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AMENABILITY AND HAHN–BANACH EXTENSION PROPERTY FOR SET VALUED MAPPINGS

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Abstract. Amenability is an important notion in harmonic analysis on groups and semigroups, and their associated Banach algebras. In this paper, we present some characterization of a semitopological semigroup S on the existence of a left invariant mean on LUC(S), AP(S) and WAP(S) in terms of the Hahn–Banach extension theorem, which extend the first author's early results in 1970s. Moreover, we refine and extend the well known Day's result and Mitchell's results on fixed point properties for set-valued mappings. As an application, we give an application of our result to a class of Banach algebras related to amenability of groups and semigroups.

1. Introduction

Throughout this paper, we assume that E is a real separated locally convex space. All topologies in this paper are assumed to be Hausdorff.

Let $A\colon E \rightrightarrows E$ be a *set-valued operator* (also known as multifunction) from E to E, i.e. for every $x \in E$, $Ax \subseteq E$, and let $\operatorname{gra} A := \{(x,y) \in E \times E \mid y \in Ax\}$ be the *graph* of A, $\operatorname{dom} A := \{x \in E \mid Ax \neq \emptyset\}$ be the *domain* of A. We say that A is a *linear relation* if $\operatorname{gra} A$ is a subspace of $E \times E$.

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Let $P\colon E \to \mathbb{R}$. We say that P is *sublinear* if $P(x+y) \le P(x) + P(y)$ and $P(\lambda x) = \lambda P(x)$ for all $x, y \in E, \lambda \ge 0$. Let C be a subset of E. Then $\operatorname{int} C$ is the *interior* of C.

Let S be a *semitopological semigroup*, i.e. S is a semigroup with Hausdorff topology such that for every $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous.

Let $\ell^\infty(S)$ denote the space of all bounded real-valued functions on S with the supremum norm: $\|\cdot\|_\infty$. For each $a \in S$ and $f \in \ell^\infty(S)$, let $l_a f$ and $r_a f$ denote the *left and right translate* of f by a respectively, i.e. $(l_a f)(s) := f(as)$ and $(r_a f)(s) := f(sa)$, for all $s \in S$. Let Y be a closed subspace of $\ell^\infty(S)$ containing constants and invariant under translations (i.e. $l_a(Y) \subseteq Y$ and $r_a(Y) \subseteq Y$, for all $a \in S$). Then a linear functional $m \in Y^*$ is called a *mean* on Y if $\|m\| = m(1) = 1$. We say that a mean m is a *left invariant mean* on Y, denoted by LIM, if

$$\langle m, l_a f\rangle = \langle m, f\rangle, \quad \text{for all } a \in S, \text{ for all } f \in Y.$$

A mean m on Y is called *multiplicative* if $m(f)\cdot m(g) = m(f\cdot g)$ for all $f, g \in Y$.

Let $\mathrm{CB}(S)$ denote the space of all bounded continuous real-valued functions on S with the supremum norm: $\|\cdot\|_\infty$. Let $\mathrm{LUC}(S)$ be the *space of bounded left uniformly continuous functions* on S, i.e. all $f \in \mathrm{CB}(S)$ such that the mappings $a \to l_a f$ from S into $\mathrm{CB}(S)$ are continuous. Note that if S is a topological group, then $\mathrm{LUC}(S)$ is precisely the *space of bounded right uniformly continuous functions* on S defined in [13]. Set $\mathcal{LO}(f) := \{l_s f \mid s \in S\}$ and $\mathcal{RO}(f) := \{r_s f \mid s \in S\}$, where $f \in \mathrm{CB}(S)$.

Let $\mathrm{AP}(S)$ and $\mathrm{WAP}(S)$ be denoted by *space of almost periodic functions* and the *space of weakly almost periodic functions* on S, respectively. More precisely, the spaces $\mathrm{AP}(S)$ and $\mathrm{WAP}(S)$ are defined by the followings:

$$\begin{aligned}
\mathrm{AP}(S) :=\ &\text{the space of all } f \in \mathrm{CB}(S) \text{ such that } \mathcal{LO}(f)\\
&(\text{or equivalently, } \mathcal{RO}(f), \text{ see [3]})\\
&\text{is relatively compact in the norm topology of } \mathrm{CB}(S);\\
\mathrm{WAP}(S) :=\ &\text{the space of all } f \in \mathrm{CB}(S) \text{ such that } \mathcal{LO}(f)\\
&(\text{or equivalently, } \mathcal{RO}(f), \text{ see [3]})\\
&\text{is relatively compact in the weak topology of } \mathrm{CB}(S).
\end{aligned}$$

In general, we have the following inclusions.

$$\mathrm{AP}(S) \subseteq \mathrm{LUC}(S) \subseteq \mathrm{CB}(S) \quad \text{and} \quad \mathrm{AP}(S) \subseteq \mathrm{WAP}(S) \subseteq \mathrm{CB}(S).$$

We say that S is *left amenable* if $\mathrm{LUC}(S)$ has a left invariant mean (LIM). In the case G is a locally compact group, this is equivalent to the space $L^\infty(G)$, the *equivalent classes of bounded measurable functions* on G, has a left invariant mean (see [12]).

Let S be a semitopological semigroup. A *set-valued action* of S on E is a set-valued mapping from $S \times E$ to E, denoted by $(s, x) \rightrightarrows s \cdot x$.

Let $e \in E$. We say that e is an *invariant element* if $e \in s \cdot e$ for every $s \in S$. Let F be a nonempty subset of E. We say that F is an *invariant set* if $(s \cdot x) \cap F \neq \emptyset$ for every $s \in S$ and every $x \in F$. We say that F is a *totally invariant set* if $s \cdot x \subseteq F$ for every $s \in S$ and every $x \in F$. Let $F \subseteq E$ be an invariant set and $f \colon F \to \mathbb{R}$. We say that f is an *invariant function* on F if for every $s \in S$ and every $x \in F$,

$$f(F \cap (s \cdot x)) = f(x), \quad \text{i.e. } f(y) = f(x), \quad \text{for all } y \in F \cap (s \cdot x).$$

Let S be a semitopological semigroup. Then *a right linear set-valued action* of S on E is a set-valued action of S on E satisfying:

(1) $(ab) \cdot x = b \cdot (a \cdot x)$, for all $a, b \in S$ and all $x \in E$, where $b \cdot (a \cdot x) := \bigcup_{z \in a \cdot x} b \cdot z$.

(2) For each $s \in S$, the map $x \rightrightarrows s \cdot x$ is a linear relation from E to E.

We denote by $\rightharpoonup$ the weak convergence of nets in E (i.e. convergence in the *weak topology* of E).

