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ON FINDING THE GROUND STATE SOLUTION TO THE LINEARLY COUPLED BREZIS–NIRENBERG SYSTEM IN HIGH DIMENSIONS: THE COOPERATIVE CASE

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Abstract. Consider the following elliptic system

$$\begin{cases} -\Delta u_i + \mu_i u_i = |u_i|^{2^*-2}u_i + \lambda \sum\limits_{j=1,\, j\neq i}^{k} u_j & \text{in } \Omega, \\ u_i = 0, \quad i = 1, \ldots, k, & \text{on } \partial\Omega, \end{cases}$$

where $k \geq 2$, $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ is a bounded domain with smooth boundary $\partial\Omega$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $\mu_i \in \mathbb{R}$ for all $i = 1, \ldots, k$ are constants and $\lambda \in \mathbb{R}$ is a parameter. By the variational method, we mainly prove that the above system has a ground state for all $\lambda > 0$. Our results reveal some new properties of the above system that imply that the parameter λ plays the same role as in the following well-known Brezis–Nirenberg equation

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and this system has a very similar structure of solutions as the above Brezis–Nirenberg equation for λ.

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1. Introduction

In this paper, we mainly consider the following elliptic system

$$
\begin{cases} -\Delta u_i + \mu_i u_i = |u_i|^{2^*-2}u_i + \lambda \sum\limits_{j=1,\, j\neq i}^{k} u_j & \text{in } \Omega, \\ u_i = 0, \quad i = 1,\dots,k & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $k \geq 2$, $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ is a bounded domain with smooth boundary $\partial\Omega$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $\mu_i \in \mathbb{R}$ for all $i = 1,\dots,k$ are constants and $\lambda \in \mathbb{R}$ is a parameter.

Over the last 25 years, owing to important applications in biology and physics in low dimensions $(1 \leq N \leq 3)$, there has been significant interest in studying the existence, multiplicity, and qualitative properties of solutions to the following elliptic system

$$
\begin{cases} -\Delta u_i + \mu_i u_i = |u_i|^{p-2}u_i + \lambda F_{u_i}(\mathbf{u}) & \text{in } \Omega, \\ u_i \in H_0^1(\Omega), \quad i = 1,\dots,k, \end{cases} \tag{1.2}
$$

where $\mathbf{u} = (u_1,\dots,u_k)$, $k \geq 2$, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a domain (bounded or unbounded), $2 < p < 2^*$ for $N = 1,2$ and $2 < p \leq 2^*$ for $N \geq 3$ with $2^* = +\infty$ for $N = 1,2$ and $2^* = 2N/(N-2)$ for $N \geq 3$ being the Sobolev critical exponent, $\mu_i \in \mathbb{R}$ for all $i = 1,\dots,k$ are constants, and $\lambda \in \mathbb{R}$ is a parameter. For example, let $k = 2$, $p = 4$, and $F(\mathbf{u}) = u_1^2u_2^2/2$, then the system (1.2) has the following nonlinearly coupled form

$$
\begin{cases} -\Delta u_1 + \mu_1 u_1 = u_1^3 + \lambda u_2^2 u_1 & \text{in } \Omega, \\ -\Delta u_2 + \mu_2 u_2 = u_2^3 + \lambda u_1^2 u_2 & \text{in } \Omega, \\ u_i \in H_0^1(\Omega), \quad i = 1,2, \end{cases} \tag{1.3}
$$

which are also known in the literature as the Gross–Pitaevskiĭ equations (see e.g. [17]). Such a system can be used to describe the Bose–Einstein condensation in two different hyperfine spin states in the Hartree–Fock theory (cf. [8]), which also arises in nonlinear optics to describe the behavior of the beam in Kerr-like photorefractive media (cf. [1]). From the viewpoint of mathematics, an important characteristic of the system (1.3) is that it is weakly coupled, that is, system (1.3) has semi-trivial solutions (the definitions are given in Definition 1.1 below). Indeed, let u_{μ_i} be the solution to the following equation

$$
\begin{cases} -\Delta u + \mu_i u = u^3 & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}
$$

Then it is easy to see that $(u_{\mu_1}, 0)$ and $(0, u_{\mu_2})$ are solutions of system (1.3). On the other hand, if we set $k = 2$, $\Omega = \mathbb{R}^N$ and $F(\mathbf{u}) = u_1u_2$, then system (1.2) has the following linearly coupled form

$$
(1.4)\qquad \begin{cases} -\Delta u_1 + \mu_1 u_1 = |u_1|^{p-2}u_1 + \lambda u_2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \mu_2 u_2 = |u_2|^{p-2}u_2 + \lambda u_1 & \text{in } \mathbb{R}^N, \\ u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \end{cases}
$$

which is also used to describe some phenomena in nonlinear optics in low dimensions ($1 \le N \le 3$) (cf. [1]). From the viewpoint of mathematics, an important characteristic of system (1.4) is that it is strongly coupled, that is, system (1.4) does not have semi-trivial solutions. Since it seems almost impossible for us to give a complete list of references, we simply refer the reader to [6], [9], [14], [23], [24], [32] and references therein for system (1.3) and [2], [3], [12], [20] and references therein for system (1.4).

Recently, system (1.2) with Sobolev critical exponent has begun to attract attention, see, for example, [10]–[12], [25], [31] and references therein. It should be pointed out that, compared with the subcritical case, the existence of a nontrivial solution is always very *fragile* for the Sobolev critical equation or system. For example, Chen and Zou considered the following critical system in [10],

$$
(1.5)\qquad \begin{cases} -\Delta u_1 + \mu_1 u_1 = |u_1|^{p_1-2}u_1 + \lambda u_2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \mu_2 u_2 = |u_2|^{p_2-2}u_2 + \lambda u_1 & \text{in } \mathbb{R}^N, \\ u_i \in H^1(\mathbb{R}^N), \quad i = 1, 2, \end{cases}
$$

where $N \ge 3$, $2 < p_1, p_2 \le 2^*$ with $2^* = 2N/(N-2)$ being the Sobolev critical exponent, $\mu_1, \mu_2 > 0$, and $0 < \lambda < \sqrt{\mu_1\mu_2}$. By using the variational method, it has been proved in [10] that system (1.5) has only zero solution with $\mu_1, \mu_2 > 0$ and $0 < \lambda < \sqrt{\mu_1\mu_2}$ for $p_1 = p_2 = 2^*$ whereas for $p_1 < p_2 = 2^*$, there exists a number $\lambda_{\mu_1,\mu_2} \in (0, \sqrt{\mu_1\mu_2}]$ such that the existence of positive ground state solutions is strongly dependent on the relation between λ and λ_{μ_1,μ_2}. Moreover, as pointed out by Chen and Zou in [10, Remark 1.1], λ_{μ_1,μ_2} can be seen as a critical value for the existence of positive ground state solutions and it remains open whether system (1.5) has a ground state solution for $\lambda = \lambda_{\mu_1,\mu_2}$. In the very recent work [25], Peng et al. studied the following critical system

$$
(1.6)\qquad \begin{cases} -\Delta u_1 + \mu_1 u_1 = |u_1|^{2^*-2}u_1 + \lambda u_2 & \text{in } \Omega, \\ -\Delta u_2 + \mu_2 u_2 = |u_2|^{2^*-2}u_2 + \lambda u_1 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \ge 3$) is a bounded domain. By using the variational method, it has been proved in [25] that system (1.6) has a positive ground state solution with $-\alpha_1 < \min\{\mu_1, \mu_2\} < 0$ and $0 < \lambda < \sqrt{(\alpha_1 + \mu_1)(\alpha_1 + \mu_2)}$, whereas system (1.6) has only zero solution with $\mu_1, \mu_2 > 0$ and $0 < \lambda < \sqrt{\mu_1\mu_2}$ if Ω is star-shaped. Moreover, some new results about the multiplicity of nontrivial solutions to system (1.6) for $\lambda > 0$ small enough were also established in [25]. Here $\alpha_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

We also note that the general k-component case of system (1.2) with

$$F(\mathbf{u}) = \frac{2}{p} \sum_{i,j=1,\, j\neq i}^{k} |u_j|^{p/2}|u_i|^{p/2}$$

have already been studied by the variational method and many results involving the critical case for $k = 2$ have been extended to the general case $k \geq 2$, see, for example, [21], [22], [29], [31] and references therein. We remark that the k-component case of system (1.2) also has a physical background and a condensation has been experimentally observed in the triplet states (cf. [26]). Moreover, from the viewpoint of mathematics, it has been observed that the general k-component case of such system with the critical Sobolev exponent may have some new phenomena and properties (cf. [31]) that are somewhat different from the 2-component case (cf. [11], [13]). Thus, inspired by the above facts, it is natural to ask *what will happen for the k-component critical system* (1.1)? *In particular, will recent results in* [25] *for $k = 2$ still hold for the general case $k \geq 2$*? *Is the general k-component case of* (1.1) *different from the* 2*-component case*? To the best of the author's knowledge, these questions have not yet been studied in the literature, thus the main purpose of the current paper is to provide an answer to these questions.

Clearly, system (1.1) has a variational structure. Indeed, for every $i = 1, \ldots, k$, let $\mathcal{H}_i$ be the Hilbert space of $H_0^1(\Omega)$ equipped with the inner product

$$\langle u, v\rangle_i = \int_\Omega \nabla u \nabla v + \mu_i uv\, dx.$$

Then if $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$, $\mathcal{H}_i$ are also the Hilbert spaces and the corresponding norms are given by $\|u\|_i = \langle u, u\rangle_i^{1/2}$, respectively. Set $\mathcal{H} = \prod_{i=1}^{k} \mathcal{H}_i$. Then $\mathcal{H}$ is a Hilbert space with the inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{k} \langle u_i, v_i\rangle_i.$$

The corresponding norm is given by $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u}\rangle^{1/2}$. Here, u_i, v_i are the ith component of $\mathbf{u}$, $\mathbf{v}$, respectively. Define

$$\mathcal{E}_\lambda(\mathbf{u}) = \sum_{i=1}^{k} \left(\frac{1}{2} \|u_i\|_i^2 - \frac{1}{2^*} \int_\Omega |u_i|^{2^*}\, dx \right) - \lambda \sum_{i,j=1,\, i<j}^{k} \int_\Omega u_i u_j\, dx. \tag{1.7}$$

Then it is easy to see that $\mathcal{E}_\lambda(\mathbf{u})$ is of C^2 in $\mathcal{H}$ and $\mathcal{E}_\lambda(\mathbf{u})$ is the corresponding functional of the system (1.1).

DEFINITION 1.1. We call $\mathbf{u}$ a nonzero solution to (1.1) if $\mathbf{u}$ is a solution to (1.1) with $\mathbf{u} \neq \mathbf{0}$. We call $\mathbf{u}$ a nontrivial solution to (1.1) if $\mathbf{u}$ is a solution to (1.1) with $u_i \neq 0$ for all $i = 1, \dots, k$. We call $\mathbf{u}$ a semi-trivial solution to (1.1) if $\mathbf{u}$ is a nonzero solution to (1.1) that is not a nontrivial solution.

Definition 1.2. We call $\mathbf{u}$ a nonnegative solution to (1.1) if $\mathbf{u}$ is a nonzero solution with $u_i \geq 0$ for all $i = 1, \dots, k$. We call $\mathbf{u}$ a positive solution to (1.1) if $\mathbf{u}$ is a nonnegative solution with $u_i > 0$ for all $i = 1, \dots, k$. We call $\mathbf{u}$ a nonpositive solution to (1.1) if $-\mathbf{u}$ is a nonnegative solution. We call $\mathbf{u}$ a negative solution to (1.1) if $-\mathbf{u}$ is a positive solution. Here, $-\mathbf{u} = (-u_1, \dots, -u_k)$. We call $\mathbf{u}$ a sign-constant solution to (1.1) if either $\mathbf{u}$ is a nonnegative solution or $\mathbf{u}$ is a nonpositive solution. We call $\mathbf{u}$ a sign-changing solution to (1.1) if $\mathbf{u}$ is a nonzero solution that is not a sign-constant solution.

Definition 1.3. We call $\mathbf{u} \in \mathcal{H}$ a ground state solution to (1.1) if $\mathbf{u}$ is a nonzero solution and $\mathcal{E}_\lambda(\mathbf{u}) \leq \mathcal{E}_\lambda(\mathbf{v})$ for all nonzero solutions $\mathbf{v}$.

Let us briefly sketch our main idea in studying (1.1). Let

$$\mathcal{F} = \operatorname{diag}\big((-\Delta + \mu_1)^{-1}, \dots, (-\Delta + \mu_k)^{-1}\big)$$

and

$$\mathcal{I} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

Then system (1.1) is equivalent to the following operator equation in $\mathcal{H}$

$$\mathbf{u} = \lambda \mathcal{T}\mathbf{u} + \mathcal{T}^*\mathbf{u}, \tag{1.8}$$

where $\mathcal{T} = \mathcal{F} \circ \mathcal{I}$ and $\mathcal{T}^* = \mathcal{F} \circ \mathcal{Z}$ with $\mathcal{Z}(\mathbf{u}) = \big(|u_1|^{2^*-2}u_1, \dots, |u_k|^{2^*-2}u_k\big)$.

By (1.8), it seems that, from the viewpoint of operators, system (1.1) has a very similar structure to the following well-known Brezis–Nirenberg equation

$$\begin{cases} \Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Note that the existence of solutions to the above well-known Brezis–Nirenberg equation is heavily dependent on the relations between λ and α_m (cf. [4], [28] and references therein). Thus, to study system (1.1) for $\lambda > 0$, it seems necessary to provide a clear understanding of the eigenvalue problem $\mathbf{u} = \lambda \mathcal{F} \circ \mathcal{I}\mathbf{u}$ corresponding to (1.1), which is equivalent to

$$\begin{cases} -\Delta u_i + \mu_i u_i = \lambda \displaystyle\sum_{j=1,\, j\neq i}^{k} u_j & \text{in } \Omega, \\ u_i = 0, \quad i = 1, \dots, k & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Here $\lambda > 0$ and $\{\alpha_m\}_{m\in\mathbb{N}}$ are the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, which are increasing for m.

Based on these observations, we first need to study the system (1.10). Clearly, system (1.10) is the linearization of (1.1) at the trivial solution $\mathbf{0}$. Let $\mathcal{N}_m$ be the corresponding eigenspace of α_m. Then our first result can be stated as follows.

