$

$$Z_2 = [(S' C(C' S(K(KI))v^{n\bar{n}})v^{n\bar{n}})/(B' C v^{n\bar{n}}v^{n\bar{n}})]\text{abs}_{x,y}(V_{y,x}^n).$$

We begin by giving some basic results on  $V_{x,y}^n$ ,  $W_{x,y}^n$ ,  $Z_1$  and  $Z_2$ .

**Lemma 4.1**  $(abs_{x,y}(V_{x,y}^n) x y)$  reduces to  $V_{x,y}^n$  in  $4n - 2$  reduction steps;  
 $(abs_{x,y}(V_{y,x}^n) x y)$  reduces to  $V_{y,x}^n$  in  $4n - 2$  reduction steps;  
 $(Z_1 x y)$  reduces to  $V_{x,y}^n$  in  $4n + 3$  reduction steps;  
 $(Z_2 x y)$  reduces to  $V_{y,x}^n$  in  $4n + 3$  reduction steps;  
 $(Z_1 x)$  and  $(Z_2 y)$  each reduces to normal form in  $2n + 1$  reduction steps.

*Proof:* These results are all immediate from the definitions of  $V_{x,y}^n$ ,  $Z_1$ , and  $Z_2$ .

**Lemma 4.2**  $red_x(V_{x,y}^n) \geq 2n - 1$ ,  $red_x(V_{y,x}^n) \geq 2n - 1$ ;  
 $red_{x,y}(V_{x,y}^n) \geq 4n - 2$ ,  $red_{x,y}(V_{y,x}^n) \geq 4n - 2$ .

*Proof:* The left-depths of  $x$  and  $y$  in  $V_{x,y}^n$  are  $2n - 1$  and  $2n - 2$  respectively, hence we get the first two inequalities, as a combinator of  $CL$  can increase the left- (or right-) depth of one of its arguments by at most 1. Let

$$X_1 \equiv [w_1/v^{\bar{n}}]([w_2/v^{2\bar{n}}]([w_3/v^{3\bar{n}}] \dots ([w_n/v^{n\bar{n}}]V_{x,y}^n) \dots)),$$

and

$$X_2 \equiv [w_1/v^{\bar{n}}]([w_2/v^{2\bar{n}}]([w_3/v^{3\bar{n}}] \dots ([w_n/v^{n\bar{n}}]V_{y,x}^n) \dots)),$$

where the  $w_i$  are distinct new variables. So  $X_1 \equiv ((w_n x)(w_n y) \dots ((w_1 x)(w_1 y)))$ , and  $v$  occurs in neither  $X_1$  nor  $X_2$ .

We note that  $red_{x,y}(X_1) = red_{x,y}(V_{x,y}^n)$ , since the right-depth of  $v^{i\bar{n}}$  in  $v^{(i+1)\bar{n}}$  is  $\bar{n}$ , and thus any attempt to utilize the fact that there exist common subexpressions of  $V_{x,y}^n$ , except the instances of  $v^{i\bar{n}}$  in  $(v^{i\bar{n}} x)$  and  $(v^{i\bar{n}} y)$  for each  $i$ , will necessitate at least  $(\bar{n} - 1)$  extra reduction steps, which is more than the number needed by  $abs_{x,y}(V_{x,y}^n)$ .  $X_2$  is treated similarly. To *create* each subexpression of the form  $(w_i x)$  or  $(w_i y)$ , an  $A$ -reduction,

$$A a_1 \dots a_r \dots a_t >_A b_1 \dots b_{r-1} \dots a_{r+1} \dots a_t \quad (r \leq t),$$

where  $a_r$  is either  $x$  or  $y$ , is needed. Each reduction step can increase the left-depth of either  $x$  or  $y$  (but not both) by at most 1. For, if it increased the left-depth of both by one, at least one more reduction step would be needed to “separate” them in order for them to be passed singly as arguments to the  $A$  combinators. We thus get  $red_{x,y}(X_1) \geq 4n - 2$ , and  $red_{x,y}(X_2) \geq 4n - 2$ . The results for  $V_{x,y}^n$  then follow.

**Lemma 4.3**  $opt_{x,y}(V_{x,y}^n) \equiv abs_{x,y}(V_{x,y}^n)$ ,  $opt_{x,y}(V_{y,x}^n) \equiv abs_{x,y}(V_{y,x}^n)$ .

*Proof:* This follows from Lemmas 4.1 and 4.2.

**Lemma 4.4**  $opt_{x,y}(W_{x,y}^n) \equiv abs_{x,y}(W_{x,y}^n)$ ,  $opt_{x,y}(W_{y,x}^n) \equiv abs_{x,y}(W_{y,x}^n)$ .

*Proof:* Since no node in  $X_1$  nor  $X_2$ , as defined in Lemma 4.2, with right-depth 0 and left-depth less than  $2n + 1$  can be shared, each reduction step in  $opt_{x,y}(W_{x,y}^n)$  may affect only the spine of  $V_{x,y}^n$  or  $V_{y,x}^n$  (but not both). So each reduction step using  $opt_{x,y}(X_1 X_2)$  can be associated with either  $V_{x,y}^n$  or  $V_{y,x}^n$ . Thus,

$$red_{x,y}(W_{x,y}^n) \geq red_{x,y}(V_{x,y}^n) + red_{y,x}(V_{x,y}^n).$$

The result then follows from lemmas 4.2 and 4.3.