We say that a set-valued action of S on E is *continuous* (resp. *weakly continuous*) if for every convergent net $s_\alpha \to s$ in S and every $x \in E$, there exists $z_\alpha \in s_\alpha \cdot x$ such that $z_\alpha \to z \in s \cdot x$ (resp. $z_\alpha \rightharpoonup z \in s \cdot x$). We shall use a right linear set-valued action of S on E, following the ideas in [34], [18], [39], to show the amenability of S with Hahn–Banach extension theorem in Section 2.

The rest of this paper are organized as follows. In Section 2, we present our first main result: Theorem 2.2. In Section 3, we establish the existence of left invariant means on the spaces $\mathrm{AP}(S)$ and $\mathrm{WAP}(S)$ as our second main result. In Section 4, we refine and extend the well known Day's result and Mitchell's result for set-valued mappings: Theorems 4.1 and 4.2 as well as Theorem 4.4. An extension of Mitchell's result in $\mathrm{WLUC}(S)$ and some characterizations of a left invariant mean on $\mathrm{AP}(S)$ are presented in Section 5. In Section 6, we present an application of our result to a class of Banach algebras related to amenability of groups and semigroups. Some open interesting problems are listed in Section 7.

The original definition of amenability, in terms of a finitely additive invariant measure (or mean) on subsets of a locally compact topological group G, was introduced by John von Neumann in 1929. In 1950, Mahlon M. Day introduced the notion of amenable semigroups (see [6], [7], [12], and [27]). The notion is later extended to Banach algebras (see [15]). A semigroup S is left amenable if and only if the Banach algebra $\ell^1(S)$ is left amenable ([21]).

2. Amenability of a semitopological semigroup S

We first introduce some preliminary properties of linear relations.

Lemma 2.1 (Cross, see [4, Chapter 1] or [40, Proposition 3.1.3, p. 15]). *Let $A\colon E \rightrightarrows E$ be a linear relation. Then the following hold*:

(a) $A0$ *is a linear subspace of* E.
(b) $Ax = y + A0$ *for all* $(x, y) \in \operatorname{gra} A$.
(c) $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ *for all* $x, y \in \operatorname{dom} A$ *and for all* $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Theorem 2.2 is our first main result which shows the amenability of a semitopological semigroup S is equivalent to Hahn–Banach extension properties, which is inspired by [18, Theorem 1].

Theorem 2.2. *Let S be a semitopological semigroup. The following conditions on S are equivalent*:

(a) *S is left amenable.*
(b) *For any continuous right linear set-valued action of S on E, if P is a continuous sublinear function on E such that* $\sup P(s \cdot x) \leq P(x)$ *for all $s \in S$, $x \in E$, and if L is an invariant linear functional on an invariant subspace F of E such that $L \leq P$ on F, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \leq P$.*
(c) *For any continuous right linear set-valued action of S on E, if U is a totally invariant open convex subset of E containing an invariant element, and M is an invariant subspace of E with $M \cap U = \emptyset$, then there exists a totally invariant closed hyperplane H of E such that $M \subseteq H$ and $H \cap U = \emptyset$.*
(d) *For any continuous right linear set-valued action of S on E with a base of the neighbourhoods of the origin consisting of totally invariant open convex sets, if M is an invariant closed subspace of E, then any two elements $x, y \in E$ can be separated by a continuous invariant linear functional on E provided $x - y \notin M$ and $x - y$ is an invariant element.*
(e) *For any continuous right linear set-valued action of S on E with a base of the neighbourhoods of the origin consisting of totally invariant open convex sets, then any two distinct points in the set E_f, the set of invariant elements in E, i.e. $E_f := \{x \in E \mid x \in s \cdot x \text{ for all } s \in S\}$ can be separated by a continuous invariant linear functional on E.*

Proof. (a) $\Rightarrow$ (b) By the assumption, $\sup P(s \cdot 0) \leq P(0) = 0$ for all $s \in S$. Lemma 2.1 (a) shows that $s \cdot 0$ is a subspace of E. Then, for every $s \in S$,

$$0 = P(0) = P(x - x) \leq P(x) + P(-x) \leq 0 + 0 = 0 \quad \text{for all } x \in s \cdot 0$$

and then $P(x) = 0$, which is

$$P(s \cdot 0) \equiv 0, \quad \text{for all } s \in S. \tag{2.1}$$

By Hahn–Banach extension theorem, there exists a linear function $\phi\colon E \to \mathbb{R}$ such that $\phi \le P$ on E and $\phi|_F = L$. By (2.1), we have $\phi(s \cdot 0) \equiv 0$, for all $s \in S$. Hence, for every $s \in S$ and $x \in E$, Lemma 2.1 (b) implies that

$$\phi(s \cdot x) = \phi(y), \quad \text{for all } y \in s \cdot x. \tag{2.2}$$

Fix $x \in E$ and define $f_x\colon S \to \mathbb{R}$ by

$$f_x(s) := \phi(s \cdot x), \quad \text{for all } s \in S.$$

By (2.2), f_x is well defined and $f_x \in \ell^\infty(S)$ since $\|f_x\|_\infty \le |P(x)| + |P(-x)|$. Now we show that $f_x \in \mathrm{LUC}(S)$. Let $s_\alpha \longrightarrow s$ in S. By the continuity of the action of S on E, there exists $z_\alpha \in s_\alpha \cdot x$ such that $z_\alpha \to z \in s \cdot x$. Then

$$\begin{aligned} f_x(s_\alpha t) - f_x(st) &= \phi(s_\alpha t \cdot x) - \phi(st \cdot x) = \phi(t \cdot (s_\alpha \cdot x)) - \phi(t \cdot (s \cdot x)) \\ &= \phi(t \cdot z_\alpha) - \phi(t \cdot z) = \phi(t \cdot (z_\alpha - z)) \end{aligned} \tag{2.3}$$

(by (2.2) and Lemma 2.1 (c))

$$\le \sup P(t \cdot (z_\alpha - z)) \le P(z_\alpha - z),$$

for all $t \in S$. Similarly,

$$f_x(st) - f_x(s_\alpha t) \le P(z - z_\alpha), \quad \text{for all } t \in S. \tag{2.4}$$

Then, combining (2.3) and (2.4),

$$\|l_{s_\alpha}(f_x) - l_s(f_x)\|_\infty = \sup_{t \in S} |f_x(s_\alpha t) - f_x(st)| \le |P(z_\alpha - z)| + |P(z - z_\alpha)| \to 0$$

(since $z_\alpha - z \to 0$). On the other hand, the continuity of ϕ shows that

$$|f_x(s_\alpha) - f_x(s)| = |\phi(s_\alpha \cdot x) - \phi(s \cdot x)| = |\phi(z_\alpha) - \phi(z)| = |\phi(z_\alpha - z)| \to 0.$$

Hence f_x is a continuous function on S and then $f_x \in \mathrm{LUC}(S)$ for all $x \in E$.