THEOREM 1.4. *Let $N \geq 1$, $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda > 0$. Then there exists a sequence $\{\lambda_m\} \subset \mathbb{R}^+$ with $\lambda_m \nearrow +\infty$ as $m \to \infty$ such that system* (1.10) *has nonzero solution if and only if $\lambda = \lambda_m$. Moreover, we also have*:

(a) *For every $m \in \mathbb{N}$, λ_m is the unique solution to the following equation*

$$\sum_{j=1}^{k} \frac{\lambda}{\alpha_m + \mu_j + \lambda} = 1. \tag{1.11}$$

(b) *Here $\mathbf{u} = (u_1, \ldots, u_k)$ is a solution to system* (1.10) *corresponding to λ_m if and only if $\mathbf{u} \in \mathcal{N}_m^* = \{\varphi \mathbf{e}_m \mid \varphi \in \mathcal{N}_m\}$, where $\mathbf{e}_m$ is the unique basic of the algebra equation $\mathcal{D}_m^* \mathbf{X} = \mathbf{0}$ with*

$$\mathcal{D}_m^* = \begin{pmatrix} \alpha_m + \mu_1 & -\lambda_m & -\lambda_m & \ldots & -\lambda_m \\ -\lambda_m & \alpha_m + \mu_2 & -\lambda_m & \ldots & -\lambda_m \\ -\lambda_m & -\lambda_m & \alpha_m + \mu_3 & \ldots & -\lambda_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda_m & -\lambda_m & -\lambda_m & \ldots & \alpha_m + \mu_k \end{pmatrix}.$$

(c) *We have $\lambda_m = \inf_{\mathbf{u}\in\mathcal{M}_{m-1}} \|\mathbf{u}\|^2/2$, where*

$$\mathcal{M}_{m-1} = \Big\{\mathbf{u} \in \big(\widetilde{\mathcal{N}}_{m-1}^*\big)^{\perp} \mid \mathcal{G}(\mathbf{u}) = 1\Big\}$$

with

$$\mathcal{G}(\mathbf{u}) = \sum_{i,j=1,\, i<j}^{k} \int_\Omega u_j u_i \, dx \quad \textit{and} \quad \big(\widetilde{\mathcal{N}}_{m-1}^*\big)^{\perp} = \bigoplus_{l=m}^{\infty} \mathcal{N}_l^*.$$

In particular, $\big(\widetilde{\mathcal{N}}_0^*\big)^{\perp} = \mathcal{H}$.

REMARK 1.5. (a) Some early studies on the eigenvalue problem related to an elliptic system that is linearly coupled are given in [7], [15], [18] and references therein. However, to the best of the author's knowledge, Theorem 1.4 seems to present the first completed results devoted to system (1.10) for all $\lambda > 0$.

(b) By Theorem 1.4, we also have the decomposition $\mathcal{H} = \bigoplus_{l=1}^{\infty} \mathcal{N}_l^*$ of the space $\mathcal{H}$. Moreover, if $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$, then $\|\mathbf{u}\|^2/2 - \lambda\mathcal{G}(\mathbf{u})$, the order-two part of the functional $\mathcal{E}_\lambda(\mathbf{u})$, is positive definite in $(\widetilde{\mathcal{N}}_m^*)^{\perp}$ and nonpositive definite in $\widetilde{\mathcal{N}}_m^*$. In particular, $\|\mathbf{u}\|^2/2 - \lambda\mathcal{G}(\mathbf{u})$ is positive definite in $\mathcal{H}$ for $0 < \lambda < \lambda_1$. These properties are very important in applying the variational method to study system (1.1).

(c) If $k = 2$, then by (1.11), $\lambda_m = \sqrt{(\alpha_m + \mu_1)(\alpha_m + \mu_2)}$ for all $m \in \mathbb{N}$.

Since the Brezis–Nirenberg equation (1.9) has only zero solution for $\lambda \leq 0$ if Ω is star-shaped, by our above observations, it is also natural to study the nonexistence result of System (1.1). Our results in this aspect can be stated as follows.

THEOREM 1.6. *Let $N \geq 3$ and $\mu_i > 0$ for all $i = 1, \ldots, k$. If Ω is also star-shaped, then system* (1.1) *only has zero solution for $0 < \lambda \leq \lambda_1^*$, where $\lambda_1^* < \lambda_1$ is the unique solution to the following equation*

$$\sum_{j=1}^{k} \frac{\lambda}{\mu_j + \lambda} = 1 \tag{1.12}$$

and λ_1 is given by Theorem 1.4.

REMARK 1.7. If $k = 2$, then by (1.12), $\lambda_1^* = \sqrt{\mu_1 \mu_2}$, which implies Theorem 1.6 for $k = 2$ is just the observation by Peng et al. in [25, Remark 1.1]. However, to the best of the author's knowledge, Theorem 1.6 for $k \geq 3$ is totally new. Moreover, we also give a uniform and precise formula to describe the number λ_1^* for all $k \geq 2$.

Since system (1.1) may only have zero solution if $\mu_i > 0$ for all $i = 1, \ldots, k$, it is natural to study the existence of nonzero solutions of system (1.1) under the condition $\min\{\mu_1, \ldots, \mu_k\} < 0$. In what follows, to state our main results about this aspect, we first introduce some notation. Let

$$\mathcal{J}_\nu(u) = \frac{1}{2}\bigg(\int_\Omega |\nabla u|^2\,dx + \nu \int_\Omega |u|^2\,dx\bigg) - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx.$$

Then it is well known that (cf. [4], [28]) in the case $N \geq 4$, $m_\nu = \mathcal{S}^{N/2}/N$ for $\nu > 0$ whereas m_ν can be attained for $\nu < 0$ in one of the following two cases:

(1) $N = 4$ and $\nu \neq -\alpha_m$ for all $m \in \mathbb{N}$,
(2) $N \geq 5$.

Moreover, we also have $0 < m_\nu < \mathcal{S}^{N/2}/N$ in these two cases. Here,

$$m_\nu = \inf_{u \in \mathcal{Q}_\nu} \mathcal{J}_\nu(u) \tag{1.13}$$

with $\mathcal{Q}_\nu = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \mathcal{J}_\nu'(u)u = 0\}$. Now, our main results can be stated as follows.

THEOREM 1.8. *Let $N \geq 4$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. If we also have $\min\{\mu_1, \ldots, \mu_k\} < 0$, then we have*:

(a) *System* (1.1) *has a positive ground state solution $\mathbf{u}_\lambda$ for $0 < \lambda < \lambda_1$. Moreover, system* (1.1) *has no sign-constant solution for $\lambda \geq \lambda_1$.*

(b) *System* (1.1) *has a ground state solution* $\widetilde{\mathbf{u}}_\lambda$ *that is also sign-changing in one of the following two cases*:

(1) $N = 4$ *and* $\lambda \geq \lambda_1$ *with* $\lambda \neq \lambda_m$ *for all* $m \in \mathbb{N}$,

(2) $N \geq 5$ *and* $\lambda \geq \lambda_1$.

Moreover, if $k = 2$ *or* $k \geq 3$ *with*

$$\mathcal{E}_\lambda(\widetilde{\mathbf{u}}_\lambda) < \min_{i,j=1,\ldots,k,\, i\neq j} \{m_{\mu_i+\lambda} + m_{\mu_j+\lambda}\}, \tag{1.14}$$

then $\widetilde{\mathbf{u}}_\lambda$ *is also nontrivial.*

Remark 1.9. (a) The existence of positive ground state solutions to system (1.1), described by (a) of Theorem 1.8, was predicted by Peng et al. in [25, Remark 1.4]. However, the novelty of (a) of Theorem 1.8 is that we give a global description of the existence and nonexistence of positive ground state solutions to system (1.1) for all $\lambda > 0$, which is based on Theorem 1.4.

(b) To the best of the author's knowledge, part (b) of Theorem 1.8 is totally new even for $k = 2$.

(c) The condition (1.14) is easy to achieve. Indeed, it has been proved in Lemma 5.7 that $\mathcal{E}_\lambda(\widetilde{\mathbf{u}}_\lambda) < \mathcal{S}^{N/2}/N$, where $\mathcal{S}$ is the best Sobolev embedding constant from $H^1(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$. On the other hand, if either $\lambda > \max\{-\mu_1, \ldots, -\mu_k\}$ or there is only one $\mu_j < 0$, then we must have

$$\min_{i,j=1,\ldots,k,\, i\neq j} \{m_{\mu_i+\lambda} + m_{\mu_j+\lambda}\} \geq \frac{1}{N}\,\mathcal{S}^{N/2},$$

which implies that the condition (1.14) holds.

(d) The condition (1.14) also implies that the case $k \geq 3$ is somewhat different from the case $k = 2$. Indeed, as we stated above, the 2-component case of system (1.1) is strongly coupled, that is, it has no semi-trivial solutions. However, the general k-component case of system (1.1) with $k \geq 3$ could be weakly coupled, that is, it may have semi-trivial solutions. For example, let $\mu_1 = \mu_2 = -\mu < 0$. Then for $\lambda < \mu$, we can see that $m_{-\mu+\lambda}$ can be attained by some $\widetilde{u}_{-\mu+\lambda}$ in one of the following two cases:

(1) $N = 4$ and $-\mu + \lambda \neq -\alpha_m$ for all $m \in \mathbb{N}$,

(2) $N \geq 5$.

Set $\widetilde{\mathbf{U}}_\lambda = (\widetilde{u}_{-\mu+\lambda}, -\widetilde{u}_{-\mu+\lambda}, 0, \ldots, 0)$. Then it is easy to see that $\widetilde{\mathbf{U}}_\lambda$ is a semi-trivial solution to system (1.1).

(e) The condition (1.14) also seems to be technical for $k \geq 3$. For example, in the case $k = 3$, we can see from Remark 5.10 that any nonzero solution must be nontrivial if $\mu_1 \neq \mu_2$, $\mu_1 \neq \mu_3$, and $\mu_2 \neq \mu_3$. Thus, it will be very interesting to discuss the most general condition to ensure that $\widetilde{\mathbf{u}}_\lambda$ is nontrivial.

(f) Recall that (cf. [4], [28]) the well-known Brezis–Nirenberg equation (1.9) has a ground state solution in one of the following two cases:

(1) $N = 4$, $\lambda > 0$ and $\lambda \neq \alpha_m$ for all $m \in \mathbb{N}$,
(2) $N \geq 5$, $\lambda > 0$.

Moreover, the ground state solution is positive for $0 < \lambda < \alpha_1$ and cannot be sign-constant for $\lambda \geq \alpha_1$. Now, by Theorem 1.8, we can see that system (1.1) has a very similar structure of solutions to the well-known Brezis–Nirenberg equation (1.9).

In this paper, we also obtain the following result.

THEOREM 1.10. *Let $\mathbf{u}_\lambda$ be the positive ground state solution to system* (1.1) *obtained by Theorem* 1.8 *for* $0 < \lambda < \lambda_1$. *Then $\mathbf{u}_\lambda \to \mathbf{0}$ strongly in $\mathcal{H}$ as $\lambda \to \lambda_1$.*

REMARK 1.11. To the best of the author's knowledge, Theorem 1.10 is also totally new for system (1.1) even for $k = 2$. Moreover, from the viewpoint of bifurcation, we can see from Theorem 1.10 that $(\mathbf{0}, \lambda_1)$ is a bifurcation point at the trivial branch $(\mathbf{0}, \lambda)$, which is also very similar to the well-known Brezis–Nirenberg equation (1.9).

REMARK 1.12. By (f) of Remark 1.9 and Remark 1.11, we call system (1.1) the linearly coupled Brezis–Nirenberg system.

Notation. Throughout this paper, C and C' are indiscriminately used to denote various absolute positive constants. We also list some notation used frequently below:

$$\begin{aligned} \mathbf{u} &= (u_1, \dots, u_k), & \mathcal{L}^r(\Omega) &= (L^r(\Omega))^k, \\ \widehat{\{\mathbf{t}, \mathbf{u}\}} &= (t_1 u_1, \dots, t_k u_k), & \mathbf{u}_{|u|} &= (|u_1|, \dots, |u_k|), \\ t\mathbf{u} &= (tu_1, \dots, tu_k), & \mathbf{u}_n &= (u_1^n, \dots, u_k^n). \end{aligned}$$

We use $O(|\mathbf{b}|)$ to denote the quantities that tend towards zero as $|\mathbf{b}| \to 0$, where $|\mathbf{b}|$ is the usual norm in $\mathbb{R}^k$ of the vector $\mathbf{b}$. We also denote the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ by $\{\alpha_m\}_{m\in\mathbb{N}}$, which are increasing for m. The corresponding eigenspaces of α_m are denoted by $\mathcal{N}_m$.

2. The spectrum of the operator $\mathcal{T}$

Recall $\mathcal{F} = \mathrm{diag}((-\Delta + \mu_1)^{-1}, \dots, (-\Delta + \mu_k)^{-1})$ and

$$\mathcal{I} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Then it is easy to see that $\mathcal{T} = \mathcal{F} \circ \mathcal{I}$ is a linear operator from $\mathcal{L}^2(\Omega)$ to $\mathcal{H}$.

Lemma 2.1. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \dots, k$. Then $\mathcal{T}$ is compact from $\mathcal{H}$ to $\mathcal{H}$.*

Proof. Let $\{\mathbf{u}_n\}$ be a bounded sequence in $\mathcal{H}$. Then without loss of generality, we may assume that $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ weakly in $\mathcal{H}$ as $n \to \infty$. By the Sobolev embedding and without loss of generality once more, we may assume that $\mathbf{u}_n \to \mathbf{u}_0$ strongly in $\mathcal{L}^2(\Omega)$ as $n \to \infty$. Denote $\mathbf{v}_n = \mathcal{T}\mathbf{u}_n$, then we have

$$\begin{cases} -\Delta v_i^n + \mu_i v_i^n = \displaystyle\sum_{j=1, j\neq i}^{k} u_j^n & \text{in } \Omega, \\ v_i^n = 0, \quad i = 1, \dots, k & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It follows that

$$\begin{aligned} \|v_i^n\|_i^2 &= \sum_{j=1,\, j\neq i}^{k} \int_\Omega u_j^n v_i^n \, dx \\ &\leq \sum_{j=1,\, j\neq i}^{k} \bigg(\int_\Omega |u_j^n|^2 \, dx\bigg)^{1/2} \bigg(\int_\Omega |v_i^n|^2 \, dx\bigg)^{1/2} \\ &\leq \frac{1}{\displaystyle\prod_{j=1}^{k} \sqrt{\mu_j}} \sum_{j=1,\, j\neq i}^{k} \|u_j^n\|_j \|v_i^n\|_i, \end{aligned}$$

which implies $\{\mathbf{v}_n\}$ is bounded in $\mathcal{H}$. Without loss of generality, we may assume that $\mathbf{v}_n \rightharpoonup \mathbf{v}_0$ weakly in $\mathcal{H}$ as $n \to \infty$. Then, by (2.1), we can see that

$$\begin{cases} -\Delta v_i^0 + \mu_i v_i^0 = \displaystyle\sum_{j=1,\, j\neq i}^{k} u_j^0 & \text{in } \Omega, \\ v_i^0 = 0, \quad i = 1, \dots, k & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Thus, $\mathbf{v}_0 = \mathcal{T}\mathbf{u}_0$. On the other hand, since $\mathbf{u}_n \to \mathbf{u}_0$ strongly in $\mathcal{L}^2(\Omega)$ as $n \to \infty$ and $\{\mathbf{v}_n\}$ is bounded in $\mathcal{H}$, we have from (2.1) once more and (2.2) that

$$\begin{aligned} \|v_i^0\|_i^2 \leq \|v_i^n\|_i^2 + o(1) &= \sum_{j=1, j\neq i}^{k} \int_\Omega u_j^n v_i^n \, dx + o(1) \\ &= \sum_{j=1, j\neq i}^{k} \bigg(\int_\Omega u_j^0 v_i^n \, dx + \int_\Omega (u_j^n - u_j^0) v_i^n \, dx\bigg) + o(1) \\ &= \sum_{j=1,\, j\neq i}^{k} \int_\Omega u_j^0 v_i^0 \, dx + o(1) = \|v_i^0\|_i^2 + o(1). \end{aligned}$$

Hence, we must have $\mathbf{v}_n \to \mathbf{v}_0$ strongly in $\mathcal{H}$ as $n \to \infty$. That is, $\mathcal{T}\mathbf{u}_n \to \mathcal{T}\mathbf{u}_0$ strongly in $\mathcal{H}$ as $n \to \infty$. $\square$

Denote the spectrum of $\mathcal{T}$ in $\mathcal{H}$ by $\sigma(\mathcal{T})$. Then by Lemma 2.1, we can see that $\sigma(\mathcal{T}) = \sigma_p(\mathcal{T})$, where $\sigma_p(\mathcal{T})$ is the point spectrum of $\mathcal{T}$ in $\mathcal{H}$. Recall that $\{\alpha_m\}_{m\in\mathbb{N}}$ are the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, which are increasing for m, and the corresponding eigenspaces of α_m are denoted by $\mathcal{N}_m$.