**Lemma 4.5**  $red_v(abs_{x,y}(V_{x,y}^n)) \leq n\bar{n} + 2\bar{n} + 8n - 3$ ,  
 $red_v(abs_{y,x}(V_{y,x}^n)) \leq n\bar{n} + 2\bar{n} + 8n - 3$ .

*Proof:* Let  $V_1 \equiv \psi_{n-1}(f, g)\bar{v} (v^{\bar{n}}) \equiv abs_{x,y}(V_{x,y}^n)$ , where

$$\begin{aligned} f \alpha \beta \gamma &> (C' S (S' C (\alpha \beta (\beta \gamma)) \gamma) \gamma), \\ g \beta \gamma &> (C' B \gamma \gamma), \text{ and} \\ \bar{v} &\equiv abs_h(\psi_{\bar{n}}(v, h)). \end{aligned}$$

Thus we have

$$\begin{aligned} f &= (C' (C' (S' (C' S))) (C' (C' (S' (S' C))) (C (S' B) I) I) I), \\ g &= (K (S (C' B) I)), \\ \bar{v} &\equiv \psi_{\bar{n}-1}((B v), v) = \psi_{\bar{n}-1}(((S B), I) v), \text{ and} \\ v^{\bar{n}} &= (\psi_{\bar{n}-1}((S I), I) v). \end{aligned}$$

Hence,

$$V_0 \equiv (S' (\psi_{n-1}(f, g)) \psi_{\bar{n}-1}((S B), I) (\psi_{\bar{n}-1}((S I), I) v))$$

reduces to normal form  $(abs_{x,y}(V_{x,y}^n))$ .

$$V_1 \equiv (S' F (\psi_{\bar{n}-1}((S B), I)) (\psi_{\bar{n}-1}((S I), I) v)),$$

where  $F$  is  $(\psi_{n-1}(f, g))$ , reduces to normal form  $(abs_{x,y}(V_{x,y}^n))$  in at most

1	because of initial $S'$
+ 9(n - 1)	because of $f$
+ 5	because of $g$
+ $\bar{n} + (\bar{n} - 1)(n - 1)$	because of $\psi_{\bar{n}-1}((S B), I)$ , since each $B$ is used for each occurrence of $f$
+ 2 $\bar{n} - 1$	because of $\psi_{\bar{n}-1}((S I), I)$

=  $n\bar{n} + 2\bar{n} + 8n - 3$  reduction steps.

The result for  $abs_{y,x}(V_{x,y}^n)$  is almost identical.

**Lemma 4.6**  $red_v(Z_1) \leq n\bar{n} + 2\bar{n} + 8n + 6$ ,  
 $red_v(Z_2) \leq n\bar{n} + 2\bar{n} + 8n + 6$ .

*Proof:* The proof is essentially the same as that for Lemma 4.5, except that

$$\begin{aligned} g \beta \gamma &> (K (K I)), \\ g &= (K (K (K (K I)))), \\ U_0 &\equiv (S' (\psi_n(f, g)) (\psi_{\bar{n}-1}((S B), I)) (\psi_{\bar{n}-1}((S I), I) v)) \end{aligned}$$

reduces to normal form  $Z_1$ ;

$$U_1 \equiv (S' F (\psi_{\bar{n}-1}((S B), I)) (\psi_{\bar{n}-1}((S I), I) v)),$$

where  $f$  is  $(\psi_n(f, g))$  reduced to normal form, reduces to normal form,  $Z_1$ , in at most  $n\bar{n} + 2\bar{n} + 8n + 6$  reduction steps. The result for  $Z_2$  is almost identical, with the  $(C' S)$  and  $(S' C)$  in  $f$  interchanged.

We now examine  $Z_1$  and  $Z_2$  more closely. First of all, by using  $Z_1$  instead of  $abs_{x,y}(V_{x,y}^n)$ , and  $Z_2$  instead of  $abs_{x,y}(V_{y,x}^n)$ , we have a structure which is more “symmetric.” The extra symmetry manifests itself in the following way.

$$\begin{aligned} abs_{x,y}(V_{x,y}^n) &= C' S (S' C (\dots (C' B v^{n\bar{n}} v^{n\bar{n}}) \dots) v^{\bar{n}}) v^{\bar{n}}, \\ Z_1 &= C' S (S' C (\dots (C' S (S' C (K (K I)) v^{n\bar{n}}) v^{n\bar{n}}) \dots) v^{\bar{n}}) v^{\bar{n}}, \end{aligned}$$

so the former contains an expression  $(C' B v^{n\bar{n}} v^{n\bar{n}})$ , corresponding to  $(B' C v^{n\bar{n}} v^{n\bar{n}})$  in  $abs_{x,y}(V_{y,x}^n)$ .