Let m be a LIM on $\mathrm{LUC}(S)$. We define $\widetilde{L}\colon E \to \mathbb{R}$ by

$$\widetilde{L}(x) := m(f_x), \quad \text{for all } x \in E.$$

By Lemma 2.1 (c), for every $x, y \in E$ and every $\gamma \in \mathbb{R}$,

$$f_{x+y}(s) = f_x(s) + f_y(s) \quad \text{and} \quad f_{\gamma x}(s) = \gamma f_x(s), \quad \text{for all } s \in S.$$

Then $\widetilde{L}$ is a linear function on E.

Let $x \in F$. We show that $\widetilde{L}(x) = L(x)$. Since L is an invariant function on the invariant subspace F and $\phi|_F = L$,

$$f_x(s) = \phi(s \cdot x) = L(F \cap (s \cdot x)) = L(x) \quad \text{for all } s \in S.$$

Hence f_x is a constant function in LUC(S). Then, for every $x \in F$,

$$\widetilde{L}(x) = m(f_x) = m(L(x)) = L(x).$$

Furthermore, let $x \in E$. We have

$$\widetilde{L}(x) = m(f_x) \le \sup_{s\in S} f_x(s) = \sup_{s\in S} \phi(s\cdot x) \le \sup_{s\in S}\{\sup P(s\cdot x)\} \le P(x).$$

Hence $\widetilde{L} \le P$. Therefore, $\widetilde{L}$ is continuous.

Next we will show that $\widetilde{L}$ is an invariant functional on E. Let $s \in S$ and $x \in E$. Let $y \in s\cdot x$. We have

$$(l_s(f_x))(t) = f_x(st) = \phi(st\cdot x) = \phi(t\cdot(s\cdot x)) = \phi(t\cdot y) = f_y(t), \tag{2.5}$$

for all $t \in S$. Thus $l_s(f_x) = f_y$ and then

$$\widetilde{L}(y) = m(f_y) = m(l_s(f_x)) = m(f_x) = \widetilde{L}(x),$$

for all $y \in s\cdot x$. Thus $\widetilde{L}$ is an invariant function. Hence $\widetilde{L}$ is a continuous invariant linear extension of L to E such that $\widetilde{L} \le P$.

(b) $\Rightarrow$ (c) Let e be an invariant element in U and set $W := U - e$. We first show that W is a totally invariant set. Let $s \in S$ and $u \in U$. Take $y \in s\cdot u$. Since $e \in s\cdot e$, Lemma 2.1 (c) shows that $y - e \in s\cdot u - s\cdot e = s\cdot(u-e)$. Thus, by Lemma 2.1 (b),

$$s\cdot(u-e) = (y-e) + s\cdot 0 = y + s\cdot 0 - e = s\cdot u - e \subseteq U - e = W.$$

Hence W is a totally invariant open convex set with $0 \in W$.

Let P be the *Minkowski functional* on E for W, i.e.

$$P(x) := \inf\{\lambda > 0 \mid x \in \lambda W\}, \quad \text{for all } x \in E.$$

Then P is sublinear, non-negative and continuous on E by [33, Theorem 1.35, p. 26]. Since W is totally invariant,

$$\sup P(s\cdot x) \le P(x), \quad \text{for all } s \in S, \text{ for all } x \in E. \tag{2.6}$$

Let F be the linear span of M and e. Then F is an invariant subspace. We define $L\colon F \to \mathbb{R}$ by (for every $x \in F$)

$$L(x) := \lambda, \quad \text{if } x = h - \lambda e \text{ with } h \in M.$$

Then $L(e) = -1$ and L is a linear functional on F. Now we claim that

$$L \le P \quad \text{on } F. \tag{2.7}$$

Let $x \in F$. Suppose to the contrary that $\lambda > P(x)$, where $\lambda := L(x)$. Thus $\lambda > 0$ and $x \in \lambda W = \lambda(U - e) = \lambda U - \lambda e$, which shows that $x/\lambda + e \in U$.

On the other hand, since $\lambda = L(x)$, there exists $h \in M$ such that

$$x = h - \lambda e \quad \text{and thus} \quad \frac{x}{\lambda} + e = \frac{h}{\lambda} \in M.$$

Hence $x/\lambda + e \in M \cap U$, which contradicts that $M \cap U = \emptyset$. Therefore, (2.7) holds.

Next we show that L is an invariant function on F. Let $s \in S$ and $y \in F$. Combining (2.6) and (2.7), we have

$$P(s \cdot 0) = 0 \quad \text{and hence} \quad L(F \cap (s \cdot 0)) = 0.$$

Thus, by Lemma 2.1 (b),

$$L(F \cap (s \cdot y)) = L(z) \quad \text{for all } z \in F \cap (s \cdot y) \tag{2.8}$$

(since $F \cap (s \cdot y) = z + F \cap (s \cdot 0)$). Set $\delta := L(y)$. Then there exists $g \in M$ such that $y = g - \delta e$. Let $v \in (s \cdot g) \cap M$. Then $v - \delta e \in F$ and Lemma 2.1 (c) shows that

$$v - \delta e \in s \cdot g - \delta(s \cdot e) = s \cdot (g - \delta e) = s \cdot y.$$

Thus (2.8) implies that

$$L(F \cap (s \cdot y)) = L(v - \delta e) = \delta = L(y).$$

Hence L is an invariant linear function on F. Thus, by (b), we can obtain a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \leq P$. Then $H := \ker \widetilde{L}$ is a closed totally invariant hyperplane of E containing M.

Now we show that $H \cap U = \emptyset$. Suppose to the contrary that there exists $x \in H \cap U$. Then we have $\widetilde{L}(x) = 0$ and $x - e \in U - e = W$. Thus

$$\widetilde{L}(x) - (-1) = \widetilde{L}(x) - L(e) = \widetilde{L}(x) - \widetilde{L}(e) = \widetilde{L}(x - e) \leq P(x - e) < 1.$$

The above inequality shows that $\widetilde{L}(x) < 0$, which contradicts that $\widetilde{L}(x) = 0$. Hence $H \cap U = \emptyset$.

(c) $\Rightarrow$ (d) Let $x, y \in E$ be with $x - y \notin M$ such that $x - y$ is an invariant element. Since M is closed, by the assumption and [33, Theorem 1.10, p. 10], there exists a totally invariant open convex set V with $0 \in V$ such that $U \cap M = \emptyset$, where $U := x - y + V$. Thus U is a totally invariant open convex set and contains an invariant element: $x - y$. So, by (c), we can obtain a totally invariant closed hyperplane H such that $x - y \notin H$. We define $L \colon E \to \mathbb{R}$, for every $z \in E$, by

$$L(z) := \lambda, \quad \text{if } z = h + \lambda(x - y) \text{ with } h \in H.$$

Then L is linear continuous, and invariant since $s \cdot 0 \subseteq H$ for all $s \in S$. Since $L(x) - L(y) = L(x - y) = 1$, $L(x) \neq L(y)$.