LEMMA 2.2. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. Then there exists a positive sequence $\{\sigma'_m\}$ with $\sigma'_m \to 0$ as $m \to \infty$ such that $\{\sigma'_m\} \subset \sigma(\mathcal{T}) \cap (0, +\infty)$.*

PROOF. For every $m \in \mathbb{N}$, let us consider the following function

$$f_m(\lambda) = \sum_{j=1}^{k} \frac{\lambda}{\alpha_m + \mu_j + \lambda}. \tag{2.3}$$

It is easy to see that $f_m(0) = 0$, $\lim_{\lambda\to+\infty} f_m(\lambda) = k$ and $f_m(\lambda)$ is strictly increasing for $\lambda > 0$. Since $k \geq 2$, there exists a unique $\lambda'_m > 0$ such that $f_m(\lambda'_m) = 1$. Let

$$\mathcal{D}'_m = \begin{pmatrix} \alpha_m + \mu_1 & -\lambda'_m & -\lambda'_m & \ldots & -\lambda'_m \\ -\lambda'_m & \alpha_m + \mu_2 & -\lambda'_m & \ldots & -\lambda'_m \\ -\lambda'_m & -\lambda'_m & \alpha_m + \mu_3 & \ldots & -\lambda'_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda'_m & -\lambda'_m & -\lambda'_m & \ldots & \alpha_m + \mu_k \end{pmatrix}.$$

Then, by a direct calculation, we can see from $\alpha_m + \mu_i + \lambda'_m > 0$ for all $m \in \mathbb{N}$ and $i = 1, \ldots, k$ that

$$\det(\mathcal{D}'_m) = \prod_{i=1}^{k} (\alpha_m + \mu_i + \lambda'_m)\bigg(1 - \sum_{i=1}^{k} \frac{\lambda'_m}{\alpha_m + \mu_i + \lambda'_m}\bigg).$$

It follows from $f_m(\lambda'_m) = 1$ that $\det(\mathcal{D}'_m) = 0$. Now, let $\mathbf{u}_m = \mathbf{b}\varphi_m$, where $\varphi_m \in \mathcal{N}_m$ and $\mathbf{b}$ is a constant vector. Since φ_m is the eigenfunction of α_m, by a direct calculation, we can see that

$$-\Delta u_i^m + \mu_i u_i^m - \lambda'_m \sum_{j=1, j\neq i}^{k} u_j^m = \bigg(b_i(\alpha_m + \mu_i) - \lambda'_m \sum_{j=1, j\neq i}^{k} b_j\bigg)\varphi_m$$

for all $i = 1, \ldots, k$. Since $\det(\mathcal{D}'_m) = 0$, there exists $\mathbf{b}_m \neq \mathbf{0}$ such that

$$-\Delta u_i^m + \mu_i u_i^m - \lambda'_m \sum_{j=1,\, j\neq i}^{k} u_j^m = 0 \quad \text{for all } i = 1, \ldots, k.$$

Let $\sigma'_m = 1/\lambda'_m$. Then $\{\sigma'_m\} \subset \sigma(\mathcal{T}) \cap (0, +\infty)$. Moreover, since $\alpha_m \nearrow +\infty$ as $m \to \infty$, we can see from $f_m(\lambda'_m) = 1$ that $\lambda'_m \nearrow +\infty$ as $m \to \infty$, which implies $\sigma'_m \searrow 0$ as $m \to \infty$. $\square$

By Lemma 2.2, we may assume that $\sigma(\mathcal{T}) \cap (0, +\infty) = \{0\} \cup \{\sigma_m\}_{m\in\mathbb{N}}$ with $\sigma_m \neq 0$ and $\sigma_m \searrow 0$ as $m \to \infty$. We also denote the corresponding eigenspace of σ_m by $\mathcal{N}_m^*$.

Proposition 2.3. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda_m = 1/\sigma_m$ for all $m \in \mathbb{N}$. Then λ_m is the unique solution to the following equation*

$$\sum_{j=1}^{k} \frac{\lambda}{\alpha_m + \mu_j + \lambda} = 1 \quad \text{for all } m \in \mathbb{N}.$$

Moreover, we also have

$$\mathcal{N}_m^* = \{\varphi \mathbf{e}_m \mid \varphi \in \mathcal{N}_m\}, \tag{2.4}$$

where $\mathbf{e}_m$ is the unique basic of the algebra equation $\mathcal{D}_m^* \mathbf{X} = \mathbf{0}$ with

$$\mathcal{D}_m^* = \begin{pmatrix} \alpha_m + \mu_1 & -\lambda_m & -\lambda_m & \ldots & -\lambda_m \\ -\lambda_m & \alpha_m + \mu_2 & -\lambda_m & \ldots & -\lambda_m \\ -\lambda_m & -\lambda_m & \alpha_m + \mu_3 & \ldots & -\lambda_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda_m & -\lambda_m & -\lambda_m & \ldots & \alpha_m + \mu_k \end{pmatrix}. \tag{2.5}$$

Proof. It is well known that $\mathcal{H}_i = \bigoplus_{m=1}^{\infty} \mathcal{N}_m$ for all $i = 1, \ldots, k$. It follows that

$$\mathcal{H} = \prod_{i=1}^{k} \mathcal{H}_i = \prod_{i=1}^{k} \bigg(\bigoplus_{m=1}^{\infty} \mathcal{N}_m \bigg) = \bigoplus_{m=1}^{\infty} (\mathcal{N}_m)^k. \tag{2.6}$$

Clearly, $\dim(\mathcal{N}_m) < \infty$ for all $m \in \mathbb{N}$. Without loss of generality and for the simplicity, we assume that $\dim(\mathcal{N}_m) = 1$ for all $m \in \mathbb{N}$ in what follows. Let $\mathbf{u} \in \mathcal{N}_m^* \setminus \{\mathbf{0}\}$, then by (2.6), we have $\mathbf{u} = \sum_{j=1}^{\infty} \{\widehat{\mathbf{a}_j, \mathbf{v}_j}\}$, where $\mathbf{a}_j \in \mathbb{R}^k$ are constant vectors and $\mathbf{v}_j \in \mathcal{N}_j^0$ with $\mathcal{N}_j^0 = (\mathcal{N}_j)^k$ for all $j \in \mathbb{N}$. That is

$$u_i = \sum_{j=1}^{\infty} a_i^j v_i^j \quad \text{for all } i = 1, \ldots, k.$$

Since we assume $\dim(\mathcal{N}_m) = 1$ for all $m \in \mathbb{N}$, we have $v_i^j = \varphi_j$ for all $i = 1, \ldots, k$. It follows from $\mathbf{u} \in \mathcal{N}_m^*$ that

$$\sum_{j=1}^{\infty} a_i^j (\alpha_j + \mu_i) \varphi_j = -\Delta u_i + \mu_i u_i = \lambda_m \sum_{l=1,\, l \neq i}^{k} u_l = \sum_{j=1}^{\infty} \bigg(\lambda_m \sum_{l=1,\, l \neq i}^{k} a_l^j \bigg) \varphi_j$$

for all $i = 1, \ldots, k$. Since φ_j are linear independent and orthorhombic in $L^2(\Omega)$, by multiplying the above equation with φ_j and integrating, we must have that

$$a_i^j (\alpha_j + \mu_i) = \lambda_m \sum_{l=1,\, l \neq i}^{k} a_l^j$$

for all $i = 1, \dots, k$ and $j \in \mathbb{N}$, which implies $\mathcal{D}_{m,j}\mathbf{a}_j = \mathbf{0}$ for all $j \in \mathbb{N}$. Here

$$\mathcal{D}_{m,j} = \begin{pmatrix} \alpha_j + \mu_1 & -\lambda_m & -\lambda_m & \dots & -\lambda_m \\ -\lambda_m & \alpha_j + \mu_2 & -\lambda_m & \dots & -\lambda_m \\ -\lambda_m & -\lambda_m & \alpha_j + \mu_3 & \dots & -\lambda_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda_m & -\lambda_m & -\lambda_m & \dots & \alpha_j + \mu_k \end{pmatrix}.$$

It follows that, for every $j \in \mathbb{N}$, either

(1) $\mathbf{a}_j = \mathbf{0}$, or
(2) $\det(\mathcal{D}_{m,j}) = 0$.

Since $\mathbf{u} \neq \{\mathbf{0}\}$, there exists $j_m \in \mathbb{N}$ such that $\det(\mathcal{D}_{m,j_m}) = 0$. By a direct calculation, we can see from $\alpha_j + \mu_i + \lambda_m > 0$, for all $j, m \in \mathbb{N}$ and $i = 1, \dots, k$, that

$$\det(\mathcal{D}_{m,j_m}) = \prod_{i=1}^{k} (\alpha_{j_m} + \mu_i + \lambda_m)\bigg(1 - \sum_{i=1}^{k} \frac{\lambda_m}{\alpha_{j_m} + \mu_i + \lambda_m}\bigg).$$

Note that $\mu_i > -\alpha_1$ for all $i = 1, \dots, k$ and $\lambda_m > 0$, thus we must have

$$\sum_{i=1}^{k} \frac{\lambda_m}{\alpha_{j_m} + \mu_i + \lambda_m} = 1. \tag{2.7}$$

Note that $\lambda_m \nearrow +\infty$ as $m \to \infty$ and $f_{j_m}(\lambda)$ is increasing for $\lambda > 0$, we also have that $\alpha_{j_{m+1}} > \alpha_{j_m}$ for all $m \in \mathbb{N}$. It follows that $j_m \geq m$ for all $m \in \mathbb{N}$. Since λ'_{j_m} is the unique solution to $f_{j_m}(\lambda) = 1$ in $(0, +\infty)$, we can see from (2.7) that $\lambda_m = \lambda'_{j_m}$. By $j_m \geq m$, we can see from the fact that λ'_m is increasing that $\lambda_m \geq \lambda'_m$ for all $m \in \mathbb{N}$. On the other hand, by $\lambda_m \nearrow +\infty$ as $m \to \infty$, we also have $\lambda_m \leq \lambda'_m$ for all $m \in \mathbb{N}$. Thus, we must have $\lambda_m = \lambda'_m$ for all $m \in \mathbb{N}$. That is, λ_m is the unique solution to the following equation

$$\sum_{j=1}^{k} \frac{\lambda}{\alpha_m + \mu_j + \lambda} = 1 \quad \text{for all } m \in \mathbb{N}. \tag{2.8}$$

It remains to show (2.4). Indeed, it suffices to show that 0 is the eigenvalue of $\mathcal{D}^*_m$ with degree 1 for all $m \in \mathbb{N}$, where $\mathcal{D}^*_m$ is given by (2.5). By a direct calculation, we can see from $\alpha_m + \mu_i + \lambda_m > 0$ f or all $m \in \mathbb{N}$ and $i = 1, \dots, k$ that

$$\det(\mathcal{D}^*_m) = \prod_{i=1}^{k} (\alpha_m + \mu_i + \lambda_m)\bigg(1 - \sum_{i=1}^{k} \frac{\lambda_m}{\alpha_m + \mu_i + \lambda_m}\bigg).$$

It follows from the fact that λ_m is the unique solution to (2.8) that $\det(\mathcal{D}^*_m) = 0$ for all $m \in \mathbb{N}$. Thus, 0 must be the eigenvalue of $\mathcal{D}^*_m$ for all $m \in \mathbb{N}$. Now, for every $m \in \mathbb{N}$, let us consider the following equation

$$\det(\mathcal{D}^*_m - \nu E) = 0,$$

where E is the identity matrix and $\nu \in \mathbb{R}$ is a constant. If $\nu \neq \alpha_m + \mu_i + \lambda_m$ for all $i = 1, \dots, k$, then by the fact that λ_m is the unique solution to (2.8), we can see that

$$
\begin{aligned}
\det(\mathcal{D}_m^* - \nu E) &= \prod_{i=1}^{k}(\alpha_m + \mu_i + \lambda_m - \nu)\left(1 - \sum_{i=1}^{k} \frac{\lambda_m}{\alpha_m + \mu_i + \lambda_m - \nu}\right) \\
&= \prod_{i=1}^{k}(\alpha_m + \mu_i + \lambda_m - \nu) \\
&\quad - \sum_{i=1}^{k} \prod_{j=1,\, j\neq i}^{k} \lambda_m(\alpha_m + \mu_j + \lambda_m - \nu) \\
&= \prod_{i=1}^{k}(\alpha_m + \mu_i + \lambda_m) - \sum_{i=1}^{k} \prod_{j=1,\, j\neq i}^{k} \lambda_m(\alpha_m + \mu_j + \lambda_m) \\
&\quad - \sum_{i=1}^{k}\Bigg(\prod_{j=1,\, j\neq i}^{k} (\alpha_m + \mu_j + \lambda_m) \\
&\quad - \sum_{j=1, j\neq i}^{k} \prod_{l=1,\, l\neq i,j}^{k} \lambda_m(\alpha_m + \mu_j + \lambda_m)\Bigg)\nu + \rho_m(\nu)\nu^2 \\
&= -\sum_{i=1}^{k}\Bigg(\prod_{j=1,\, j\neq i}^{k} (\alpha_m + \mu_j + \lambda_m) \\
&\quad - \sum_{j=1, j\neq i}^{k} \prod_{l=1,\, l\neq i,j}^{k} \lambda_m(\alpha_m + \mu_j + \lambda_m)\Bigg)\nu + \rho_m(\nu)\nu^2,
\end{aligned} \tag{2.9}
$$

where $\rho_m(\nu)$ is a polynomial of degree at most $k-2$. For every $i = 1, \dots, k$, by the fact that λ_m is the unique solution to (2.8) once more, we have

$$
\begin{aligned}
&\prod_{j=1, j\neq i}^{k} (\alpha_m + \mu_j + \lambda_m) - \sum_{j=1,\, j\neq i}^{k} \prod_{l=1,\, l\neq i,j}^{k} \lambda_m(\alpha_m + \mu_j + \lambda_m) \\
&\quad = \prod_{j=1,\, j\neq i}^{k} (\alpha_m + \mu_j + \lambda_m)\left(1 - \sum_{l=1, l\neq i}^{k} \frac{\lambda_m}{\alpha_m + \mu_l + \lambda_m}\right) \\
&\quad = \frac{\lambda_m}{\alpha_m + \mu_i + \lambda_m} \prod_{j=1,\, j\neq i}^{k} (\alpha_m + \mu_j + \lambda_m) > 0.
\end{aligned}
$$