*Note carefully the ordering of the subscripts  $x$  and  $y$ .* Apart from the interchange of  $(C' B)$  and  $(B' C)$ ,  $abs_{x,y}(V_{x,y}^n)$  and  $abs_{x,y}(V_{y,x}^n)$  can be interconverted merely by swapping occurrences of  $(C' S)$  and  $(S' C)$ . It is not necessary also to swap occurrences of  $(C' B)$  and  $(B' C)$  in the  $Z_i$ .

Consider the proof of Lemma 4.5. Since code which reduces to  $Z_1$  can be created by swapping the occurrences of  $(S' C)$  and  $(C' S)$  in the definition of  $f$ , we may replace  $(C' S)$  and  $(S' C)$  in  $U_0$  by variables  $t_1$  and  $t_2$  respectively and abstract them out. Thus  $f$  would become

$$(C' (C' (S' t_1))(C' (C' (S' t_2)) (C (S' B) I) I) I).$$

After  $U_0$  had then been reduced to normal form we would have

$$U'_0 = t_1 (t_2 (t_1 (t_2 \dots (t_1 (t_2 (K (K I)) v^{n\bar{n}}) v^{n\bar{n}}) \dots) v^{2\bar{n}}) v^{2\bar{n}}) v^{\bar{n}}) v^{\bar{n}}.$$

Abstracting  $t_1$  and  $t_2$  from this expression yields  $8n$  new combinators, since

$$\begin{aligned} U'_0 &= (U'_1 t_1 t_2) \text{ and } U'_0 = (U'_2 t_2 t_1) \text{ where} \\ U'_1 &\equiv C (S C' (C' C (B' S I(\dots)) v^{\bar{n}})) v^{\bar{n}}, \\ U'_2 &\equiv C' C (B' S I (C (S C' (\dots)) v^{\bar{n}})) v^{\bar{n}}, \end{aligned}$$

and so an extra  $16n$  reduction steps, as each combinator must be used twice.

**Lemma 4.7**  $opt_{t_1,t_2}(U'_0) = U'_1$ ,  $opt_{t_2,t_1}(U'_0) = U'_2$ .

*Proof:* We examine the first case; the second is almost identical. As in Lemma 4.2, we are unable to utilize the code-sharing possibilities offered by the  $v^{i\bar{n}}$ , and the other internal nodes of  $U'_0$  cannot be shared. Due to the symmetry of  $U'_0$ , we are interested in code  $\bar{U}$  and  $\bar{U}'$  such that  $(\bar{U} t_1 t_2)$  reduces to  $(t_1 (t_2 (\bar{U}' t_1 t_2) v^{\bar{n}}) v^{\bar{n}})$  in the minimal number of reduction steps. Each combinator  $A$  occurring in  $\bar{U}$  must take as its last argument precisely one of  $t_1$  or  $t_2$ . It is then straightforward to enumerate the possible  $\bar{U}$ , and the result follows.

However, we cannot simply abstract the  $t_i$  from  $U'_0$ . We would, as in Lemma 4.6, need to consider  $red_v(U'_1)$  and  $red_v(U'_2)$ .

**Lemma 4.8**  $red_{w,v}(w Z_1 Z_2) \leq n\bar{n} + 2\bar{n} + 28n + 17$ .

*Proof:* Replace in  $U'_0$  above  $t_1$  by  $(B C (S C'))$ , and  $t_2$  by  $(B (C' C) (B' S I))$ , thus obtaining  $U''_0$ , where  $(U''_0 (C' S) (S' C)) = Z_1$  and  $(U''_0 (S' C) (C' S)) = Z_2$ . Thus we have introduced 10 combinators to create each  $Z_i$  (total of  $20n$  reduction steps).

We have also  $red_v(Z_1) = red_v(U'_0)$ , since the structures of  $Z_1$  and  $U'_0$  are essentially identical. So

$$(C' B (C (C S' (C (C I (C' S)) (S' C))) (C (C I (S' C)) (C' S))) opt_v(U'_0)) w v$$

reduces to  $(w Z_1 Z_2)$  in at most  $red_v(U'_0) + 11 + 20n$  reduction steps. Apply Lemma 4.6.

**Lemma 4.9**  $red_{w,v}(w abs_{x,y}(V_{x,y}^n) abs_{x,y}(V_{y,x}^n)) \geq red_{w,v}(w Z_1 Z_2)$ .

*Proof:* Clear by symmetry.

**Lemma 4.10**  $opt_v(v^{\bar{n}}) \equiv \psi_{\bar{n}-1}((S I), I)$ .

*Proof:* Clear by inspection.