(d) $\Rightarrow$ (e) Let $x, y \in E_f$ with $x \neq y$. We have $M := \{0\}$ is an invariant closed subspace of E. Clearly, $x - y$ is an invariant element by Lemma 2.1 (c) and $x - y \notin M$. Then, applying (d), x and y can be separated by a continuous invariant linear functional on E.

(e) $\Rightarrow$ (a) Apply [18, Theorem 1 (a), (d)]. $\square$

COROLLARY 2.3. *Let S be a semitopological semigroup. If S is abelian, a solvable group, or a compact semigroup with finite intersection property for closed right ideals, then S has properties* (b)–(e) *of Theorem 2.2.*

PROOF. In all above cases, S is left amenable (see [9], [10]). Then apply Theorem 2.2 directly. $\square$

Similarly, we have the following result for $\ell^\infty(S)$. Note that it is traditional to assume that an algebraic semigroup S to be a topological semigroup with the discrete topology. Then, S left amenable is equivalent to that $\ell^\infty(S)$ has a left invariant mean.

THEOREM 2.4. *Let S be algebraically a semigroup. The following conditions on S are equivalent*:

(a) *S is left amenable.*

(b) *For any right linear set-valued action of S on E, if P is a sublinear function on E such that* $\sup P(s\cdot x) \le P(x)$ *for all $s \in S$, $x \in E$, and if L is an invariant linear functional on an invariant subspace F of E such that $L \le P$ on F, then there exists an invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \le P$.*

PROOF. (a) $\Rightarrow$ (b) Follow the proof of Theorem 2.2, (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) We define a right linear action of S on $\ell^\infty(S)$ by

$$s \cdot f := l_s f, \quad \text{for all } s \in S, \text{ for all } f \in \ell^\infty(S).$$

Then the above the right linear action is well defined. Set $P := \|\cdot\|_\infty$ on $\ell^\infty(S)$. So we have P is a sublinear function and

$$P(s\cdot f) = P(l_s f) = \|l_s f\|_\infty \le \|f\|_\infty = P(f), \quad \text{for all } s \in S, \text{ for all } f \in \ell^\infty(S).$$

Let F be the subspace of $\ell^\infty(S)$, which consists of all constant functions. Thus F is an invariant subspace. Fix $a \in S$. Define $L\colon F \to \mathbb{R}$ by

$$L(f) := f(a), \quad \text{for all } f \in F.$$

Thus L is an invariant linear functional on F and $L(f) = f(a) \le \|f\|_\infty = P(f)$ for all $f \in F$. Then, by (b), there exists an invariant linear extension function $\widetilde{L}$ of L on $\ell^\infty(S)$ such that $\widetilde{L} \le P = \|\cdot\|_\infty$. Hence $\widetilde{L}$ is continuous and thus $\widetilde{L} \in (\ell^\infty(S))^*$ with $\|\widetilde{L}\| \le 1$. We also have $\widetilde{L}(1) = L(1) = 1$. Then $m := \widetilde{L}$ is a left invariant mean of $\ell^\infty(S)$. Hence S is left amenable. $\square$

3. Existence of a LIM on AP(S) and WAP(S)

In this section, we will follow the ideas in [20], [11] to present some results on the existence of a left invariant mean on AP(S) and WAP(S).

Let S be a semitopological semigroup. We say that a set-valued action of S on E is *almost periodic* (resp. *weakly almost periodic*) of S on E if for each $x \in E$, there exists $z_s \in s \cdot x$ with $s \in S$ such that the set $\{z_s \mid s \in S\}$ is relatively compact in the topology of E (resp. weak topology).

Theorem 3.1 is inspired by [11, Theorems 1 and 3] by Fan and [20, Theorem 1], which is our second main result and also extends [34, Theorem 15.A] by Silverman.

THEOREM 3.1. *Let S be a semitopological semigroup. The following conditions on S are equivalent*:

(a) $\mathrm{AP}(S)$ *has a left invariant mean.*

(b) *For any almost periodic weakly continuous right linear set-valued action of S on E, if P is a continuous sublinear function on E such that $\sup P(s \cdot x) \leq P(x)$ for all $s \in S$, $x \in E$, and if L is an invariant linear functional on an invariant subspace F of E such that $L \leq P$ on F, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \leq P$.*

(c) *For any almost periodic weakly continuous right linear set-valued action of S on E, if F is an invariant subspace of E and U is a nonempty convex subset of E such that $U - e$ is a totally invariant set for some point $e \in F \cap \operatorname{int} U$, then for each invariant linear functional L on F such that $L(x) \leq \alpha$ for all $x \in F \cap U$ and some fixed real number α, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L}(x) \leq \alpha$ for all $x \in U$.*

PROOF. (a) $\Rightarrow$ (b) By the assumption,

$$\sup P(s \cdot 0) \leq P(0) = 0, \quad \text{for all } s \in S. \tag{3.1}$$

By Hahn–Banach extension theorem, there exists a linear function $\phi\colon E \to \mathbb{R}$ such that $\phi \leq P$ on E and $\phi|_F = L$. Since $s \cdot 0$ is a subspace (see Lemma 2.1 (a)), (3.1) and Lemma 2.1 (b) show that, for all $s \in S$ and $x \in E$,

$$\phi(s \cdot 0) \equiv 0 \quad \text{and} \quad \phi(s \cdot x) = \phi(y), \quad \text{for all } y \in s \cdot x. \tag{3.2}$$

Fix $x \in E$ and define $f_x\colon S \to \mathbb{R}$ by

$$f_x(s) := \phi(y) = \phi(s \cdot x) \quad (\text{for all } y \in s \cdot x), \quad \text{for all } s \in S.$$

Then f_x is well defined by (3.2) and $f_x \in \ell^\infty(S)$ since $\|f_x\|_\infty \leq |P(x)| + |P(-x)|$. Now we first show that $f_x \in \mathrm{CB}(S)$. Let $s_\alpha \to s$ in S. By the weak continuity of the action of S on E, there exists $z_\alpha \in s_\alpha \cdot x$ such that $z_\alpha \rightharpoonup z \in s \cdot x$. Then the continuity of ϕ shows that

$$|f_x(s_\alpha) - f_x(s)| = |\phi(s_\alpha \cdot x) - \phi(s \cdot x)| = |\phi(z_\alpha) - \phi(z)| = |\phi(z_\alpha - z)| \to 0.$$

Hence $f_x \in \mathrm{CB}(S)$.