Therefore, by (2.9), we can see that 0 is the eigenvalue of $\mathcal{D}_m^*$ with degree 1 for all $m \in \mathbb{N}$. $\square$

3. A variational characteristic of λ_m

Let $\mathcal{M}_0 = \{\mathbf{u} \in \mathcal{H} \mid \mathcal{G}(\mathbf{u}) = 1\}$, where

$$\mathcal{G}(\mathbf{u}) = \sum_{i,j=1,\, i<j}^{k} \int_\Omega u_j u_i \, dx. \tag{3.1}$$

Set

$$\lambda_1^* = \inf_{\mathbf{u} \in \mathcal{M}_0} \|\mathbf{u}\|^2.$$

Lemma 3.1. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. Then λ_1^* is attained and $\lambda_1^* = 2\lambda_1$.*

Proof. Clearly, $\lambda_1^* \geq 0$. Let $\{\mathbf{u}_n\} \subset \mathcal{M}_0$ be a minimizing sequence. Then it is easy to see that $\{\mathbf{u}_n\}$ is bounded in $\mathcal{H}$. It follows from the Sobolev embedding that $\mathbf{u}_n \to \mathbf{u}_0$ strongly in $\mathcal{L}^2(\Omega)$ as $n \to \infty$, which implies that $\mathcal{G}(\mathbf{u}_0) = 1$. Thus, by the weakly semi-continuity of the norm $\|\cdot\|$, we have $\|\mathbf{u}_0\|^2 = \lambda_1^*$. It remains to show that $\lambda_1^* = 2\lambda_1$. Indeed, by Proposition 2.3, we can see that

$$\varphi \mathbf{e}_1 = \lambda_1 \mathcal{T} \varphi \mathbf{e}_1,$$

where φ is the eigenfunction corresponding to α_1 and α_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. It follows from $\mathcal{T} = \mathcal{F} \circ \mathcal{I}$ that

$$\begin{cases} e_i^1(-\Delta\varphi + \mu_i\varphi) = \lambda_1 \displaystyle\sum_{j=1,\, j\neq i}^{k} e_j^1 \varphi & \text{in } \Omega, \\ \varphi = 0, \quad i = 1, \ldots, k & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Multiplying (3.2) with $e_i^1\varphi$ and integrating by parts, we have

$$\|\varphi\, \mathbf{e}_1\|^2 = 2\lambda_1 \mathcal{G}(\varphi\, \mathbf{e}_1).$$

Let $\widetilde{\mathbf{u}} = \varphi\, \mathbf{e}_1 / \sqrt{\mathcal{G}(\varphi\, \mathbf{e}_1)}$. Then, by the fact that $\mathcal{G}(\varphi\, \mathbf{e}_1)$ and $\|\varphi\, \mathbf{e}_1\|^2$ have the same order, we can see that $\widetilde{\mathbf{u}} \in \mathcal{M}_0$, which implies

$$2\lambda_1 = \frac{\|\varphi\, \mathbf{e}_1\|^2}{\mathcal{G}(\varphi\, \mathbf{e}_1)} = \|\widetilde{\mathbf{u}}\|^2 \geq \lambda_1^*.$$

On the other hand, it is easy to see that $\mathcal{M}_0$ is a C^1 manifold in $\mathcal{H}$. Thus, by the method of Lagrange multipliers, there exists $\delta \in \mathbb{R}$ such that

$$\mathbf{u}_0 - \delta\, \mathcal{G}'(\mathbf{u}_0) = \mathbf{0}, \tag{3.3}$$

Since $\mathcal{G}'(\mathbf{u}_0)\, \mathbf{u}_0 = 2\, \mathcal{G}(\mathbf{u}_0) = 2$, by multiplying (3.3) with $\mathbf{u}_0$, we can see that $\delta = \lambda_1^*/2$. It follows from $\mathcal{T} = \mathcal{F} \circ \mathcal{I}$ that

$$\mathbf{u}_0 = \frac{1}{2}\, \lambda_1^* \mathcal{T} \mathbf{u}_0.$$

Thus $2/\lambda_1^*$ is a eigenvalue of $\mathcal{T}$. Note that $\sigma_1 > \sigma_m$ for all $m \geq 2$, we must have $\lambda_1^* \geq 2\lambda_1$. Hence, we obtain that $\lambda_1^* = 2\lambda_1$. □

By Proposition 2.3 and (2.6), we have the following decomposition of $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{m=1}^{\infty} \mathcal{N}_m^*. \tag{3.4}$$

Let

$$\widetilde{\mathcal{N}}_m^* = \bigoplus_{k=1}^{m} \mathcal{N}_k^* \quad \text{and} \quad \big(\widetilde{\mathcal{N}}_m^*\big)^{\perp} = \bigoplus_{k=m+1}^{\infty} \mathcal{N}_k^*. \tag{3.5}$$

We also define

$$\lambda_2^* = \inf_{\mathbf{u}\in\mathcal{M}_1} \|\mathbf{u}\|^2, \quad \text{where } \mathcal{M}_1 = \Big\{\mathbf{u} \in \big(\widetilde{\mathcal{N}}_1^*\big)^{\perp} \mid \mathcal{G}(\mathbf{u}) = 1\Big\}$$

with $\mathcal{G}(\mathbf{u})$ given by (3.1).

Lemma 3.2. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. Then λ_2^* is attained and $\lambda_2^* = 2\lambda_2$.*

Proof. Since the proof is similar to that of Lemma 3.1, we only sketch it and point out the differences. Indeed, by a similar argument as used in the proof of Lemma 3.1, we can see that there exists $\mathbf{u}_1 \in \mathcal{M}_1$ such that $\|\mathbf{u}_1\|^2 = \lambda_2^*$. In what follows, we prove that $\lambda_2^* = 2\lambda_2$. In fact, by (3.4), we can see that $\mathcal{N}_2^* \subset (\widetilde{\mathcal{N}}_1^*)^{\perp}$. Now, also by a similar argument as used in the proof of Lemma 3.1, we can show that $2\lambda_2 \geq \lambda_2^*$. On the other hand, since it is easy to see that $\mathcal{M}_1$ is a C^1 manifold in the space $(\widetilde{\mathcal{N}}_1^*)^{\perp}$, also by a similar argument as used in the proof of Lemma 3.1, we have

$$\mathbf{u}_1 = \frac{1}{2}\,\lambda_2^* \mathcal{T}\mathbf{u}_1 \quad \text{in } \big(\widetilde{\mathcal{N}}_1^*\big)^{\perp},$$

which, together with Proposition 2.3, implies

$$\mathbf{u}_1 = \frac{1}{2}\,\lambda_2^* \mathcal{T}\mathbf{u}_1.$$

Thus $2/\lambda_2^*$ is a eigenvalue of $\mathcal{T}$. Similar to that of Lemma 3.1, we must have $\lambda_2^* \geq 2\lambda_2$. Hence, we obtain that $\lambda_2^* = 2\lambda_2$. □

Now, by iteration, we define λ_m^* ($m \geq 3$) as

$$\lambda_m^* = \inf_{\mathbf{u}\in\mathcal{M}_{m-1}} \|\mathbf{u}\|^2, \quad \text{where } \mathcal{M}_{m-1} = \Big\{\mathbf{u} \in \big(\widetilde{\mathcal{N}}_{m-1}^*\big)^{\perp} \mid \mathcal{G}(\mathbf{u}) = 1\Big\}.$$

Then by a similar argument as used for Lemma 3.2, we have the following lemma.

Lemma 3.3. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. Then λ_m^* is attained and $\lambda_m^* = 2\lambda_m$ for all $m \geq 3$.*

Combining Lemmas 3.1–3.3, we actually have the following result.

Proposition 3.4. *Let $N \geq 1$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. Then λ_m^* is attained and $\lambda_m^* = 2\lambda_m$ for all $m \geq 1$.*

Proof of Theorem 1.4 follows immediately from Propositions 2.3 and 3.4.

4. The nonexistence result

Define

$$f_0(\lambda)=\sum_{j=1}^{k}\frac{\lambda}{\mu_j+\lambda}. \tag{4.1}$$

If $\mu_i>0$ for all $i=1,\dots,k$, then by a similar argument as used for $f_m(\lambda)$, which is given by (2.3), we can see that $f_0(\lambda)$ is increasing for $\lambda>0$ with $\lim_{\lambda\to 0^+} f_0(\lambda)=0$ and $\lim_{\lambda\to+\infty} f_0(\lambda)=k$. Thus, there exists unique $\lambda_1^*>0$ such that $f_0(\lambda_1^*)=1$. Moreover, since $\alpha_1>0$, it is also easy to see from the fact that $f_1(\lambda)$ is increasing for $\lambda>0$ that $\lambda_1^*<\lambda_1$.

LEMMA 4.1. *Let $N\geq 1$ and $\mu_i>0$ for all $i=1,\dots,k$. Then*

$$\int_\Omega\bigg(\sum_{j=1}^{k}\mu_j|u_j|^2-2\lambda\sum_{i,l=1,\,i<l}^{k}u_iu_l\bigg)\,dx\geq 0 \tag{4.2}$$

for all $\mathbf{u}\in\mathcal{H}$ if and only if $0<\lambda\leq\lambda_1^*$.

PROOF. Let

$$\mathcal{U}=\begin{pmatrix}\mu_1 & -\lambda & -\lambda & \dots & -\lambda\\ -\lambda & \mu_2 & -\lambda & \dots & -\lambda\\ -\lambda & -\lambda & \mu_3 & \dots & -\lambda\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\lambda & -\lambda & -\lambda & \dots & \mu_k\end{pmatrix}.$$

For every $\gamma\in\mathbb{R}$, by a direct calculation, we have

$$\det(\mathcal{U}-\gamma E)=\begin{cases}\displaystyle\prod_{j=1}^{k}(\mu_j+\lambda-\gamma)\bigg(1-\sum_{j=1}^{k}\frac{\lambda}{\mu_j+\lambda-\gamma}\bigg), & \gamma\neq\mu_j+\lambda,\ \text{for all } j,\\ \displaystyle\lambda^{2-i}\prod_{j=1,\,j\neq i}^{k}(\mu_j+\lambda-\gamma), & \gamma=\mu_i+\lambda,\ \text{for some } i.\end{cases}$$

Let

$$g(\gamma)=1-\sum_{j=1}^{k}\frac{\lambda}{\mu_j+\lambda-\gamma}.$$

Then $g(0)=1-f_0(\lambda)$. It follows that $g(0)\geq 0$ if and only if $0<\lambda\leq\lambda_1^*$. Note that $g(\gamma)$ is decreasing for $\gamma<0$ with $\lim_{\gamma\to-\infty}g(\gamma)=1$, thus there exists a unique $\gamma_*<0$ such that $g(\gamma_*)=0$ if and only if $\lambda>\lambda_1^*$. It follows that $\mathcal{U}$ has a negative eigenvalue if and only if $\lambda>\lambda_1^*$. On the other hand, since $g(\gamma)$ is decreasing in $(\mu_j+\lambda,\mu_{j+1}+\lambda)$ with $\lim_{\gamma\to(\mu_j+\lambda)^-}g(\gamma)=-\infty$ and $\lim_{\gamma\to(\mu_j+\lambda)^+}g(\gamma)=+\infty$, $k-1$ eigenvalues of $\mathcal{U}$ must lie in $[\lambda+\mu_1,\lambda+\mu_k]$. Hence, the eigenvalues of $\mathcal{U}$ have the following two properties:

(1) all k eigenvalues lies in $[0,\lambda+\mu_k]$ if $0<\lambda\leq\lambda_1^*$,

(2) there is a unique negative eigenvalue and other $k-1$ eigenvalues lie in $[\lambda+\mu_1,\lambda+\mu_k]$ if $\lambda>\lambda_1^*$.

Therefore, $\mathcal{U}$ is nonnegative definite if and only if $0<\lambda\le\lambda_1^*$, which implies (4.2) holds if and only if $0<\lambda\le\lambda_1^*$. □

We close this section with a proof.

Proof of Theorem 1.6. Let $\mathbf{u}\in\mathcal{H}$ be a solution to system (1.1), then by the classical regularity theories, $u_i\in C^2(\Omega)$ for all $i=1,\ldots,k$. Now, by the Pohozaev identity, we can see that

$$\frac{N-2}{2N}\sum_{j=1}^k\int_\Omega|\nabla u_j|^2\,dx+\frac{1}{2N}\sum_{j=1}^k\int_{\partial\Omega}(x,n)|\nabla u_j|^2\,ds$$
$$=-\frac{1}{2}\int_\Omega\bigg(\sum_{j=1}^k\mu_j|u_j|^2-2\lambda\sum_{i,l=1,\,i<l}^k u_iu_l\bigg)\,dx+\frac{N-2}{2N}\sum_{j=1}^k\int_\Omega|u_j|^{2^*}\,dx,$$

where n is the unit outer normal vector of Ω. It follows from $\mathbf{u}\in\mathcal{H}$ being a solution to system (1.1) that

$$\frac{1}{2N}\sum_{j=1}^k\int_{\partial\Omega}(x,n)|\nabla u_j|^2\,ds=-\frac{1}{N}\int_\Omega\bigg(\sum_{j=1}^k\mu_j|u_j|^2-2\lambda\sum_{i,l=1,\,i\neq l}^k u_iu_l\bigg)\,dx.$$

Since Ω is star-shaped, we must have from Lemma 4.1 that $\mathbf{u}=\mathbf{0}$ for $0<\lambda\le\lambda_1^*$. □

5. Existence of ground states

5.1. The definite case $0<\lambda<\lambda_1$.

Let

$$\mathcal{P}_0=\big\{\mathbf{u}\in\mathcal{H}\setminus\{\mathbf{0}\}\mid\mathcal{E}'_\lambda(\mathbf{u})\mathbf{u}=0\big\}.\tag{5.1}$$

Then $\mathcal{P}_0$ is the well-known Nehari manifold of $\mathcal{E}_\lambda(\mathbf{u})$, where $\mathcal{E}_\lambda(\mathbf{u})$ is given by (1.7). Since

$$\mathcal{E}'_\lambda(\mathbf{u})\mathbf{u}=\|\mathbf{u}\|^2-2\lambda\mathcal{G}(\mathbf{u})-\sum_{i=1}^k\int_\Omega|u_i|^{2^*}\,dx,$$

it is easy to see that $\mathcal{P}_0$ is a C^1 manifold in $\mathcal{H}$, where $\mathcal{G}(\mathbf{u})$ is given by (3.1). Let

$$c_\lambda=\inf_{\mathbf{u}\in\mathcal{P}_0}\mathcal{E}_\lambda(\mathbf{u}).\tag{5.2}$$

Lemma 5.1. *Let $N\ge4$, $\mu_i>-\alpha_1$ for all $i=1,\ldots,k$ and $0<\lambda<\lambda_1$. If we also have $\min\{\mu_1,\ldots,\mu_k\}<0$, then $0<c_\lambda<\mathcal{S}^{N/2}/N$, where $\mathcal{S}$ is the best Sobolev embedding constant from $H^1(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$.*

Proof. Let $\mathbf{u}\in\mathcal{P}_0$. Then, by the Sobolev embedding, we have from $p>2$ that

$$\|\mathbf{u}\|^2-2\lambda\mathcal{G}(\mathbf{u})=\sum_{i=1}^k\int_\Omega|u_i|^{2^*}\,dx\le C\sum_{i=1}^k\|u_i\|_i^{2^*}\le C'\|\mathbf{u}\|^{2^*}.\tag{5.3}$$

It follows from Proposition 3.4 that

$$\left(1-\frac{\lambda}{\lambda_1}\right)\leq C'\|\mathbf{u}\|^{2^*-2}, \tag{5.4}$$

which, together with $0<\lambda<\lambda_1$ and Proposition 3.4 once more, implies

$$\mathcal{E}_\lambda(\mathbf{u})=\mathcal{E}_\lambda(\mathbf{u})-\frac{1}{2^*}\,\mathcal{E}'_\lambda(\mathbf{u})\mathbf{u}=\frac{1}{N}\big(\|\mathbf{u}\|^2-2\lambda\mathcal{G}(\mathbf{u})\big)\geq\frac{1}{N}\left(1-\frac{\lambda}{\lambda_1}\right)\|\mathbf{u}\|^2. \tag{5.5}$$

Since $\mathbf{u}\in\mathcal{P}_0$ is arbitrary, we must have from (5.4) that

$$c_\lambda\geq\frac{1}{N}\left(1-\frac{\lambda}{\lambda_1}\right)^2C'.$$

It remains to show that $c_\lambda<\mathcal{S}^{N/2}/N$. Recall that m_ν can also be attained by some u_ν for $-\alpha_1<\nu<0$, where m_ν is given by (1.13). Now, without loss of generality, we assume $-\alpha_1<\mu_1<0$ and set $\mathbf{U}_{\mu_1}=(u_{\mu_1},0,\dots,0)$. Then it is easy to see that $\mathbf{U}_{\mu_1}\in\mathcal{P}_0$. It follows that

$$c_\lambda\leq\mathcal{E}_\lambda(\mathbf{U}_{\mu_1})=\mathcal{J}_{\mu_1}(u_{\mu_1})=m_{\mu_1}<\frac{1}{N}\,\mathcal{S}^{N/2},$$

which completes the proof. □

Now, we can obtain the following.