**Lemma 4.11**  $red_v(Z_1) \geq n\bar{n} + 2\bar{n} + 8n - 10$ .

*Proof:* We count the minimum number of combinators needed in  $opt_v(Z_1)$ . We note first that it will be necessary to share certain sections of code. The occurrences of  $v^{\bar{n}}$  must be shared, and by Lemma 4.10  $red_v(v^{\bar{n}}) = 2\bar{n} - 1$ . Since the expressions  $v^{i\bar{n}}$  must be shared there will be a function  $h : v^{i\bar{n}} \rightarrow v^{(i+1)\bar{n}}$  which must be executed  $(n - 1)$  times. Each execution of  $h$  must require at least  $\bar{n} - 1$  reduction steps, as the depth of  $v^{i\bar{n}}$  in  $v^{(i+1)\bar{n}}$  is  $\bar{n}$ . Since the right-depth of  $\psi_{\bar{n}}(v, x)$  is  $\bar{n}$ , at least  $\bar{n} - 1$  reduction steps will be needed to create  $h$  initially. We are using the “simplest” method for obtaining each  $v^{i\bar{n}}$ . We thus need an expression  $\bar{Z}$  which will take as arguments  $h$  and  $v^{i\bar{n}}$ , returning an expression of the form

$$(C' S (S' C \bar{Z}' v^{i\bar{n}}) v^{i\bar{n}}),$$

where  $\bar{Z}'$  is  $\bar{Z}$  with arguments  $h$  and  $(hv^{i\bar{n}})$ . So

$$\bar{Z} = S' (C' (C' S)) (S' (C' (S' C)) (S' B \bar{Z}' I) I) I.$$

This code is optimal. We get  $9(n - 1)$  extra reduction steps from the  $\bar{Z}$ , and the result follows. Note the effects at the “top” and “bottom” of  $Z_1$  have been ignored and will introduce (a few) extra combinators.

**Lemma 4.12**  $red_{w,v}(w Z_1 Z_2) \geq n\bar{n} + 2\bar{n} + 25n - 11$ .

*Proof:* We note first of all that the only differences between  $Z_1$  and  $Z_2$  are the left-most  $(C' S)$  and  $(S' C)$  expression referred to at the start of the subsection. Thus the “obvious” way to achieve the expression  $opt_{w,v}(w Z_1 Z_2)$  is to use a strategy similar to that outlined in Lemma 4.8. Such a strategy involves replacing  $t_1$  and  $t_2$  in  $U'_0$  by expressions consisting only of combinators such that the resulting expression ( $U'_0$ , say) acts as if  $t_1$  and  $t_2$  had been abstracted out, yet is still of the same essential structure as  $U_0$ . Thus  $(U'_0 t_1 t_2)$  reduces to  $U'_0$ . If such a strategy is adopted, the replacements for  $t_1$  and  $t_2$  previously given are optimal. Unlike the previous lemma, it is not obvious that this reduction strategy is optimal. However, it is sufficiently close to optimal for our purposes.

There are two other possible reduction strategies. The first involves creating some (and by symmetry this implies *all*) of the  $v^{i\bar{n}}$  and passing them as arguments

to code representing  $Z_1$  and  $Z_2$ . This would require  $O(n^2)$  extra reduction steps, so such a strategy is unacceptable.

The second involves amending the definition of  $f$  so that the number of reduction steps needed to abstract the  $t_i$  is less. For instance,

$$f \alpha \beta \gamma > C (S' C (C' C (B' S I (\alpha \beta (\beta \gamma))) \gamma)) \gamma$$

would implement the optimal abstraction of  $t_1$  and  $t_2$  from  $U'_0$  given earlier. Now suppose that we had decided on another, more efficient, abstraction of  $t_1$  and  $t_2$  from  $U'_0$ . The corresponding  $f$  will be such that  $f \alpha \beta \gamma > F$ , where  $\alpha$ ,  $\beta$  and  $\gamma$  occur in  $F$ , but the depth of  $\alpha$  in  $F$  is increased by at least one, and thus abstracting  $\alpha$ ,  $\beta$  and  $\gamma$  from  $F$  will yield at least one extra combinator, hence a total of  $n - 1$  extra reduction steps. The optimal number of combinators introduced to abstract  $t_1$  and  $t_2$  from  $U'_0$  is  $8n$ , hence

$$\begin{aligned} \text{red}_{w,v}(w Z_1 Z_2) &\geq 2\text{red}_{t_1,t_2}(U'_0) + (n - 1) + \text{red}_v(Z_1) \\ &= 2(8n) + (n - 1) + (n\bar{n} + 2\bar{n} + 8n - 10) \end{aligned}$$

(by Lemma 4.11).

**Lemma 4.13** *Let  $s_{n,m} = \max_E |\text{abs}_{x_1, \dots, x_m}(E)|$  as  $E$  ranges over expressions in CL with  $|E| = n$ . Then  $s_{n,m} < 2mn$ .*

*Proof:* See Joy [9] or Kennaway [11].

**5 The transformation** Given an instance,  $I$ , of HS, we construct an instance  $f(I)$  of OP as follows.

We assume  $m = |S|$ ,  $r = |C|$  and  $c_i = \{c_{i,1}, c_{i,2}\}$ , then  $f(I)$  comprises a combinator expression  $E$ , containing variables  $v, d_1, \dots, d_m$  (all distinct), defined by

$$E \equiv (W_{c_{1,1}c_{1,2}}^n \cdots W_{c_{r,1}c_{r,2}}^n),$$

where  $n = 100r^3$ , and an integer  $k' = 30r(m + r) + 4n(r + k) + (n\bar{n} + 2\bar{n} + 28n)$ . Note that the  $f$  so constructed is injective and that the size of the instance of OP is polynomial in the size of the instance of HS. We see also that  $m \leq 2r$ . We shall assume that  $r$  is large, for instance  $r \geq 100$ . To compute the transformation, we need to show that  $I$  is a YES-instance of HS iff  $f(I)$  is a YES-instance of OP. Before doing this we motivate our definition and establish two further lemmas.