Next we will show that $f_x \in \mathrm{AP}(S)$. Let $s \in S$. Following the proof of (2.5), we have $l_s(f_x) = f_y$, for all $y \in s \cdot x$. Since the action of S on E is almost periodic, there exists $z_s \in s \cdot x$ such that $\{z_s \mid s \in S\}$ is relatively compact in E. Hence the above equation implies that

$$\mathcal{L}O(f_x) = \{l_s(f_x) \mid s \in S\} = \{f_{z_s} \mid s \in S\}. \tag{3.3}$$

Next we claim that

$$\|f_v - f_w\|_\infty \le |P(v-w)| + |P(w-v)|, \quad \text{for all } v, w \in E. \tag{3.4}$$

Let $v, w \in E$. By (3.2) and Lemma 2.1 (c), for all $s \in S$,

$$\begin{aligned} f_v(s) - f_w(s) &= \phi(s \cdot v) - \phi(s \cdot w) \\ &= \phi(s \cdot (v-w)) \le \sup P(s \cdot (v-w)) \le P(v-w). \end{aligned} \tag{3.5}$$

Similarly, $f_w(s) - f_v(s) \le P(w-v)$, for all $s \in S$. Thus combining with (3.5), we have (3.4) holds.

Now combining (3.3) and (3.4), we have $\mathcal{L}O(f_x)$ is relatively compact and then $f_x \in \mathrm{AP}(S)$ for all $x \in E$.

Let m be a LIM on $\mathrm{AP}(S)$. We define $\widetilde{L}\colon E \to \mathbb{R}$ by

$$\widetilde{L}(x) := m(f_x), \quad \text{for all } x \in E.$$

Similarly to the corresponding part in the proof of Theorem 2.2, (a) $\Rightarrow$ (b), $\widetilde{L}$ is a continuous invariant linear extension of L to E such that $\widetilde{L} \le P$.

(b) $\Rightarrow$ (c) Clearly, it holds when $L \equiv 0$ on F. Now we suppose that $L \not\equiv 0$ on F. We first claim that $L(e) < \alpha$.

Since $L \not\equiv 0$ on F and L is linear, there exists $v \in F$ such that $L(v) > 0$. By $e \in F \cap \operatorname{int} U$, there exists $t > 0$ such that $e + tv \in F \cap U$. Since $L \le \alpha$ on $F \cap U$, the above equation shows that

$$L(e) < L(e) + tL(v) = L(e + tv) \le \alpha.$$

Hence $L(e) < \alpha$.

Now we define $L_0\colon F \to \mathbb{R}$ by

$$L_0(x) := \frac{L(x)}{\alpha - L(e)}, \quad \text{for all } x \in F.$$

Then L_0 is an invariant linear function on F since L is an invariant linear function on F.

Set $W := U - e$. Let $P\colon E \to \mathbb{R}$ be defined by

$$P(x) := \inf\{\lambda > 0 \mid x \in \lambda W\}, \quad \text{for all } x \in E.$$

We have P is sublinear, non-negative and continuous on E by [33, Theorem 1.35, p. 26]. Since W is a totally invariant set,

$$\sup P(s \cdot x) \le P(x), \quad \text{for all } s \in S, \text{ for all } x \in E. \tag{3.6}$$

Now we show that

$$L_0 \le P \quad \text{on } F. \tag{3.7}$$

Let $x \in F$. Suppose to the contrary that $\lambda - \delta > P(x)$, where $\lambda := L_0(x)$ and $\delta > 0$. Thus $\lambda - \delta > 0$ and then

$$P\biggl(\frac{x}{\lambda-\delta}\biggr) = \frac{1}{\lambda-\delta}P(x) < 1.$$

This implies that $x/(\lambda-\delta) \in W = U - e$. Hence $x/(\lambda-\delta) + e \in U \cap F$ since $e, x \in F$. Thus

$$\frac{1}{\lambda-\delta}L(x) + L(e) = L\biggl(\frac{x}{\lambda-\delta} + e\biggr) \le \alpha$$

and then

$$\frac{L(x)}{\alpha - L(e)} \le \lambda - \delta = L_0(x) - \delta,$$

which is a contradiction with that $L(x)/(\alpha - L(e)) = L_0(x)$. Hence (3.7) holds. Then, applying (b), there exists a continuous invariant linear extension $\widetilde{L_0}$ of L_0 to E such that $\widetilde{L_0} \le P$. Therefore, $\widetilde{L_0}(y) \le P(y) \le 1$ for all $y \in W = U - e$. Then, for every $x \in U$,

$$\begin{aligned}\widetilde{L_0}(x) &= \widetilde{L_0}(x) - \widetilde{L_0}(e) + \widetilde{L_0}(e)\\ &= \widetilde{L_0}(x-e) + L_0(e) \le 1 + \frac{L(e)}{\alpha - L(e)} = \frac{\alpha}{\alpha - L(e)}.\end{aligned}$$

Now define $\widetilde{L}\colon E \to \mathbb{R}$ by $\widetilde{L} := (\alpha - L(e))\widetilde{L_0}$. Then $\widetilde{L}$ is a continuous invariant linear extension of L to E such that $\widetilde{L}(x) \le \alpha$ for all $x \in U$.

(c) $\Rightarrow$ (a) Apply [20, Theorem 1 (b) and (a)] directly. □

With a proof similar to that of Theorem 3.1, we can have the following result for WAP(S).

THEOREM 3.2. *Let S be a semitopological semigroup. The following conditions on S are equivalent*:

(a) WAP(S) *has a left invariant mean.*

(b) *For any weakly almost periodic weakly continuous right linear set-valued action of S on E, if P is a continuous sublinear function on E such that $\sup P(s \cdot x) \le P(x)$ for all $s \in S$, $x \in E$, and if L is an invariant linear functional on an invariant subspace F of E such that $L \le P$ on F, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \le P$.*

(c) *For any weakly almost periodic weakly continuous right linear set-valued action of S on E, if F is an invariant subspace of E and U is a nonempty convex subset of E such that $U - e$ is a totally invariant set for some point $e \in F \cap \operatorname{int} U$, then for each invariant linear functional L on F*

such that $L(x) \leq \alpha$ for all $x \in F \cap U$ and some fixed real number α, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L}(x) \leq \alpha$ for all $x \in U$.

A semitopological semigroup is *left reversible* if any two closed right ideals $\overline{aS}$ and $\overline{bS}$ have non-void intersection (for every $a, b \in S$) (see [14]).

Corollary 3.3. *If S is a left reversible semitopological semigroup and normal, then* (b) *and* (c) *of Theorem* 3.2 *hold* (*see* [17]).

Remark 3.4. Some other generalizations of Hahn–Banach extension theorems can be found in [28], [29], [16], [14], [12], [23].

4. An extension of Day's fixed point properties

In this section, we will refine and extend the well known Day's results (see [8], [1], [26]) for set-valued mappings: Theorems 4.1 and 4.4.

Let S be a semitopological semigroup. Then *a linear set-valued action* of S on E is a set-valued action of S on E satisfying:

(1) $(ab) \cdot x = a \cdot (b \cdot x)$, for all $a, b \in S$ and all $x \in E$.
(2) For each $s \in S$, the map $x \rightrightarrows s \cdot x$ is a linear relation from E to E.