PROPOSITION 5.2. *Let $N\geq4$, $\mu_i>-\alpha_1$ for all $i=1,\dots,k$ and $0<\lambda<\lambda_1$. If we also have $\min\{\mu_1,\dots,\mu_k\}<0$, then there exists $\mathbf{u}_\lambda\in\mathcal{P}_0$ such that $\mathbf{u}_\lambda$ is a positive ground state solution to system* (1.1).

PROOF. Let

$$\mathcal{A}_\lambda(\mathbf{u})=\mathcal{E}'_\lambda(\mathbf{u})\mathbf{u}. \tag{5.6}$$

Then it is easy to see that $\mathcal{A}_\lambda(\mathbf{u})$ is C^1 in $\mathcal{H}$. Moreover, for every $\mathbf{u}\in\mathcal{P}_0$, we have from Theorem 1.4 and (5.4) that

$$\begin{aligned}\mathcal{A}'_\lambda(\mathbf{u})\mathbf{u}&=2\big(\|\mathbf{u}\|^2-2\lambda\mathcal{G}(\mathbf{u})\big)-2^*\sum_{i=1}^k\int_\Omega|u_i|^{2^*}\,dx\\&=(2-2^*)\big(\|\mathbf{u}_\lambda\|^2-2\lambda\mathcal{G}(\mathbf{u}_\lambda)\big)\leq-C.\end{aligned} \tag{5.7}$$

Thus, by applying Ekeland's variational principle and the implicit function theorem in a standard way, we can obtain a (PS) sequence of $\mathcal{E}_\lambda(\mathbf{u})$ in $\mathcal{P}_0$, denoted by $\{\mathbf{u}_n\}$, at the energy level c_λ. By (5.5), we can see that $\{\mathbf{u}_n\}$ is bounded in $\mathcal{H}$. Without loss of generality and by the Sobolev embedding, we may assume that $\mathbf{u}_n\rightharpoonup\mathbf{u}_0$ weakly in $\mathcal{H}$ and $\mathbf{u}_n\to\mathbf{u}_0$ strongly in $\mathcal{L}^q(\Omega)$ for all $1\leq q<2^*$ as $n\to\infty$. Clearly, $\mathcal{E}'_\lambda(\mathbf{u}_0)=0$. If $\mathbf{u}_0=\mathbf{0}$, then we have

$$\int_\Omega|\nabla u_i^n|^2\,dx=\int_\Omega|u_i^n|^{2^*}\,dx+o(1) \tag{5.8}$$

for all $i = 1, \ldots, k$. Note that by (5.4), we have

$$\sum_{i=1}^{k} \int_{\Omega} |\nabla u_i^n|^2\, dx \geq C + o(1).$$

Thus, there is at least one $i_0 \in \{1, \ldots, k\}$ such that

$$\int_{\Omega} |\nabla u_{i_0}^n|^2\, dx \geq C' + o(1).$$

It follows from (5.8) and Sobolev's inequality that

$$\int_{\Omega} |\nabla u_{i_0}^n|^2\, dx \geq \mathcal{S}^{N/2} + o(1),$$

which together with (5.8) once more, implies

$$\begin{aligned} c_\lambda + o(1) = \mathcal{E}_\lambda(\mathbf{u}_n) &= \mathcal{E}_\lambda(\mathbf{u}_n) - \frac{1}{2}\,\mathcal{E}_\lambda'(\mathbf{u}_n)\mathbf{u}_n \\ &= \frac{1}{N}\sum_{j=1}^{k} \int_{\Omega} |u_j^n|^{2^*}\, dx \geq \frac{1}{N}\,\mathcal{S}^{N/2} + o(1). \end{aligned} \tag{5.9}$$

It contradicts to Lemma 5.1. Thus, we must have that $\mathbf{u}_0 \in \mathcal{H} \setminus \{\mathbf{0}\}$, which implies $\mathbf{u}_0 \in \mathcal{P}_0$. Hence

$$c_\lambda + o(1) = \mathcal{E}_\lambda(\mathbf{u}_n) \geq \mathcal{E}_\lambda(\mathbf{u}_0) + o(1) \geq c_\lambda + o(1).$$

Therefore, c_λ is attained by $\mathbf{u}_0$. Let $u_i^* = |u_i^0|$ for all $i = 1, \ldots, k$. Then it is easy to see that $\mathcal{G}(\mathbf{u}_0) \leq \mathcal{G}(\mathbf{u}_*)$, where $\mathbf{u}_* = (u_1^*, \ldots, u_k^*)$. Since $\lambda > 0$, we can see from a standard argument that there exists $t_\lambda \in (0, 1]$ such that $t_\lambda \mathbf{u}_* \in \mathcal{P}_0$. A standard argument also implies $\mathcal{E}_\lambda(\mathbf{u}_0) \geq \mathcal{E}_\lambda(t\mathbf{u}_0)$ for all $t \geq 0$. Thus, by $\lambda > 0$, we have

$$c_\lambda = \mathcal{E}_\lambda(\mathbf{u}_0) \geq \mathcal{E}_\lambda(t_\lambda \mathbf{u}_0) \geq \mathcal{E}_\lambda(t_\lambda \mathbf{u}_*) \geq c_\lambda.$$

Let $\mathbf{u}_\lambda = t_\lambda \mathbf{u}_*$. Then c_λ is attained by $\mathbf{u}_\lambda$ with $u_i^\lambda \geq 0$ for all $i = 1, \ldots, k$. It remains to show that $\mathbf{u}_\lambda$ is a nontrivial solution to system (1.1). Indeed, since $\mathcal{P}_0$ is C^1 in $\mathcal{H}$, by the method of Lagrange multipliers, there exists $\delta \in \mathbb{R}$ such that

$$\mathcal{E}_\lambda'(\mathbf{u}_\lambda) - \delta \mathcal{A}_\lambda'(\mathbf{u}_\lambda) = \mathbf{0}. \tag{5.10}$$

By multiplying (5.10) with $\mathbf{u}_\lambda$, we can see from (5.7) that $\delta = 0$ and $\mathcal{E}_\lambda'(\mathbf{u}_\lambda) = \mathbf{0}$, which implies $\mathbf{u}_\lambda$ is a solution to system (1.1) with $u_i^\lambda \geq 0$ for all $i = 1, \ldots, k$. By the maximum principle, we have that either $u_i > 0$ or $u_i = 0$ for all $i = 1, \ldots, k$. Now, suppose $\mathbf{u}_\lambda$ is not a nontrivial solution, then there exists $j \in \{1, \ldots, k\}$ such that $u_j^\lambda = 0$. Since $\mathbf{u}_\lambda \neq \mathbf{0}$, without loss of generality, we assume that $u_i^\lambda > 0$ for $i = 1, \ldots, i_0$ and $u_i^\lambda = 0$ for $i = i_0 + 1, \ldots, k$ with some $i_0 \in \{1, \ldots, k-1\}$.

Since $\mathbf{u}_\lambda$ is a solution to system (1.1), we have from $u_i^\lambda = 0$ for $i = i_0 + 1, \ldots, k$ that $\mathbf{u}_\lambda$ is also a solution to the following system

$$\begin{cases} -\Delta u_i + \mu_i u_i = |u_i|^{2^*-2} u_i + \lambda \displaystyle\sum_{j=1,\, j\neq i}^{i_0} u_j & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{i_0} u_i = 0 & \text{in } \Omega, \\ u_i = 0, \quad i = 1, \ldots, i_0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

It is impossible since $u_i^\lambda > 0$ for $i = 1, \ldots, i_0$. $\square$

5.2. The indefinite case $\lambda \geq \lambda_1$. Without loss of generality, we may assume $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$ by Theorem 1.4.

PROPOSITION 5.3. *Let $N \geq 3$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. Suppose $\mathbf{u}$ is a nonzero solution to system* (1.1), *then $\mathbf{u}$ must be a sign-changing solution to system* (1.1).

PROOF. Suppose the contrary, then $\mathbf{u}$ is either nonnegative or nonpositive. Without loss of generality, we assume that $\mathbf{u}$ is nonnegative. Now, multiplying system (1.1) with $\mathbf{v}_1$ and integrating by parts, where $\mathbf{v}_1 \in \mathcal{N}_1^* = \{\varphi\, \mathbf{e}_1 \mid \varphi \in \mathcal{N}_1\}$ is the corresponding eigenfunction of λ_1 given by Theorem 1.4, we have from $\lambda \geq \lambda_m$ that

$$\begin{aligned} \lambda_1 \mathcal{G}'(\mathbf{u})\mathbf{v}_1 &= \lambda_1 \mathcal{G}'(\mathbf{v}_1)\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \\ &= \sum_{j=1}^{k} \int_\Omega |u_j|^{2^*-2} u_j v_j^1 \, dx + \lambda \mathcal{G}'(\mathbf{u})\mathbf{v}_1 > \lambda_1 \mathcal{G}'(\mathbf{u})\mathbf{v}_1, \end{aligned}$$

which is impossible. Thus, $\mathbf{u}$ is a sign-changing solution to system (1.1). $\square$

For simplicity, we assume that $\dim(\mathcal{N}_m) = 1$ for all $m \in \mathbb{N}$ in what follows. Now, by Theorem 1.4 once more, we also have that $\dim(\mathcal{N}_m^*) = 1$ for all $m \in \mathbb{N}$. Let

$$\mathcal{F}_\lambda(\mathbf{u}) = (\mathcal{A}_\lambda(\mathbf{u}), \mathcal{E}_\lambda'(\mathbf{u})\mathbf{w}_1, \ldots, \mathcal{E}_\lambda'(\mathbf{u})\mathbf{w}_m),$$

where $\mathbf{w}_i \in \mathcal{N}_i^*$ for all $i = 1, \ldots, m$ and $\mathcal{A}_\lambda(\mathbf{u})$ is given by (5.6). Then it is easy to see that $\mathcal{F}_\lambda(\mathbf{u})$ is C^1 in $\mathcal{H}$.

LEMMA 5.4. *Let $N \geq 3$, $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. Then $\mathcal{P}_m$ is a C^1 manifold in $\mathcal{H}$ with codimension $m+1$, where*

$$\mathcal{P}_m = \big\{\mathbf{u} \in \mathcal{H} \setminus \widetilde{\mathcal{N}}_m^* \mid \mathcal{F}_\lambda(\mathbf{u}) = \mathbf{0}\big\} \tag{5.12}$$

and $\widetilde{\mathcal{N}}_m^*$ is given by (3.5).

Proof. Since $\mathcal{F}_\lambda(\mathbf{u})$ is C^1 in $\mathcal{H}$, $\mathcal{P}_m$ is a C^1 manifold in $\mathcal{H}$. It remains to show that the codimension of $\mathcal{P}_m$ is $m+1$. For every $\mathbf{u} \in \mathcal{P}_m$, we set $\mathbf{z} = \sum_{i=1}^m a_i\mathbf{w}_i + s\mathbf{u}$. Then

$$\mathcal{F}'_\lambda(\mathbf{u})\mathbf{z} = \big(\mathcal{A}'_\lambda(\mathbf{u})\mathbf{z}, \mathcal{F}'_{\lambda,1}(\mathbf{u})\mathbf{z}, \ldots, \mathcal{F}'_{\lambda,m}(\mathbf{u})\mathbf{z}\big) \in \mathbb{R}^{m+1},$$

where $\mathcal{F}_{\lambda,i}(\mathbf{u}) = \mathcal{E}'_\lambda(\mathbf{u})\mathbf{w}_i$ for all $i = 1, \ldots, m$. By a direct calculation, we have

$$\mathcal{A}'_\lambda(\mathbf{u})\mathbf{z} = 2(\langle \mathbf{u}, \mathbf{z}\rangle - \lambda\mathcal{G}'(\mathbf{u})\mathbf{z}) - 2^* \sum_{j=1}^k \int_\Omega |u_j|^{2^*-2} u_j z_j \, dx, \tag{5.13}$$

where $\mathcal{G}(\mathbf{u})$ is given by (3.1). On the other hand, for every $i = 1, \ldots, m$, we have

$$\mathcal{F}'_{\lambda,i}(\mathbf{u})\mathbf{z} = \langle \mathbf{z}, \mathbf{w}_i\rangle - \lambda\mathcal{G}'(\mathbf{z})\mathbf{w}_i - (2^*-1)\sum_{j=1}^k \int_\Omega |u_j|^{2^*-2} z_j w_j^i \, dx. \tag{5.14}$$

Set $\mathbf{t} = (s, a_1, \ldots, a_m)$. Then by (5.13) and (5.14), we can see from $\mathbf{u} \in \mathcal{P}_m$ that

$$\begin{aligned}(\mathcal{F}'_\lambda(\mathbf{u})\mathbf{z}) \cdot \mathbf{t} = &\sum_{i=1}^m \big(\|a_i\mathbf{w}_i\|^2 - 2\lambda\mathcal{G}(a_i\mathbf{w}_i)\big) \\ &- \sum_{j=1}^k \int_\Omega |u_j|^{2^*-2}\bigg(\sum_{i=1}^m a_i w_j^i\bigg)^2 dx \\ &- (2^*-2)\sum_{j=1}^k \int_\Omega |u_j|^{2^*-2}(su_j + \sum_{i=1}^m a_i w_j^i)^2 \, dx.\end{aligned} \tag{5.15}$$

Here $\mathbf{t} \cdot \mathbf{s}$ is the usual inner product in $\mathbb{R}^{m+1}$. Since $\mathbf{u} \in \mathcal{H} \setminus \widetilde{\mathcal{N}}^*_m$ and $\sum_{i=1}^m a_i w_j^i \in \widetilde{\mathcal{N}}^*_m$, we can see from Theorem 1.4 that

$$\int_\Omega |u_j|^{2^*-2}\bigg(su_j + \sum_{i=1}^m a_i w_j^i\bigg)^2 dx > 0 \quad \text{for all } j = 1, \ldots, k.$$

Thus, by (5.15), we have from Theorem 1.4 once more that $(\mathcal{F}'_\lambda(\mathbf{u})\mathbf{z}) \cdot \mathbf{t} < 0$ for all $\mathbf{t} \neq \mathbf{0}$. Thus, for every $\mathbf{u} \in \mathcal{P}_m$, $\mathcal{F}'_\lambda(\mathbf{u})$ is onto. It follows that

$$\mathcal{H} = \widetilde{\mathcal{N}}^*_m \oplus \mathbb{R}\mathbf{u} \oplus T_{\mathbf{u}}\mathcal{P}_m$$

for all $\mathbf{u} \in \mathcal{P}_m$, where $T_{\mathbf{u}}\mathcal{P}_m$ is the tangent space of $\mathcal{P}_m$ at $\mathbf{u}$. Therefore, the codimension of $\mathcal{P}_m$ is $m+1$. □

We also need the following two technique lemmas.