Let  $\Gamma$  be the set of all functions from  $\{1, \dots, r\} \rightarrow \{1, 2\}$ . Thus  $\Gamma$  represents the possibilities for ordering the elements of the  $c_i$  as the suffices of the  $W$ s. Fix some  $\varphi \in \Gamma$ , and let  $a_i = c_{i,3-\varphi(i)}$ . Let  $b_1, \dots, b_q$  be an enumeration of the  $a_i$ , so we have not presupposed an ordering on the  $a_i$ , and

$$\begin{aligned} X_i^1 &\equiv [(S (C (K I) (v^{n\bar{n}} c_{i,3-\varphi(i)})))/(B (v^{n\bar{n}} c_{i,3-\varphi(i)} v^{n\bar{n}}))] \text{abs}_{c_{i,\varphi(i)}}(V_{c_{i,1},c_{i,2}}^n), \\ X_i^2 &\equiv [(C (S (K I) v^{n\bar{n}}) (v^{n\bar{n}} c_{i,3-\varphi(i)})))/(C v^{n\bar{n}} (v^{n\bar{n}} c_{i,3-\varphi(i)}))] \text{abs}_{c_{i,\varphi(i)}}(V_{c_{i,2},c_{i,1}}^n), \end{aligned}$$

thus  $(Z_j c_{i,\varphi(i)})$  reduces to  $X_i^j$ .

Let  $Y_1, \dots, Y_{2p}$  be an enumeration of the  $X_i^1$  and  $X_i^2$ , where we note that, due to the symmetry of the  $X_i^j$  there must be an even number of  $Y_i$ . Let  $x_i^1, x_i^2$ , and  $y_j$  be variables which will correspond with  $X_i^1, X_i^2$ , and  $Y_j$  respectively.

$$\begin{aligned}\bar{E}_1 &\equiv \text{abs}_{y_1, \dots, y_{2p}, b_1, \dots, b_q}(\bar{E}_2), \\ \bar{E}_2 &\equiv ((x_1^1 a_1)(x_1^2 a_1) \dots (x_r^1 a_r)(x_r^2 a_r)), \\ \bar{E}_3 &\equiv \bar{E}_1 Y_1 \dots Y_{2p} b_1 \dots b_q.\end{aligned}$$

Thus the choice of  $\varphi(i)$  corresponds with code-sharing variables  $v$  and  $c_{i, \varphi(i)}$  in  $W_{x,y}^n$ , and  $p$  will correspond to  $k$  in HS.

$\bar{E}_3$  reduces to  $E$  in  $e_1 + e_x$  reduction steps, where, by Lemma 4.13,

$$e_1 < 2(2p + q)(4r) \leq 24r^2$$

(since  $q \leq r$  and  $p \leq r$ ).  $e_1$  is the number of combinators introduced by  $\text{abs}$  in  $\bar{E}_1$ , and, by Lemma 4.1,  $e_x = 2r(2n + 2)$  is the number of reduction steps for the  $X_i^1$  and  $X_i^2$ .

So we now have a situation where we have taken  $E$  and abstracted two variables (one of them being  $v$ ) from each  $W_{x,y}^n$  in  $E$  (we remember that there are three variables occurring in  $W_{x,y}^n$ ). By introducing the  $b_i$  we have ensured that no predefined ordering has been specified for the abstraction of the variables in  $E$  different to  $v$ . This has led to code-sharing; thus if, for instance,  $W_{x,y}^n$  and  $W_{z,x}^n$  are in  $E$ , then we *may* have chosen to code-share the occurrences of  $(v^{i\bar{n}} x)$  in  $W_{x,y}^n$  and  $W_{z,x}^n$ .

Now,  $Y_i = (Z^i y_i)$ , where  $Z^i \in \{Z_1, Z_2\}$ ,  $y_i$  occurs in  $Y_i$ , and  $y_i \neq v$  ( $1 \leq i \leq 2p$ ).

Let

$$\bar{E}_4 = (\text{abs}_{z_1, z_2, d_1, \dots, d_m}(\bar{E}_1(z^1 y_1) \dots (z^{2p} y_{2p}) b_1 \dots b_q)),$$

where the  $z^i$  are variables corresponding to the  $Z^i$ , and  $z_1, z_2$  is the enumeration of the  $z^i$  corresponding to  $Z_1, Z_2$ , and we note that each of  $Z_1$  and  $Z_2$  contains precisely *one* variable  $v$ .  $(\bar{E}_4 Z_1 Z_2 d_1 \dots d_m)$  reduces to  $\bar{E}_3$  in  $e_4 + e_y$  reduction steps, where  $e_4 < 2(m + 2)(4p + q + 2) < 24mr$ , by Lemma 4.13.  $e_4$  is the number of combinators introduced by  $\text{abs}$  in  $\bar{E}_4$ ,  $e_y = 2p(2n + 1)$  is the number of reduction steps for the  $Z_1$  and  $Z_2$ . Now, let

$$\begin{aligned}\bar{Z} &= \text{opt}_{w,v}(w Z_1 Z_2) \text{ and} \\ e_z &= \text{red}_{w,v}(w Z_1 Z_2).\end{aligned}$$

We have, by Lemmas 4.8 and 4.12,  $n\bar{n} + 2\bar{n} + 25n - 11 \leq e_z < n\bar{n} + 2\bar{n} + 28n + 17$ , and  $(\bar{Z} \bar{E}_4 v d_1 \dots d_m)$  reduces to  $\bar{E}_3$  in  $e_4 + e_y + e_z$  reduction steps.