Let $C \subseteq E$ be nonempty. We say a set-valued action of S on E is *right closed on the set C* if for each $s \in S$, the set $\{(x, y) \in E \times E \mid y \in s \cdot x\} \cap C \times C$ is closed in $E \times E$.

We say a set-valued action of S on E is *left closed on the set C* if, for each $x \in C$, the implication

$$\text{if } s_\alpha \to s \text{ and } \ z_\alpha \in (s_\alpha \cdot x) \cap C \text{ with } z_\alpha \to z \text{ then } z \in s \cdot x$$

holds. A set-valued action of S on E is *separately closed on the set C* if it is left closed on the set C and right closed on the set C. We say a set-valued action of S on E is *jointly closed on the set C* if for every convergent net $s_\alpha \to s$ in S and every convergent net $x_\alpha \to x$ in C such that the implication

$$\text{if } z_\alpha \in (s_\alpha \cdot x_\alpha) \cap C \text{ with } z_\alpha \to z \text{ then } z \in s \cdot x$$

holds.

Let $s \in S$. The linear functional $\boldsymbol{\delta}_s \colon \ell^\infty(S) \to \mathbb{R}$ is defined by

$$\boldsymbol{\delta}_s(f) := f(s), \quad \text{for all } f \in \ell^\infty(S).$$

Then we have $\boldsymbol{\delta}_s \in (\ell^\infty(S))^*$.

Let $C \subseteq E$ be a nonempty convex set and $f \colon C \to \mathbb{R}$. We say that f is *an affine function* on C if, for every $x, y \in C$,

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \quad \text{for all } t \in [0, 1].$$

THEOREM 4.1. *Let S be a semigroup. The following conditions on S are equivalent*:

(a) *S is left amenable.*

(b) *For any linear set-valued action of S on E such that $(st)\cdot 0 = s\cdot 0$ (for all $s,t\in S$), if $C\subseteq E$ is an invariant convex compact set and the action of S on E is right closed on C, then $E_f := \{x\in C \mid x\in s\cdot x$, for all $s\in S\}\neq\emptyset$.*

PROOF. (a) $\Rightarrow$ (b) Let the space Y be defined as

$Y :=$ the space of all affine continuous real-valued functions on the set C.

Fix $c\in C$. Since C is an invariant subset of E, there exists $c_s\in(s\cdot c)\cap C$ for every $s\in S$. We define the mapping $L^c\colon Y\to\ell^\infty(S)$ by (for every $f\in Y$)

$$(L^c f)(s) := f(c_s), \quad \text{for all } s\in S.$$

Since S is left amenable with the discrete topology, $\ell^\infty(S)$ has a left invariant mean. Let u be a left invariant mean of $\ell^\infty(S)$. By Day's result (see [7]), there exists a weak* convergent net $(u_\alpha)_{\alpha\in\Gamma}$ in $(\ell^\infty(S))^*$ such that

$$u_\alpha \xrightarrow{\text{w}^*} u, \tag{4.1}$$

where $u_\alpha := \sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}\boldsymbol{\delta}_{s_{\alpha,i}}$ with $\lambda_{\alpha,i}>0$, $n_\alpha\in\mathbb{N}$, $s_{\alpha,i}\in S$ and $\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}=1$. Since $\left(\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}c_{s_{\alpha,i}}\right)_{\alpha\in\Gamma}$ is in the convex compact set C, there exists a convergent subnet of $\left(\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}c_{s_{\alpha,i}}\right)_{\alpha\in\Gamma}$, still denoted by $\left(\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}c_{s_{\alpha,i}}\right)_{\alpha\in\Gamma}$ for convenience, such that

$$\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}c_{s_{\alpha,i}} \to c_\infty \in C. \tag{4.2}$$

Let $f\in Y$. Then by (4.1), we have

$$\begin{aligned}\langle u, L^c f\rangle &= \lim_\alpha \langle u_\alpha, L^c f\rangle = \lim_\alpha \left\langle \sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}\boldsymbol{\delta}_{s_{\alpha,i}}, L^c f\right\rangle \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}\boldsymbol{\delta}_{s_{\alpha,i}}(L^c f) = \lim_\alpha \sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}(L^c f)s_{\alpha,i} \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}f(c_{s_{\alpha,i}}) = \lim_\alpha f\left(\sum_{i=1}^{n_\alpha}\lambda_{\alpha,i}c_{s_{\alpha,i}}\right)\end{aligned} \tag{4.3}$$

(since f is affine)

$$= f(c_\infty) \quad \text{(by (4.2))}.$$

On the other hand, let $s \in S$. By (4.1) again, we have

$$\begin{aligned} (4.4)\qquad \langle u, l_s(L^c f)\rangle &= \lim_\alpha \langle u_\alpha, l_s(L^c f)\rangle = \lim_\alpha \Big\langle \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}\boldsymbol{\delta}_{s_{\alpha,i}}, l_s(L^c f)\Big\rangle \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}\langle \boldsymbol{\delta}_{s_{\alpha,i}}, [l_s(L^c f)]\rangle = \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}[l_s(L^c f)]s_{\alpha,i} \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}(L^c f)(ss_{\alpha,i}) = \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(c_{ss_{\alpha,i}}) \\ &= \lim_\alpha f\bigg(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{ss_{\alpha,i}}\bigg) \end{aligned}$$

(since f is affine). By Lemma 2.1 (b) and (c), we have

$$\begin{aligned} (4.5)\qquad s\cdot(s_{\alpha,i}\cdot c) &= s\cdot(c_{s_{\alpha,i}} + s_{\alpha,i}\cdot 0) = s\cdot c_{s_{\alpha,i}} + s\cdot(s_{\alpha,i}\cdot 0) \\ &= s\cdot c_{s_{\alpha,i}} + (ss_{\alpha,i})\cdot 0 = s\cdot c_{s_{\alpha,i}} + s\cdot 0 \end{aligned}$$

(by the assumption)

$$= s\cdot(c_{s_{\alpha,i}} + 0) = s\cdot c_{s_{\alpha,i}}.$$

Note that $c_{ss_{\alpha,i}} \in (ss_{\alpha,i})\cdot c = s\cdot(s_{\alpha,i}\cdot c) = s\cdot c_{s_{\alpha,i}}$ by (4.5). Hence

$$(4.6)\qquad \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{ss_{\alpha,i}} \in \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} s\cdot c_{s_{\alpha,i}} = s\cdot\bigg(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\bigg)$$

(by Lemma 2.12.1). Since C is convex compact and $\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{ss_{\alpha,i}} \in C$, the right closeness of the action of S on C and (4.6) imply that all cluster points of $\Big(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{ss_{\alpha,i}}\Big)_\alpha$ are in the set $C \cap (s\cdot c_\infty)$ by (4.2). Thus, by (4.4),

$$(4.7)\qquad \langle u,\, l_s(L^c f)\rangle \le \sup\langle f, C\cap(s\cdot c_\infty)\rangle.$$

Since u is a left invariant mean of $\ell^\infty(S)$, (4.3) and (4.7) give us

$$f(c_\infty) \le \sup\langle f, C\cap(s\cdot c_\infty)\rangle, \quad \text{for all } f \in Y.$$

Hence $c_\infty \in C\cap(s\cdot c_\infty)$ by the Separation Theorem and then

$$c_\infty \in C\cap(s\cdot c_\infty), \quad \text{for all } s \in S.$$

Thus $E_f \neq \emptyset$.