Lemma 5.5. *Let $N \geq 1$, $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. Then, for every $\mathbf{u} \in \mathcal{P}_m$, we have*

$$\mathcal{E}_\lambda(\mathbf{u}) \geq \mathcal{E}_\lambda\bigg(t\mathbf{u} + \sum_{i=1}^m a_i\mathbf{w}_i\bigg) \tag{5.16}$$

for all $t \geq 0$ and $a_i \in \mathbb{R}$, where $\mathcal{P}_m$ is given by (5.12) *and $\mathbf{w}_i \in \mathcal{N}_i^*$. Moreover,* (5.16) *is equal if and only if $t = 1$ and $a_i = 0$ for all $i = 1, \ldots, m$.*

PROOF. Let $\mathbf{z} = \sum_{i=1}^{m} a_i \mathbf{w}_i$. Then by $\mathbf{u} \in \mathcal{P}_m$ and Proposition 3.4, we have

$$
\begin{aligned}
(5.17)\quad \mathcal{E}_\lambda(t\mathbf{u}+\mathbf{z}) - \mathcal{E}_\lambda(\mathbf{u}) &= \frac{(t^2-1)}{2}\|\mathbf{u}\|^2 + t\langle \mathbf{u}, \mathbf{z}\rangle \\
&\quad + \frac{1}{2}\|\mathbf{z}\|^2 - \lambda\bigg((t^2-1)\mathcal{G}(\mathbf{u}) - t\sum_{j,l=1,\, l<j}^{k} \int_\Omega u_j z_l\,dx\bigg) \\
&\quad - \frac{1}{2^*}\sum_{j=1}^{k}\int_\Omega \big(|tu_j+z_j|^{2^*} - |u_j|^{2^*}\big)\,dx \\
&\leq \sum_{j=1}^{k}\int_\Omega \frac{t^2-1}{2}|u_j|^{2^*} - \frac{1}{2^*}\big(|tu_j+z_j|^{2^*} - |u_j|^{2^*} - 2^*|u_j|^{2^*-2}tu_jz_j\big)\,dx,
\end{aligned}
$$

where $\mathcal{G}(\mathbf{u})$ is given by (3.1). For every $j = 1, \ldots, k$, we consider the following function

$$
f_j(t) = \frac{t^2-1}{2}|u_j|^{2^*} - \frac{1}{2^*}\big(|tu_j+z_j|^{2^*} - |u_j|^{2^*} - 2^*|u_j|^{2^*-2}tu_jz_j\big).
$$

If there exists $t_0 \geq 0$ such that $f_j'(t_0) = 0$, then we must have

$$
\big(|u_j|^{2^*-2} - |t_0u_j+z_j|^{2^*-2}\big)(t_0u_j+z_j)u_j = 0. \tag{5.18}
$$

Since $\mathbf{u} \in \mathcal{H} \setminus \widetilde{\mathcal{N}}_m^*$ and $\mathbf{z} \in \widetilde{\mathcal{N}}_m^*$, we can see from (5.18) that $u_j = 0$, where $\widetilde{\mathcal{N}}_m^*$ is given by (3.5). It follows that $f_j(t) = -|z_j|^{2^*}/2^* \leq 0$. Note that $f_j(t) \to -\infty$ as $t \to +\infty$ if $u_j \neq 0$ whereas $f_j(t) \equiv -|z_j|^{2^*}/2^* \leq 0$ if $u_j = 0$, thus we must have that $f_j(t) \leq 0$ for all $t \geq 0$, which, together with (5.17), implies (5.16). Moreover, since $\mathbf{u} \in \mathcal{H} \setminus \widetilde{\mathcal{N}}_m^*$ and $\mathbf{z} \in \widetilde{\mathcal{N}}_m^*$, it is also easy to see from (5.17) that (5.16) is equal if and only if $t = 1$ and $a_i = 0$ for all $i = 1, \ldots, m$. □

LEMMA 5.6. *Let $N \geq 3$, $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. Then, for every $\mathbf{u} \in \mathcal{H} \setminus \widetilde{\mathcal{N}}_m^*$, there exist unique $t_{\mathbf{u}} > 0$ and $\mathbf{v}_{\mathbf{u}} \in \widetilde{\mathcal{N}}_m^*$ such that $t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}} \in \mathcal{P}_m$, where $\widetilde{\mathcal{N}}_m^*$ is given by* (3.5) *and $\mathcal{P}_m$ is given by* (5.12).

PROOF. Let $\mathbf{u} \in \mathcal{H} \setminus \widetilde{\mathcal{N}}_m^*$ and consider the following function

$$
f(\mathbf{t}) = \mathcal{E}_\lambda\bigg(t\mathbf{u} + \sum_{i=1}^{m} a_i\mathbf{w}_i\bigg),
$$

where $\mathbf{t}=(t,a_1,\dots,a_m)\in\mathbb{R}^+\times\mathbb{R}^m$. By Theorem 1.4, from $\lambda_m\le\lambda<\lambda_{m+1}$ we have that

$$f(\mathbf{t})\le|\mathbf{t}|^2\big(\|\mathbf{u}^\perp\|^2-2\lambda\mathcal{G}(\mathbf{u}^\perp)\big)\\ -|\mathbf{t}|^{2^*}\sum_{j=1}^k\frac{1}{2^*}\int_\Omega\bigg|\frac{1}{|\mathbf{t}|}\bigg(tu_j+\sum_{i=1}^m a_iw_j^i|\mathbf{t}|\bigg)\bigg|^{2^*}dx, \tag{5.19}$$

where $\mathbf{u}=\check{\mathbf{u}}+\mathbf{u}^\perp$ with $\check{\mathbf{u}}\in\widetilde{\mathcal{N}}_m^*$ and $\mathbf{u}^\perp\in\big(\widetilde{\mathcal{N}}_m^*\big)^\perp$. Since $\mathbf{u}\in\mathcal{H}\setminus\widetilde{\mathcal{N}}_m^*$ and $a_i\mathbf{w}_i\in\mathcal{N}_i^*$ for all $i=1,\dots,m$, by the Lebesgue dominated convergence theorem and Theorem 1.4 once more, there exists $R>0$ such that

$$\inf_{|\mathbf{t}|\ge R}\int_\Omega\bigg|\frac{1}{|\mathbf{t}|}\bigg(tu_j+\sum_{i=1}^m a_iw_j^i\bigg)\bigg|^{2^*}dx\ge C, \tag{5.20}$$

which together with (5.19) and $2^*>2$, implies $f(\mathbf{t})\to-\infty$ as $|\mathbf{t}|\to+\infty$. On the other hand, since $\lambda_m\le\lambda<\lambda_{m+1}$, we have from Theorem 1.4 and a standard argument that $f(\mathbf{t})\le 0$ if $t=0$ and $f(\mathbf{t})>0$ if $a_i=0$ for all $i=1,\dots,m$ and $t>0$ small enough. Thus, there exists $t_{\mathbf{u}}>0$ and $a_{i,\mathbf{u}}\in\mathbb{R}$ for all $i=1,\dots,m$ such that $f(\mathbf{t_u})=\max_{\mathbf{t}\in\mathbb{R}^+\times\mathbb{R}^m}f(\mathbf{t})$, where $\mathbf{t_u}=(t_{\mathbf{u}},a_{1,\mathbf{u}},\dots,a_{m,\mathbf{u}})$. It follows that

$$t_{\mathbf{u}}\mathbf{u}+\sum_{i=1}^m a_{i,\mathbf{u}}\mathbf{w}_i\in\mathcal{P}_m.$$

Thanks to Lemma 5.5, $\mathbf{t_u}$ must be unique. □

Set

$$\widetilde{c}_\lambda=\inf_{\mathbf{u}\in\mathcal{P}_m}\mathcal{E}_\lambda(\mathbf{u}), \tag{5.21}$$

where $\mathcal{P}_m$ is given by (5.12).

LEMMA 5.7. *Let $N\ge4$, $\mu_i>-\alpha_1$ for all $i=1,\dots,k$ and $\lambda_m\le\lambda<\lambda_{m+1}$ for some $m\in\mathbb{N}$. If $\min\{\mu_1,\dots,\mu_k\}<0$ and either*

(a) *$N=4$, $\lambda_m<\lambda<\lambda_{m+1}$, or*
(b) *$N\ge5$,*

then $0<\widetilde{c}_\lambda<\mathcal{S}^{N/2}/N$.

PROOF. Let $\mathbf{u}\in\mathcal{P}_m$. Then $\mathbf{u}=\check{\mathbf{u}}+\mathbf{u}^\perp$, where $\check{\mathbf{u}}\in\widetilde{\mathcal{N}}_m^*$ and $\mathbf{u}^\perp\in\big(\widetilde{\mathcal{N}}_m^*\big)^\perp$. Now, by Lemma 5.5, we have $\mathcal{E}_\lambda(\mathbf{u})\ge\mathcal{E}_\lambda(t\mathbf{u}^\perp)$ for $t>0$ small enough. Since $\lambda_m\le\lambda<\lambda_{m+1}$, a standard argument implies $\mathcal{E}_\lambda(\mathbf{u})\ge C$. Note that $\mathbf{u}\in\mathcal{P}_m$ is arbitrary, we must have $\widetilde{c}_\lambda>0$. We next show that $\widetilde{c}_\lambda<\mathcal{S}^{N/2}/N$. Let

$$V_\varepsilon(x)=\bigg(\frac{N(N-2)\varepsilon^2}{(\varepsilon^2+|x|^2)^2}\bigg)^{(N-2)/4}. \tag{5.22}$$

Then it is well known that V_ε is the unique positive solution to the following equation up to a translation

$$-\Delta u = |u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N), \tag{5.23}$$

where $D^{1,2}(\mathbb{R}^N) = \big\{u \in L^{2^*}(\mathbb{R}^N) \mid |\nabla u| \in L^2(\mathbb{R}^N)\big\}$. Moreover, we have

$$\int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |V_\varepsilon|^{2^*} dx = \mathcal{S}^{\frac{N}{2}},$$

where $\mathcal{S}$ is the best Sobolev embedding constant from $H_0^1(\Omega)$ to $L^{2^*}(\mathbb{R}^N)$. Let $\varphi(r)$ be a nonnegative smooth radial cut-off function on $[0,+\infty)$ such that $\varphi(r) = 1$ on a ball contained in Ω and supp$\varphi \subset \Omega$. Define $v_\varepsilon(x) = V_\varepsilon(x)\varphi(|x|)$, then we have from [28, Lemma 3.4] that

$$\int_\Omega |v_\varepsilon|^{2^*} dx = \mathcal{S}^{N/2} + O\big(\varepsilon^N\big), \qquad \int_\Omega |\nabla v_\varepsilon|^2 dx = \mathcal{S}^{N/2} + O\big(\varepsilon^{N-2}\big), \tag{5.24}$$

$$\int_\Omega |v_\varepsilon|^{2^*-1} dx = O\big(\varepsilon^{(N-2)/2}\big), \qquad \int_\Omega |\nabla v_\varepsilon| \, dx = O\big(\varepsilon^{(N-2)/2}\big), \tag{5.25}$$

and

$$\int_\Omega |v_\varepsilon|^2 dx \geq \begin{cases} C\varepsilon^2|\ln(\varepsilon)| + O\big(\varepsilon^2\big) & \text{for } N = 4, \\ C\varepsilon^2 + O\big(\varepsilon^{N-2}\big) & \text{for } N \geq 5. \end{cases} \tag{5.26}$$

Without loss of generality, we may assume that $\mu_1 < 0$. Now, set $\mathbf{V}_\varepsilon = (v_\varepsilon, 0, \ldots, 0)$. Since $\Omega \setminus \operatorname{supp}\varphi$ is a nonempty open set in Ω, by [28, Lemma 3.3], $v_\varepsilon \in H_0^1(\Omega) \setminus \widetilde{\mathcal{N}}_m$, where $\widetilde{\mathcal{N}}_m = \bigoplus_{i=1}^{m} \mathcal{N}_i$. It follows that $\mathbf{V}_\varepsilon \in \mathcal{H} \setminus \widetilde{\mathcal{N}}_m^*$, where $\widetilde{\mathcal{N}}_m^*$ is given by (3.5). By Lemma 5.6, it suffices to show that

$$\sup_{t\geq 0,\, \mathbf{w}\in\widetilde{\mathcal{N}}_m^*} \mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) < \frac{1}{N}\,\mathcal{S}^{N/2}.$$

Indeed, let $\Omega_* = \Omega \setminus \operatorname{supp}\varphi$, then, by the convexity, we have that, for every $t > 0$ and $\mathbf{w} \in \widetilde{\mathcal{N}}_m^*$,

$$\int_\Omega |tv_\varepsilon + w_1|^{2^*} dx \geq \int_\Omega (tv_\varepsilon)^{2^*} dx + 2^* \int_\Omega (tv_\varepsilon)^{2^*-1} w_1 \, dx + \int_{\Omega_*} |w_1|^{2^*} dx, \tag{5.27}$$

which together with $\dim(\widetilde{\mathcal{N}}_m^*) < \infty$ implies

$$\int_\Omega |tv_\varepsilon + w_1|^{2^*} dx \geq \int_\Omega (tv_\varepsilon)^{2^*} dx + 2^* \int_\Omega (tv_\varepsilon)^{2^*-1} w_1 \, dx + C\|w_1\|_1^{2^*}.$$