We note that, by using  $Z_1$  instead of  $\text{abs}_{x,y}(V_{x,y}^n)$  and  $Z_2$  instead of  $\text{abs}_{x,y}(V_{y,x}^n)$  we have introduced *at most*  $12r$  extra reduction steps from using the optimal code for each individual  $V_{x,y}^n$ , and have got improved code for the abstraction of  $v$  from  $V_{x,y}^n$ , by Lemma 4.9.

**Lemma 5.1** *There exists an expression  $\bar{E}_5$  containing no variables such that (recalling that  $v, d_1, \dots, d_m$  is our enumeration of the variables occurring in  $E$ )  $\bar{E}_5 v d_1 \dots d_m$  reduces to  $E$  in  $e$  steps, where*

$$e < 30r(m + r) + 4n(r + p) + (n\bar{n} + 2\bar{n} + 28n).$$

*Proof:* From the above discussion, let  $\bar{E}_5 = (\bar{Z} \bar{E}_4)$ .

$$\begin{aligned}
e &= e_1 + e_x + e_4 + e_y + e_z \\
&< 24r^2 + 2r(2n + 2) + 24mr + 2p(2n + 1) + e_z \\
&< 27r(m + r) + 4n(r + p) + e_z, \text{ since } 4r + 2p < 3r^2 \\
&< 30r(m + r) + 4n(r + p) + (n\bar{n} + 2\bar{n} + 28n), \text{ since } 17 < 3r^2.
\end{aligned}$$

We assume that  $n$  is “large” (though only polynomially so) compared to  $r$  and  $m$ . We have found an expression  $\bar{E}_5$  which after suitable arguments have been added reduces to  $E$  in  $4n(r + p) + e_z + O(r^2)$  reduction steps. We associate  $p$  in this with  $k$  in HS. We next show that “optimal” code representing  $E$  reduces in approximately  $4n(r + p) + e_z$  reduction steps.

We know the value of  $e_z$  to within (approximately)  $3n$ . Thus we know the “optimal” size of code, and we have an algorithm for getting to within narrow bounds of such code and certainly to sufficient accuracy to evaluate the value of  $k$  necessary to furnish a solution of HS. Thus we argue that, if we can find code representing  $E$  of size at most  $30r(m + r) + 4nr + 4nk + (n\bar{n} + 2\bar{n} + 28n)$  for  $E$  in polynomial time, we can solve HS in polynomial time also.

**Lemma 5.2**  $red_{v,d_1,\dots,d_m}(E) \geq (4n - 2)(r + p) + (n\bar{n} + 2\bar{n} + 25n - 11)$ .

*Proof:* Since the depth of  $V_{x,y}^n$  is greater than  $n\bar{n}$ , optimal code to represent  $V_{x,y}^n$  reduces in at least  $n\bar{n}$  steps, and by Lemma 4.5 there exists code representing  $E$  which reduces in less than  $2n\bar{n}$  steps. Thus to produce optimal code for  $E$  some code-sharing will be necessary. An “obvious” strategy would be to share as many common subexpressions as possible, in particular all occurrences of  $v^{i\bar{n}}$  and of  $(v^{i\bar{n}} z)$ , where  $z \in \{d_1, \dots, d_m\}$ . This does not, however, yield a *strategy* for producing optimal code, since we may assume only that *most* of these subexpressions must be shared, and we have not exhibited an optimal method for generating them.

Consider  $V_{x,y}^n$  and  $V_{a,b}^n$ , where  $x, y, a$ , and  $b$  are distinct. The only sharable subexpressions are those containing only occurrences of  $v$ , that is the  $v^{i\bar{n}}$ . Suppose we require to find code  $\bar{V}$  such that  $(\bar{V} v x y a b)$  reduces to  $(V_{x,y}^n V_{a,b}^n)$  in the minimum number of steps. Then we may assume that we share code  $X$  where  $(X x y)$  reduces to  $V_{x,y}^n$ . For, if we share code that allows more complicated arguments, we do *not* improve the code we produce, since we still require a similar amount of work for each  $V_{x,y}^n$ , of which less may be shared. By Lemma 4.3, we may assume that  $X \equiv abs_{x,y}(V_{x,y}^n)$ . If  $a \equiv x$  then this allows us the possibility of sharing the instances of  $v^{i\bar{n}}$  also.