(b) $\Rightarrow$ (a) We define a linear action of S on $(\ell^\infty(S))^*$ by

$$s\cdot u := l_s^* u, \quad \text{for all } s \in S, \text{ for all } u \in (\ell^\infty(S))^*,$$

where l_s^* is the adjoint of l_s. Then the above linear action is well defined since l_s is a continuous linear operator from $\ell^\infty(S)$ to $\ell^\infty(S)$. Let C be set of all means on $\ell^\infty(S)$. Thus we have C is a weak* compact convex set and C is invariant.

Clearly, the linear action of S is right closed on C in the weak* topology of $(\ell^\infty(S))^*$ by the continuity of l_s^*. By (b), there exists $m \in C$, i.e. m is a mean on $\ell^\infty(S)$, such that

$$m = s \cdot m = l_s^* m, \quad \text{for all } s \in S.$$

Hence m is a left invariant mean of $\ell^\infty(S)$. Therefore, S is left amenable. $\square$

Theorem 4.2 is inspired by [26, Theorem 2] by Mitchell, and we shall refine and extend it for set-valued mappings.

THEOREM 4.2. *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *S is left amenable.*
(b) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s \cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is jointly closed on C, then $E_f := \{x \in C \mid x \in (s \cdot x) + F, \text{ for all } s \in S\} \neq \emptyset$.*
(c) *For any linear set-valued action of S on E such that $s \cdot 0 = t \cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is jointly closed on C, then $E_f := \{x \in C \mid x \in s \cdot x, \text{ for all } s \in S\} \neq \emptyset$.*

PROOF. (a) $\Rightarrow$ (b) Let the space Y be defined as $Y := \{x^* \in E^* \mid \langle x^*, F\rangle \equiv 0\}$. Fix $c \in C$. Since C is an invariant subset of E, there exists $c_s \in (s \cdot c) \cap C$ for every $s \in S$. By the definition of the space Y, for all $s \in S$ and $f \in Y$, we have $f(s \cdot 0) \equiv 0$ since $s \cdot 0 \subseteq F$.

We define the mapping $L^c \colon Y \to \ell^\infty(S)$, for every $f \in Y$, by

$$(L^c f)(s) := f(c_s) = f(c_s) + f(s \cdot 0) = f\left(c_s + s \cdot 0\right) = f(s \cdot c), \quad \text{for all } s \in S$$

(by Lemma 2.1 (b)). Let $f \in Y$. Now we show that $L^c f \in \mathrm{LUC}(S)$. Let $s_\gamma \to s$ in S. We first show that

$$|(L^c f)(s_\gamma) - (L^c f)(s)| \to 0. \tag{4.8}$$

Suppose to the contrary that there exists $\lambda > 0$ and a subnet of $(s_\gamma)_\gamma$, still denoted by $(s_\gamma)_\gamma$ for convenience, such that

$$|(L^c f)(s_\gamma) - (L^c f)(s)| = |f(c_{s_\gamma}) - f(s \cdot c)| > \lambda. \tag{4.9}$$

Since $c_{s_\gamma} \in C$ and C is compact, there exists a convergent subnet of $(c_{s_\gamma})_\gamma$, still denoted by $(c_{s_\gamma})_\gamma$ for convenience, such that

$$c_{s_\gamma} \to z. \tag{4.10}$$

Since $c_{s_\gamma} \in s_\gamma \cdot c$ and the action of S on E is jointly closed on C, we have $z \in s \cdot c$ by (4.10). Then by the continuity of f, $f(c_{s_\gamma}) \to f(z) = f(s \cdot c)$, which contradicts (4.9). Hence (4.8) holds and thus $L^c f$ is a continuous function on S. Then $L^c f \in \mathrm{CB}(S)$.

Next we shall show that

$$\|l_{s_\gamma}(L^c f)) - l_s(L^c f))\|_\infty = \sup_{t\in S} |(L^c f)(s_\gamma t) - (L^c f)(st)| \to 0. \tag{4.11}$$

Suppose to the contrary that there exist $\delta > 0$ and a subnet of $(s_\gamma)_\gamma$, still denoted by $(s_\gamma)_\gamma$ for convenience, and $t_\gamma \in S$ such that

$$|(L^c f)(s_\gamma t_\gamma) - (L^c f)(st_\gamma)| > \delta.$$

Thus, by the definition of $L^c f$, we have

$$\begin{aligned} \delta < |(L^c f)(s_\gamma t_\gamma) - (L^c f)(st_\gamma)| &= |f(s_\gamma \cdot (t_\gamma \cdot c)) - f(s \cdot (t_\gamma \cdot c))| \\ &= |f(s_\gamma \cdot c_{t_\gamma}) - f(s \cdot c_{t_\gamma})| \end{aligned} \tag{4.12}$$

by Lemma 2.1 (b). Since $c_{t_\gamma} \in C$ and C is compact, there exists a subnet of $(c_{t_\gamma})_\gamma$, still denote by $(c_{t_\gamma})_\gamma$, such that

$$c_{t_\gamma} \to d \in C. \tag{4.13}$$

Since C is an invariant set, there exists $v_\gamma \in (s_\gamma \cdot c_{t_\gamma}) \cap C$. By the compactness of C, there exists a subnet of $(v_\gamma)_\gamma$, still denote by $(v_\gamma)_\gamma$ for convenience, such that

$$v_\gamma \to v \in C. \tag{4.14}$$

Since the action of S on E is jointly closed on C, (4.13) implies that $v \in s \cdot d$.

On the other hand, since C is an invariant set, there also exists w_γ in $(s \cdot c_{t_\gamma}) \cap C$. By the compactness of C, there exists a subnet of $(w_\gamma)_\gamma$, still denote by $(w_\gamma)_\gamma$ for convenience, such that

$$w_\gamma \to w \in C. \tag{4.15}$$

Since the action of S on E is jointly closed on C, $w \in s \cdot d$ by (4.13) again. Thus combining (4.12), (4.14) and (4.15), we have

$$\delta < \big|f(v_\gamma) - f(w_\gamma)\big| \to |f(v) - f(w)| = |f(v) - f(v)| = 0$$

(since $v, w \in s \cdot d$), which is a contradiction. Hence (4.11) holds. Combining the above results, we have $L^c f \in \mathrm{LUC}(S)$ for all $f \in Y$.