It follows from the Hölder inequality, the Sobolev embedding and $\dim\big(\widetilde{\mathcal{N}}_m^*\big) < \infty$ that

$$
\begin{aligned}
\mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) \le{}& \frac{1}{2}\|tv_\varepsilon\|_1^2 + \frac{1}{2}\|\mathbf{w}\|^2 - \lambda\mathcal{G}(\mathbf{w}) + \langle tv_\varepsilon, w_1\rangle_1 \\
& - \lambda\int_\Omega tv_\varepsilon \sum_{j=2}^k w_j\,dx - \frac{1}{2^*}\sum_{j=2}^k \int_{\Omega_*} |w_j|^{2^*}\,dx \\
& - \frac{1}{2^*}\int_\Omega |tv_\varepsilon|^{2^*}\,dx - \int_\Omega (tv_\varepsilon)^{2^*-1} w_1\,dx - C\|w_1\|_1^{2^*} \\
\le{}& \frac{1}{2}\|tv_\varepsilon\|_1^2 - \frac{1}{2^*}\int_\Omega |tv_\varepsilon|^{2^*}\,dx + \frac{1}{2}\|\mathbf{w}\|^2 - \lambda\mathcal{G}(\mathbf{w}) \\
& + \sum_{j=2}^k \bigg(C\|w_j\|_j \int_\Omega |\nabla tv_\varepsilon|\,dx - C'\|w_j\|_j^{2^*}\bigg) \\
& + \|w_1\|_1\bigg(\int_\Omega |\nabla tv_\varepsilon|\,dx + \int_\Omega |tv_\varepsilon|^{2*-1}\,dx\bigg) - C'\|w_1\|_1^{2^*},
\end{aligned}
$$

which, together with (5.24), implies

$$
\begin{aligned}
\mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) \le{}& \frac{1}{2}\|tv_\varepsilon\|_1^2 - \frac{1}{2^*}\int_\Omega |tv_\varepsilon|^{2^*}\,dx + \frac{1}{2}\|\mathbf{w}\|^2 - \lambda\mathcal{G}(\mathbf{w}) \\
& + \sum_{j=1}^k \big(\big(t + t^{2^*-1}\big)O\big(\varepsilon^{(N-2)/2}\big)\|w_j\|_j - C\|w_j\|_j^{2^*}\big).
\end{aligned} \tag{5.28}
$$

We claim that there exists $R_0 > 0$ independent of $\varepsilon > 0$ small enough such that

$$
\mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) \le 0 \quad \text{for } t^2 + \|\mathbf{w}\|^2 \ge R_0^2.
$$

Indeed, we redefine $w_j = a_j\widetilde{w}_j$, where $\|w_j\|_j = 1$. We also use the notation $\mathbf{s} = (t, a_1, \dots, a_k) \in \mathbb{R}^+ \times \mathbb{R}^k$ and $R = \sqrt{t^2 + \sum_{j=1}^k a_j^2}$. Since $|a_j/R| \le 1$, by (5.24)–(5.25) and (5.27), we have from the Sobolev embedding that

$$
\begin{aligned}
& R^{-2^*}\bigg(\int_\Omega |tv_\varepsilon + w_1|^{2^*}\,dx + \sum_{j=2}^k \int_\Omega |w_j|^{2^*}\,dx\bigg) \\
&\quad = \int_\Omega \left|\frac{t}{R}\,v_\varepsilon + \frac{a_1}{R}\,\widetilde{w}_1\right|^{2^*} dx + \sum_{j=2}^k \int_\Omega \left|\frac{a_j}{R}\,\widetilde{w}_j\right|^{2^*} dx \\
&\quad \ge \left(\frac{t}{R}\right)^{2^*} \mathcal{S}^{N/2} + O\big(\varepsilon^{(N-2)/2}\big) + \sum_{j=1}^k \left(\frac{a_j}{R}\right)^{2^*} C\int_\Omega |\widetilde{w}_j|^{2^*}\,dx,
\end{aligned} \tag{5.29}
$$

where $O\big(\varepsilon^{(N-2)/2}\big)$ is independent of t, a_j, and R. Since $|t/R| \le 1$, we may assume that $t/R \to t_0$ as $R \to +\infty$. If $t_0 = 0$, then by $R = \sqrt{t^2 + \sum_{j=1}^k a_j^2}$, we must have from $\dim\big(\widetilde{\mathcal{N}}_m^*\big) < \infty$ that

$$\sum_{j=1}^{k}\left(\frac{a_j}{R}\right)^{2^*} C\int_\Omega |\widetilde{w}_j|^{2^*}\,dx \geq C' > 0$$

for R large enough. Otherwise, we have $t_0 > 0$. It follows that $(t/R)^{2^*}\mathcal{S}^{N/2} > C'$ for R large enough. Thus, by (5.29), there exists $R' > 0$ independent of $\varepsilon > 0$ small enough such that

$$R^{-2^*}\left(\int_\Omega |tv_\varepsilon + w_1|^{2^*}\,dx + \sum_{j=2}^{k}\int_\Omega |w_j|^{2^*}\,dx\right) \geq C' \tag{5.30}$$

for $\varepsilon > 0$ small enough and $R \geq R'$, where $C' > 0$ is independent of ε. Thanks to $2^* > 2$, we have from (5.30) that there exists $R_0 \geq R'$ independent of $\varepsilon > 0$ small enough such that

$$\mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) < 0 \quad \text{for } R \geq R_0.$$

Now, let $t \leq R_0$. If $N \geq 5$, then by (5.25), (5.26), and (5.28), we have from Theorem 1.4, $\mu_1 < 0$ and a similar calculation as used in the proof of [28, Lemma 3.5] that

$$\mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) \leq \frac{1}{N}\mathcal{S}^{N/2} - C\varepsilon^2 + O\big(\varepsilon^{N(N-2)/(N+2)}\big) < \frac{1}{N}\mathcal{S}^{N/2}$$

with $\varepsilon > 0$ small enough. If $N = 4$ and $\lambda_m < \lambda < \lambda_{m+1}$, then, by Theorem 1.4, we have

$$\frac{1}{2}\|\mathbf{w}\|^2 - \lambda\mathcal{G}(\mathbf{w}) \leq -C\|\mathbf{w}\|^2.$$

Thus, also by (5.25), (5.26), and (5.28), we have from $\mu_1 < 0$ and a similar calculation as used in the proof of [28, Lemma 3.5] once more that

$$\mathcal{E}_\lambda(t\mathbf{V}_\varepsilon + \mathbf{w}) \leq \frac{1}{4}\mathcal{S}^2 - C\varepsilon^2|\ln(\varepsilon)| + O\big(\varepsilon^2\big) < \frac{1}{4}\mathcal{S}^2$$

with $\varepsilon > 0$ small enough. Hence, we must have $\widetilde{c}_\lambda < \mathcal{S}^{N/2}/N$. $\square$

Let $\mathbb{B}_{1,m}^+ = \big\{\mathbf{u} \in \big(\widetilde{\mathcal{N}}_m^*\big)^\perp \mid \|\mathbf{u}\| = 1\big\}$, where $\big(\widetilde{\mathcal{N}}_m^*\big)^\perp$ is given by (3.5). For every $\mathbf{u} \in \mathbb{B}_{1,m}^+$, by Lemma 5.6, there exist unique $t_{\mathbf{u}} > 0$ and $\mathbf{v}_{\mathbf{u}} \in \widetilde{\mathcal{N}}_m^*$ such that $t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}} \in \mathcal{P}_m$, where $\mathcal{P}_m$ is given by (5.12).

Now, let us consider the functional $\Psi_\lambda\colon \mathbb{B}_{1,m}^+ \to \mathbb{R}$ given by

$$\Psi_\lambda(\mathbf{u}) = \mathcal{E}_\lambda(\mathfrak{m}(\mathbf{u})), \tag{5.31}$$

where $\mathfrak{m}(\mathbf{u}) = t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}}$.

LEMMA 5.8. *Let $N \geq 3$, $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. Then we have*:

(a) $\Psi_\lambda(\mathbf{u})$ *is of C^1 on $\mathbb{B}_{1,m}^+$. Moreover,*

$$\Psi_\lambda'(\mathbf{u})\mathbf{w} = \mathcal{E}_\lambda'(\mathfrak{m}(\mathbf{u}))[t_{\mathbf{u}}\mathbf{w}] \quad \textit{for all } \mathbf{u}, \mathbf{w} \in \mathbb{B}_{1,m}^+. \tag{5.32}$$

(b) $\widehat{c}_\lambda = \widetilde{c}_\lambda$, *where* $\widehat{c}_\lambda = \inf_{\mathbb{B}^+_{1,m}} \Psi_\lambda(\mathbf{u})$ *and* $\widetilde{c}_\lambda$ *is given by* (5.21).

(c) $\{\mathbf{u}_n\}$ *is a* (PS) *sequence of* $\Psi_\lambda(\mathbf{u})$ *if and only if* $\{\mathfrak{m}(\mathbf{u}_n)\}$ *is a* (PS) *sequence of* $\mathcal{E}_\lambda(\mathbf{u})$.

PROOF. (a) We first assert that $\mathfrak{m}(\mathbf{u})$ is continuous on $\mathbb{B}^+_{1,m}$. Let $\{\mathbf{u}_n\} \subset \mathbb{B}^+_{1,m}$ such that $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathcal{H}$ as $n \to \infty$. Then, by a similar argument as used for (5.20), we can see that $\{\mathfrak{m}(\mathbf{u}_n)\}$ is bounded in $\mathcal{H}$. Since $\dim(\widetilde{\mathcal{N}}^*_m) < \infty$, we may assume that $t_{\mathbf{u}_n} \to t_0$ and $\mathbf{v}_{\mathbf{u}_n} \to \mathbf{v}_0$ strongly in $\mathcal{H}$ as $n \to \infty$. It follows that $t_0\mathbf{u} + \mathbf{v}_0 \in \mathcal{P}_m$, where $\mathcal{P}_m$ is given by (5.12). Thanks to Lemma 5.6, we must have $t_0\mathbf{u} + \mathbf{v}_0 = \mathfrak{m}(\mathbf{u})$. Thus, $\mathfrak{m}(\mathbf{u})$ is continuous on $\mathbb{B}^+_{1,m}$. Let $\mathbf{u}, \mathbf{w} \in \mathbb{B}^+_{1,m}$ and use the notation $\mathbf{u}_s = \mathbf{u} + s\mathbf{w}$, where $s \in \mathbb{R}$. Now, since $\mathbf{v}_{\mathbf{u}_s}, \mathbf{v}_{\mathbf{u}} \in \widetilde{\mathcal{N}}^*_m$, we have from Taylor's expansion and the definition of $\mathcal{P}_m$ that

$$\begin{aligned}\Psi_\lambda(\mathbf{u}_s) - \Psi_\lambda(\mathbf{u}) &= \mathcal{E}_\lambda(\mathfrak{m}(\mathbf{u}_s)) - \mathcal{E}_\lambda(\mathfrak{m}(\mathbf{u})) = \mathcal{E}_\lambda(t_{\mathbf{u}_s}\mathbf{u}_s + \mathbf{v}_{\mathbf{u}_s}) - \mathcal{E}_\lambda(t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}_s})\\ &\le \mathcal{E}_\lambda(t_{\mathbf{u}_s}\mathbf{u}_s + \mathbf{v}_{\mathbf{u}_s}) - \mathcal{E}_\lambda(t_{\mathbf{u}_s}\mathbf{u} + \mathbf{v}_{\mathbf{u}_s}) = \mathcal{E}'_\lambda(t_{\mathbf{u}_s}\mathbf{u} + \mathbf{v}_{\mathbf{u}_s})[t_{\mathbf{u}_s}s\mathbf{w}] + o(s)\end{aligned}$$

and

$$\begin{aligned}\Psi_\lambda(\mathbf{u}_s) - \Psi_\lambda(\mathbf{u}) &= \mathcal{E}_\lambda(\mathfrak{m}(\mathbf{u}_s)) - \mathcal{E}_\lambda(\mathfrak{m}(\mathbf{u})) = \mathcal{E}_\lambda(t_{\mathbf{u}_s}\mathbf{u}_s + \mathbf{v}_{\mathbf{u}_s}) - \mathcal{E}_\lambda(t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}})\\ &\ge \mathcal{E}_\lambda(t_{\mathbf{u}}\mathbf{u}_s + \mathbf{v}_{\mathbf{u}}) - \mathcal{E}_\lambda(t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}}) = \mathcal{E}'_\lambda(t_{\mathbf{u}}\mathbf{u} + \mathbf{v}_{\mathbf{u}})[t_{\mathbf{u}}s\mathbf{w}] + o(s).\end{aligned}$$

By the continuity of $\mathfrak{m}(\mathbf{u})$, we can see that $t_{\mathbf{u}_s} \to t_{\mathbf{u}}$ and $\mathbf{v}_{\mathbf{u}_s} \to \mathbf{v}_{\mathbf{u}}$ as $s \to 0$. Thus

$$\frac{\partial\Psi_\lambda(\mathbf{u})}{\partial\mathbf{w}} = \lim_{s\to 0}\frac{\Psi_\lambda(\mathbf{u}_s) - \Psi_\lambda(\mathbf{u})}{s} = \mathcal{E}'_\lambda(\mathfrak{m}(\mathbf{u}))[t_{\mathbf{u}}\mathbf{w}].$$

Since $\partial\Psi_\lambda(\mathbf{u})/\partial\mathbf{w}$ is continuous for $\mathbf{u}, \mathbf{w}$ and is linear for $\mathbf{w}$, by [30, Proposition 1.3], $\Psi'_\lambda(\mathbf{u})$ exists and (5.32) holds.

(b) By the definition of $\widehat{c}_\lambda$ and $\widetilde{c}_\lambda$, it is easy to see that $\widehat{c}_\lambda \ge \widetilde{c}_\lambda$. On the other hand, by Lemma 5.5, for every $\mathbf{u} \in \mathcal{P}_m$, we have $\mathbf{u}^\perp/\|\mathbf{u}^\perp\| \in \mathbb{B}^+_{1,m}$, where $\mathbf{u} = \check{\mathbf{u}} + \mathbf{u}^\perp$ with $\check{\mathbf{u}} \in \widetilde{\mathcal{N}}^*_m$ and $\mathbf{u}^\perp \in (\widetilde{\mathcal{N}}^*_m)^\perp$. By Lemmas 5.5 and 5.6, we can see that there exists unique $t = \|\mathbf{u}^\perp\|$ and $\mathbf{v} = \check{\mathbf{u}}$ such that $\mathfrak{m}(\mathbf{u}^\perp/\|\mathbf{u}^\perp\|) = \mathbf{u} \in \mathcal{P}_m$. It follows that

$$\mathcal{E}_\lambda(\mathbf{u}) = \mathcal{E}_\lambda\left(\mathfrak{m}\left(\frac{1}{\|\mathbf{u}^\perp\|}\mathbf{u}^\perp\right)\right) = \Psi_\lambda\left(\frac{1}{\|\mathbf{u}^\perp\|}\mathbf{u}^\perp\right) \ge \widehat{c}_\lambda.$$

Thus, we also have $\widehat{c}_\lambda \le \widetilde{c}_\lambda$, which implies $\widehat{c}_\lambda = \widetilde{c}_\lambda$.

(c) $\mathbb{B}_1 = \{\mathbf{u} \in \mathcal{H} \mid \|\mathbf{u}\| = 1\}$. It is easy to see that $\mathbb{B}_1 = (\widetilde{\mathcal{N}}^*_m \cap \mathbb{B}_1) \oplus \mathbb{B}^+_{1,m}$.