Consider now  $V_{x,y}^n$  and  $V_{a,y}^n$ , where  $x, y$ , and  $a$  are distinct. Suppose we wish to find code  $\bar{V}$  such that  $(\bar{V} v x y a)$  reduces to  $(V_{x,y}^n V_{a,y}^n)$  in the minimum number of steps. If we share code as  $X$  above, we lose the possibility of sharing expressions  $(v^{i\bar{n}} y)$  containing  $y$ . However, if we have created  $abs_{x,y}(V_{y,x}^n)$ , we *will* be able to share those expressions. By symmetry, for a nontrivial  $E$  it will be necessary to create both  $abs_{x,y}(V_{x,y}^n)$  and  $abs_{x,y}(V_{y,x}^n)$ , hence we will need at least  $e_z$  reduction steps to perform that creation. At this point we note that, by Lemma 4.4, we would not be better off treating each  $W_{x,y}^n$  as a single unit rather than a combination of  $V_{x,y}^n$  and  $V_{y,x}^n$ .

Each occurrence of  $abs_{x,y}(V_{x,y}^n)$  or  $abs_{x,y}(V_{y,x}^n)$  will, by Lemma 4.1, require  $4n - 2$  reduction steps, of which  $2n - 1$  *cannot* be shared (viz. the second arguments), and  $2n - 1$  *may* be shared (the first arguments), thus yielding  $4n - 2$  for each  $W_{x,y}^n$  (total  $(4n - 2)r$ ) and  $4n - 2$  for each shared expression (total  $(4n - 2)p$ ). We also have, by Lemma 4.12,  $e_z \geq n\bar{n} + 2\bar{n} + 25n - 11$ .

Using  $W_{x,y}^n$  ensures that, if at any point we introduce  $abs_{x,y}(V_{x,y}^n)$ , we must also introduce  $abs_{y,x}(V_{x,y}^n)$ , thus ensuring symmetry. Use of  $Z_1$  and  $Z_2$  in the previous analysis serves to iron out the asymmetry which is introduced at the “bottom” of  $V_{x,y}^n$  when applying  $abs$ .

**Theorem 5.3** *HS transforms to OP.*

*Proof:* We have, from Lemmas 5.1 and 5.2:

1. A map  $f$  from an instance of HS to an instance of OP which can be evaluated in polynomial time, and which is injective;
2. An algorithm which will find code for an instance OP which reduces (after suitable arguments have been added) to  $E$  in  $e$  steps, where  $e < k' = 30r(m + r) + 4n(r + k) + (n\bar{n} + 2\bar{n} + 28n)$ ; and
3. A proof that  $red_{v,d_1,\dots,d_m}(E) \geq (4n - 2)(r + k) + (n\bar{n} + 2\bar{n} + 25n - 11)$ .

The difference between these two bounds is  $30r(m + r) + 2(r + k) + 3n - 11$ , which is less than the change in value of either of them if  $k$  is altered by 1 (viz.  $4n$ ), since  $n = 100r^3$ . If we produce code which reduces in  $k'$  reduction steps, we can find a value for  $k$  which is uniquely determined, which will solve the corresponding instance of HS.

**Lemma 5.4** *Suppose expression  $E_1$ , which contains only combinators, is such that  $(E_1 x_1 \dots x_m)$  reduces to  $E_3$  in  $p$  reduction steps, where  $E_3$  contains no combinators and the  $x_i$  are distinct variables. Then there exists an expression  $E_2$ , containing only combinators, such that  $(E_2 x_1 \dots x_m)$  reduces to  $E_3$  in at most  $p$  steps, with  $|E_2| \leq (|E_3| + 4p - m)^2$ .*

*Proof:* Let the combinators for the  $p$  reduction steps when  $(E_1 x_1 \dots x_m)$  is reduced be (in order)  $c_1, \dots, c_p$ . We can construct the expression  $E_2$  by working “backwards” from the graph representing  $E_3$ , effectively “mimicking” the original reduction in reverse. Where necessary we insert a “dummy” symbol, which is then replaced by an expression when appropriate.

When  $(E_2 x_1 \dots x_m)$  is finally constructed, each remaining dummy symbol is replaced by a single combinator, as this atom will have been “deleted” when  $(E_2 x_1 \dots x_m)$  is reduced.

Since the original reduction was normal order,  $E_2$  will be in normal form. However there may be some code-sharing in  $E_2$ , but since this expression is in normal form the number of reduction steps for  $(E_2 x_1 \dots x_m)$  will be the same as if  $E_2$  were considered as a tree with the shared subgraphs copied. This is because no shared node in  $E_2$  will be overwritten.

At no point must we introduce more than 4 extra symbols at any one step (for example, suppose our expression after reduction step  $n$  is  $\alpha$  and an  $S'$  reduction is used

at reduction step  $n$ , then the expression after step  $(n - 1)$  would be:  $(S' \kappa \beta \gamma \delta)$ , i.e., symbol  $\alpha$  has been replaced by expression  $(\kappa (\beta \delta)(\gamma \delta))$ .

We are constructing a graph, therefore the number of leaf nodes in  $E_2$  will be at most  $(|E_3| + 4p - m)$ . Since this graph may contain shared nodes,

$$|E_2| \leq (|E_3| + 4p - m)^2.$$

Note that this construction is nondeterministic: it assumes one has been able to choose which subexpression of an intermediate expression to rewrite in order to mimic the original reduction.