Since $\mathcal{E}'_\lambda(\mathfrak{m}(\mathbf{u})) = 0$ in $\widetilde{\mathcal{N}}^*_m \oplus \mathbb{R}\mathbf{u}$ by the definition of $\mathcal{P}_m$, we have from (5.32) that

$$\begin{aligned}\|\Psi'_\lambda(\mathbf{u})\| &= \sup_{\mathbf{w}\in\mathbb{B}^+_{1,m}} \Psi'_\lambda(\mathbf{u})\mathbf{w} = \sup_{\mathbf{w}\in\mathbb{B}^+_{1,m}} \mathcal{E}'_\lambda(\mathfrak{m}(\mathbf{u}))[t_{\mathbf{u}}\mathbf{w}]\\ &= t_{\mathbf{u}}\sup_{\mathbf{z}\in\mathbb{B}_1} \mathcal{E}'_\lambda(\mathfrak{m}(\mathbf{u}))\mathbf{z} = t_{\mathbf{u}}\|\mathcal{E}'_\lambda(\mathfrak{m}(\mathbf{u}))\|.\end{aligned} \tag{5.33}$$

Since $\widehat{c}_\lambda = \widetilde{c}_\lambda > 0$ by Lemma 5.7, we can see that $\{t_{\mathbf{u}_n}\}$ is bounded away from 0 if $\{\mathbf{u}_n\}$ is a (PS) sequence of $\Psi_\lambda(\mathbf{u})$. On the other hand, if $\{\mathfrak{m}(\mathbf{u}_n)\}$ is a (PS) sequence of $\mathcal{E}_\lambda(\mathbf{u})$, then by Lemma 5.7, we can apply a similar argument as used for (5.9) to show that $\{\mathfrak{m}(\mathbf{u}_n)\}$ is bounded in $\mathcal{L}^{2^*}(\Omega)$. Recall that $\mathfrak{m}(\mathbf{u}_n) = t_{\mathbf{u}_n}\mathbf{u}_n + \mathbf{v}_{\mathbf{u}_n}$ with $\{\mathbf{v}_{\mathbf{u}_n}\} \subset \widetilde{\mathcal{N}}_m^*$ and $\dim(\widetilde{\mathcal{N}}_m^*) < +\infty$, thus, $\{\mathbf{v}_{\mathbf{u}_n}\}$ is also bounded, which together with $\{\mathfrak{m}(\mathbf{u}_n)\} \subset \mathcal{P}_m$ and $\{\mathbf{u}_n\} \subset \mathbb{B}^+_{1,m}$, implies $\{t_{\mathbf{u}_n}\}$ is bounded. Thus, by (5.33), $\{\mathbf{u}_n\}$ is a (PS) sequence of $\Psi_\lambda(\mathbf{u})$ if and only if $\{\mathfrak{m}(\mathbf{u}_n)\}$ is a (PS) sequence of $\mathcal{E}_\lambda(\mathbf{u})$. □

Recall that in the case $N \geq 4$, $m_\nu = \mathcal{S}^{N/2}/N$ for $\nu > 0$, whereas $0 < m_\nu < \mathcal{S}^{N/2}/N$ can be attained for $\nu < 0$ in one of the following two cases:

(1) $N = 4$ and $\nu \neq -\alpha_m$ for all $m \in \mathbb{N}$,

(2) $N \geq 5$,

where m_ν is given by (1.13).

Proposition 5.9. *Let $N \geq 4$, $\mu_i > -\alpha_1$ for all $i = 1, \dots, k$ and $\lambda_m \leq \lambda < \lambda_{m+1}$ for some $m \in \mathbb{N}$. If $\min\{\mu_1, \dots, \mu_k\} < 0$ and either*

(a) *$N = 4$, $\lambda_m < \lambda < \lambda_{m+1}$, or*

(b) *$N \geq 5$,*

then there exists $\widehat{\mathbf{u}}_\lambda \in \mathcal{P}_m$ such that $\widehat{\mathbf{u}}_\lambda$ is a ground state solution to system (1.1) that is also sign-changing. Moreover, if $k = 2$ or $k \geq 3$ with

$$\widetilde{c}_\lambda < \min_{i,j=1,\dots,k,\, i\neq j}\{m_{\mu_i+\lambda} + m_{\mu_j+\lambda}\},$$

then $\widehat{\mathbf{u}}_\lambda$ is also nontrivial.

Proof. Since $\mathbb{B}^+_{1,m}$ is a natural constraint in $(\widetilde{\mathcal{N}}_m^*)^\perp$, by applying Ekeland's variational principle and the implicit function theorem, we can see that $\Psi_\lambda(\mathbf{u})$ has a (PS) sequence $\{\mathbf{u}_n\}$ at the energy level $\widehat{c}_\lambda$, where $(\widetilde{\mathcal{N}}_m^*)^\perp$ is given by (3.5). By Lemma 5.7, we can apply a similar argument as used for (5.9) to show that $\{\mathfrak{m}(\mathbf{u}_n)\}$ is bounded in $\mathcal{L}^{2^*}(\Omega)$. Recall that $\mathfrak{m}(\mathbf{u}_n) = t_{\mathbf{u}_n}\mathbf{u}_n + \mathbf{v}_{\mathbf{u}_n}$ with $\{\mathbf{v}_{\mathbf{u}_n}\} \subset \widetilde{\mathcal{N}}_m^*$ and $\dim(\widetilde{\mathcal{N}}_m^*) < +\infty$, thus $\{\mathbf{v}_{\mathbf{u}_n}\}$ is also bounded, which together with $\{\mathfrak{m}(\mathbf{u}_n)\} \subset \mathcal{P}_m$ and $\{\mathbf{u}_n\} \subset \mathbb{B}^+_{1,m}$, implies $\{t_{\mathbf{u}_n}\}$ is bounded. Hence, $\{\mathfrak{m}(\mathbf{u}_n)\}$ is bounded in $\mathcal{H}$. For simplicity, we denote $\mathfrak{m}(\mathbf{u}_n)$ by $\mathbf{w}_n$. By the Sobolev embedding theorem and without loss of generality, we may assume that $\mathbf{w}_n \rightharpoonup \mathbf{w}_0$ weakly in $\mathcal{H}$ and $\mathbf{w}_n \to \mathbf{w}_0$ strongly in $\mathcal{L}^q(\Omega)$ for all $1 \leq q < 2^*$ as $n \to \infty$. Thanks to Lemma 5.7, by a similar argument as used in the proof of Proposition 5.2, we must have $\mathbf{w}_0 \neq \mathbf{0}$. Clearly, $\mathcal{E}_\lambda'(\mathbf{w}_0) = \mathbf{0}$.

Let $\widehat{\mathbf{w}}_n = \mathbf{w}_n - \mathbf{w}_0$. Then $\widehat{\mathbf{w}}_n \rightharpoonup \mathbf{0}$ weakly in $\mathcal{H}$ and $\widehat{\mathbf{w}}_n \to \mathbf{0}$ strongly in $\mathcal{L}^q(\Omega)$ for all $1 \leq q < 2^*$ as $n \to \infty$. It follows from the Brezis–Lieb lemma that

$$\mathcal{E}_\lambda(\mathbf{w}_n) = \mathcal{E}_\lambda(\mathbf{w}_n) + \sum_{j=1}^k \left(\frac{1}{2}\int_\Omega |\nabla \widehat{w}_j^n|^2\,dx - \frac{1}{2^*}\int_\Omega |\widehat{w}_j^n|^{2^*}\,dx\right) + o(1)$$

and

$$\mathcal{E}'_\lambda(\mathbf{w}_n)\mathbf{w}_n^j = \mathcal{E}'_\lambda(\widehat{\mathbf{w}}_n)\widehat{\mathbf{w}}_n^j + \mathcal{E}'_\lambda(\mathbf{w}_0)\mathbf{w}_0^j + o(1) \quad \text{for all } j = 1, \dots, k,$$

where $\mathbf{u}^1 = (u_1, 0, \dots, 0)$ and $\mathbf{u}^j = (0, \dots, u_j, 0, \dots, 0)$ for $j = 2, \dots, k$. It follows that

$$\int_\Omega \left|\nabla \widehat{w}_j^n\right|^2 dx = \int_\Omega \left|\widehat{w}_j^n\right|^{2^*} + o(1)$$

for all $j = 1, \dots, k$. If $\|\widehat{\mathbf{w}}_n\| \geq C + o(1)$, then by a standard argument, we must have that

$$\sum_{j=1}^k \left(\frac{1}{2}\int_\Omega \left|\nabla \widehat{w}_j^n\right|^2 dx - \frac{1}{2^*}\int_\Omega \left|\widehat{w}_j^n\right|^{2^*} dx\right) \geq \frac{1}{N}\,\mathcal{S}^{N/2} + o(1),$$

which contradicts Lemma 5.7 and the fact that $\mathcal{E}_\lambda(\mathbf{w}_0) \geq 0$. Hence, $\mathbf{w}_n \to \mathbf{w}_0$ strongly in $\mathcal{H}$ as $n \to \infty$. Denote $\mathbf{w}_0$ by $\widehat{\mathbf{u}}_\lambda$, then by Proposition 5.3, $\widehat{\mathbf{u}}_\lambda$ is a ground state solution to system (1.1) that is also sign-changing. It remains to show that $\widehat{\mathbf{u}}_\lambda$ is also nontrivial. If $k = 2$, then by the fact that system (1.1) is strong coupled for $k = 2$, it is easy to see that $\widehat{\mathbf{u}}_\lambda$ is nontrivial. Let us show that $\widehat{\mathbf{u}}_\lambda$ is also nontrivial for $k \geq 3$ with

$$\widetilde{c}_\lambda < \min_{i,j=1,\dots,k,\, i\neq j} \left\{m_{\mu_i+\lambda} + m_{\mu_j+\lambda}\right\}. \tag{5.34}$$

Suppose the contrary, then there exists $j \in \{1, \dots, k\}$ such that $\widehat{u}_j^\lambda = 0$. Without loss of generality, we assume that $\widehat{u}_i^\lambda \neq 0$ for $i = 1, \dots, i_0$ and $\widehat{u}_i^\lambda = 0$ for $i = i_0 + 1, \dots, k$ with some $i_0 \in \{1, \dots, k-1\}$. By the fact that system (1.1) is strong coupled for $k = 2$, it is easy to see $i_0 \geq 2$. On the other hand, since $\widehat{\mathbf{u}}_\lambda$ is a solution to system (1.1), we have from $\widehat{u}_i^\lambda = 0$ for $i = i_0 + 1, \dots, k$ that $\widehat{\mathbf{u}}_\lambda$ is also a solution to system (5.11), which implies that $\widehat{u}_i^\lambda$ $(i = 1, \dots, i_0)$ are also solutions to the following equation:

$$\begin{cases} -\Delta u_i + (\mu_i + \lambda)u_i = |u_i|^{2^*-2}u_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

By the definition of $m_{\mu_i+\lambda}$ given by (1.13), we must have from $u_i \neq 0$ for all $i = 1, \dots, i_0$ that

$$\mathcal{J}_{\mu_i+\lambda}\left(\widehat{u}_i^\lambda\right) \geq m_{\mu_i+\lambda}. \tag{5.35}$$

Note that by (5.11) once more, we have

$$\mathcal{G}(\mathbf{u}) = \sum_{i,j=1\,,i\neq j}^{i_0} \int_\Omega u_i u_j\, dx = -\sum_{i=1}^{i_0} \mathcal{B}_{u_i,2}^2,$$

which together with (5.35) and $i_0 \geq 2$, implies

$$\mathcal{E}_\lambda\left(\widehat{\mathbf{u}}_\lambda\right) = \sum_{i=1}^{i_0} \mathcal{J}_{\mu_i+\lambda}\left(\widehat{u}_i^\lambda\right) \geq \min_{i,j=1,\dots,k,\, i\neq j} \left\{m_{\mu_i+\lambda} + m_{\mu_j+\lambda}\right\}.$$

This contradicts (5.34), which implies $\widehat{\mathbf{u}}_\lambda$ must be nontrivial if (5.34) holds. □

REMARK 5.10. In the case $k = 3$, we can show that any nonzero solution $\mathbf{u}$ must be nontrivial if $\mu_1 \neq \mu_2$, $\mu_1 \neq \mu_3$ and $\mu_2 \neq \mu_3$. Indeed, suppose the contrary, then as in the proof of Proposition 5.9, we must have $i_0 = 2$ and $\mathbf{u}$ satisfies:

$$\begin{cases} -\Delta u_1 + \mu_1 u_1 = |u_1|^{p-2}u_1 + \lambda u_2 & \text{in } \Omega, \\ -\Delta u_2 + \mu_2 u_2 = |u_2|^{p-2}u_2 + \lambda u_1 & \text{in } \Omega, \\ u_1 + u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that $(\mu_2 - \mu_1)u_2 = 0$ in Ω, which contradicts $u_2 \neq 0$ and $\mu_1 \neq \mu_2$.

Proof of Theorem 1.8 follows immediately from Propositions 5.2–5.9.

6. The concentration behavior of $\mathbf{u}_\lambda$ as $\lambda \to \lambda_1$

Recall that $\mathbf{u}_\lambda$ is the ground state solution obtained by Theorem 1.8 for $0 < \lambda < \lambda_1$ such that $\mathbf{u}_\lambda \in \mathcal{P}_0$ and $\mathcal{E}_\lambda(\mathbf{u}_\lambda) = c_\lambda$, where $\mathcal{P}_0$ and c_λ are respectively given by (5.1) and (5.2).

PROPOSITION 6.1. *Let $N \geq 4$ and $\mu_i > -\alpha_1$ for all $i = 1, \ldots, k$. If we also have $\min\{\mu_1, \ldots, \mu_k\} < 0$, then for every $\{\beta_n\} \subset (0, \lambda_1)$ with $\beta_n \to \lambda_1$ as $n \to \infty$, there exists a subsequence, which still denoted by $\{\beta_n\}$, such that $\mathbf{u}_{\beta_n} \to 0$ strongly in $\mathcal{H}$ as $n \to \infty$.*

PROOF. Let $\{\beta_n\} \subset (0, \lambda_1)$ with $\beta_n \to \lambda_1$ as $n \to \infty$. Without loss of generality, we may also assume that $\beta_n \nearrow \lambda_1$ as $n \to \infty$. Recall (5.2), we can see from a similar argument as used in the proof of [19, Lemma 5.1] that c_λ is nonincreasing for $\lambda \in (0, \lambda_1)$. It follows from (5.5) that $\{\mathbf{u}_{\beta_n}\}$ is bounded in $\mathcal{H}$. Without loss of generality, we may assume that $\mathbf{u}_{\beta_n} \rightharpoonup \mathbf{u}_0$ weakly in $\mathcal{H}$ as $n \to \infty$. Since $\sup\limits_{n\in\mathbb{N}} c_{\beta_n} < \mathcal{S}^{N/2}/N$, by a similar argument as used in the proof of Proposition 5.2, we can show that $\mathbf{u}_0 \neq \mathbf{0}$ if $\lim\limits_{n\to\infty} c_{\beta_n} > 0$, which contradicts Proposition 5.3 owing to the fact that $\mathbf{u}_{\beta_n}$ are positive. Now, the conclusion follows immediately from (5.5) once more. □

Finally, we get proof of Theorem 1.10. Since Proposition 6.1 holds, we can obtain the conclusion by a standard arguments.

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