**Theorem 5.5** *The Optimization Problem is in NP.*

*Proof:* From Lemma 5.4, we need only generate expressions  $E'$  nondeterministically, with  $|E'| \leq (|E| + 4k' - m)^2$ , such that the only atomic subexpressions of  $E'$  are combinators and  $(E' x_1 \dots x_m)$  reduces to  $E$  in at most  $k'$  steps. We note here that we have already produced an expression  $E'$  which such that  $(E' x_1 \dots x_m)$  reduces to  $E$  in at most  $k'$  steps (see Lemma 5.1 above), and we may without loss of generality assume that  $k < k'$ . The steps necessary from creating the expression  $E'$  to deciding whether  $E'$  is a suitable expression can clearly be completed in polynomial time.

**Theorem 5.6** *The Optimization Problem is NP-Complete*

*Proof:* This is a consequence of Theorems 5.3 and 5.5.

**6 Final observations** If we restrict our attention to a subset of combinators, a *subbase*, and the corresponding set of functions which are representable using them, then the problem of producing optimal code *may* be simplified, as Batini and Pettorossi [2] show for the subbase  $\{B\}$ .

However, it is reasonable to assume that the result we have given is true if we do not restrict the functions we allow, provided that we use only a finite set of combinators. Our proof is specific to one particular set of combinators (it would, for example, fail at Lemmas 4.5 and 4.6 for a different set of combinators). A general proof is required.

**Acknowledgments** We are grateful to Warren Burton, Mike Paterson, Alan Gibbons and Tom Axford for comments on previous versions of this document, and to the United Kingdom Science and Engineering Research Council for funding the initial research.

## REFERENCES

- [1] Barendregt, H. P., *The Lambda Calculus, its Syntax and Semantics*, North-Holland, Amsterdam, 1981. [Zbl 0467.03010](#) [MR 83b:03016](#) [1](#)
- [2] Batini, C., and A. Pettorossi, *Some Properties of Subbases in Weak Combinatory Logic*, Report 75-04, Istituto di Automatica, Roma, 1975. [6](#)
- [3] Clarke, T. J. W., P. J. S. Gladstone, C. D. MacLean and A. C. Norman, "SKIM—The S,K,I Reduction Machine" in *Conference Record of the 1980 LISP Conference*, Stanford University, 1980. [1](#)

- [4] Curry, H. B., W. Craig and R. C. Feys, *Combinatory Logic*, vol. 1, North-Holland, Amsterdam, 1958. [Zbl 0081.24104](#) [MR 20:817](#) 1
- [5] Curry, H. B., J. R. Hindley and J. P. Seldin, *Combinatory Logic*, vol. 2, North-Holland, Amsterdam, 1972. [Zbl 0242.02029](#) 1
- [6] Garey, M. R., and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman, San Francisco, 1979. [Zbl 0411.68039](#) [MR 80g:68056](#) 2
- [7] Glaser, H., C. Hankin and D. Till, *Principles of Functional Programming*, Prentice-Hall, Englewood Cliffs, 1984. [Zbl 0649.68002](#) [MR 88i:68007](#) 1
- [8] Hindley, J. R., and J. P. Seldin, *Introduction to Combinators and  $\lambda$ -Calculus*, Cambridge University Press, Cambridge, 1986. [Zbl 0614.03014](#) [MR 88j:03009](#) 1
- [9] Joy, M. S., *On the Efficient Implementation of Combinators as an Object Code for Functional Programs*, PhD Thesis, University of East Anglia, Norwich, 1985. 1, 4
- [10] Joy, M. S., V. J. Rayward-Smith and F. W. Burton, "Efficient combinator code," *Computer Languages*, vol. 10 (1985), pp. 221–224. 1
- [11] Kennaway, J. R., "The complexity of a translation of  $\lambda$ -calculus to combinators," Internal Report CS/82/023/E, University of East Anglia, Norwich, 1982. 4
- [12] Klop, J. W., *Combinatory Reduction Systems*, Mathematisch Centrum, Amsterdam, 1980. [Zbl 0466.03006](#) [MR 83e:03026](#) 3.1
- [13] Stoye, W. R., "The implementation of functional languages using custom hardware," Technical Report 81, University of Cambridge Computer Laboratory, Cambridge, 1985. 1
- [14] Turner, D. A., "Another algorithm for bracket abstraction," *The Journal of Symbolic Logic*, vol. 44, (1979), pp. 67–70. [Zbl 0408.03013](#) [MR 80f:03021](#) 1, 3.2
- [15] Turner, D. A., "A new implementation technique for applicative languages," *Software—Practice and Experience*, vol. 9 (1979), pp. 31–49. [Zbl 0386.68009](#) 3.1
- [16] Turner, D. A., "Combinator reduction machines," in *Proceedings of the International Workshop on High Level Computer Architecture*, Los Angeles, 1984. 1

*Department of Computer Science*  
*University of Warwick*  
*Coventry, CV4 7AL*  
*U.K.*  
*email: M.S.Joy@dcs.warwick.ac.uk*

*School of Information Systems*  
*University of East Anglia*  
*Norwich, NR4 7TJ*  
*U.K.*  
*email: V.Rayward-Smith@sys.uea.ac.uk*