Let u be a left invariant mean of $\mathrm{LUC}(S)$. By Day's result (see [7]), there exists a weak* convergent net $(u_\alpha)_{\alpha\in\Gamma}$ in $(\mathrm{LUC}(S))^*$ such that

$$u_\alpha \xrightarrow{\mathrm{w}^*} u, \tag{4.16}$$

where $u_\alpha := \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \boldsymbol{\delta}_{s_{\alpha,i}}$ with $\lambda_{\alpha,i} > 0$, $n_\alpha \in \mathbb{N}$, $s_{\alpha,i} \in S$ and $\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} = 1$.

Since $\left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\right)_{\alpha\in\Gamma}$ is in the convex compact set C, there exists a convergent subnet of $\left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\right)_{\alpha\in\Gamma}$, still denoted by $\left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\right)_{\alpha\in\Gamma}$ for convenience, such that

$$\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}} \to c_\infty \in C. \tag{4.17}$$

Let $f \in Y$. Then following the corresponding lines in the proof of Theorem 4.1 (a) $\Rightarrow$ (b), we have

$$\langle u,\, L^c f\rangle = \lim_\alpha \langle u_\alpha, L^c f\rangle = \langle f, c_\infty\rangle. \tag{4.18}$$

Let $s \in S$. By Lemma 2.1 (b) and (c), we have

$$\begin{aligned} s\cdot(s_{\alpha,i}\cdot c) &= s\cdot(c_{s_{\alpha,i}} + s_{\alpha,i}\cdot 0) = s\cdot c_{s_{\alpha,i}} + s\cdot(s_{\alpha,i}\cdot 0) \\ &= s\cdot c_{s_{\alpha,i}} + (ss_{\alpha,i})\cdot 0 \subseteq s\cdot c_{s_{\alpha,i}} + F \end{aligned} \tag{4.19}$$

(by the assumption). Then $c_{ss_{\alpha,i}} \in (ss_{\alpha,i})\cdot c = s\cdot(s_{\alpha,i}\cdot c) \subseteq s\cdot c_{s_{\alpha,i}} + F$ by (4.19). Hence, by Lemma 2.1 (c),

$$\begin{aligned} \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{ss_{\alpha,i}} &\in \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i}(s\cdot c_{s_{\alpha,i}} + F) \\ &\subseteq \left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} s\cdot c_{s_{\alpha,i}}\right) + F = s\cdot\left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\right) + F. \end{aligned} \tag{4.20}$$

We have

$$\begin{aligned} \langle u, l_s(L^c f)\rangle &= \lim_\alpha \langle u_\alpha, l_s(L^c f)\rangle = \lim_\alpha \left\langle \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \boldsymbol{\delta}_{s_{\alpha,i}}, l_s(L^c f)\right\rangle \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \langle \boldsymbol{\delta}_{s_{\alpha,i}}, l_s(L^c f)\rangle = \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} (L^c f)(ss_{\alpha,i}) \\ &= \lim_\alpha \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \langle f, c_{ss_{\alpha,i}}\rangle = \lim_\alpha \left\langle f, \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{ss_{\alpha,i}}\right\rangle \\ &\leq \limsup_\alpha \sup \left\langle f, s\cdot\left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\right) + F\right\rangle \quad (\text{by } (4.20)) \\ &= \limsup_\alpha \left\langle f, s\cdot\left(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\right)\right\rangle \qquad (\text{since } \langle f, F\rangle \equiv 0) \\ &\leq \langle f, (s\cdot c_\infty)\rangle \end{aligned} \tag{4.21}$$

(by the closeness of the action on C, the compactness of C and (4.17)).

Since u is a left invariant mean of $\mathrm{LUC}(S)$, (4.18) and (4.21) show that

$$\langle f,\ c_\infty\rangle \leq \langle f, (s\cdot c_\infty)\rangle, \quad \text{for all } f \in Y. \tag{4.22}$$

Since $(s \cdot c_\infty) + F = y + s \cdot 0 + F = y + F$ is closed (for every $y \in s \cdot c_\infty$) by Lemma 2.1 (b), (4.22) shows that $c_\infty \in (s \cdot c_\infty) + F$ by the Separation Theorem. Since $c_\infty \in C$ by (4.17),

$$c_\infty \in ((s \cdot c_\infty) + F) \cap C, \quad \text{for all } s \in S.$$

Hence $E_f \neq \emptyset$.

(b) $\Rightarrow$ (c) Let $F := s \cdot 0$ (for all $s \in S$). Then F is a closed subspace by the assumption. By (b), there exists $c_0 \in C$ such that $c_0 \in s \cdot c_0 + F$, for all $s \in S$. Thus Lemma 2.1 (b) shows that

$$c_0 \in s \cdot c_0 + F = s \cdot c_0 + s \cdot 0 = s \cdot c_0, \quad \text{for all } s \in S.$$

Hence $E_f \neq \emptyset$.

(c) $\Rightarrow$ (a) Apply the proof of [26, Theorem 2, (F2) $\Rightarrow$ (P2)]. □

Theorem 4.3 is inspired by [26, Theorem 1] by Mitchell.

THEOREM 4.3. *Let S be a semitopological semigroup. The following conditions on S are equivalent*:

(a) $\mathrm{LUC}(S)$ *has a multiplicative left invariant mean.*

(b) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s \cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant compact set and the action of S on E is jointly closed on C, then*

$$E_f := \{x \in C \mid x \in (s \cdot x) + F, \ \textit{for all } s \in S\} \neq \emptyset.$$

(c) *For any linear set-valued action of S on E such that $s \cdot 0 = t \cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant compact set and the action of S on E is jointly closed on C, then*

$$E_f := \{x \in C \mid x \in s \cdot x, \ \textit{for all } s \in S\} \neq \emptyset.$$

PROOF. (a) $\Rightarrow$ (b) Let the space Y be defined as

$$Y := \{x^* \in E^* \mid \langle x^*, F \rangle \equiv 0\}.$$

Fix $c \in C$. Since C is an invariant subset of E, there exists $c_s \in (s \cdot c) \cap C$ for every $s \in S$. By the definition of the space Y, for all $s \in S$ and $f \in Y$, we have $f(s \cdot 0) \equiv 0$ since $s \cdot 0 \subseteq F$.

We define the mapping $L^c \colon Y \to \ell^\infty(S)$, for every $f \in Y$, by

$$(L^c f)(s) := f(c_s) = f(c_s) + f(s \cdot 0) = f(c_s + s \cdot 0) = f(s \cdot c), \quad \text{for all } s \in S.$$

Following the corresponding lines in the proof of Theorem 4.2, (a) $\Rightarrow$ (b), we have $L^c f \in LUC(S)$, for all $f \in Y$.

Let u be a multiplicative left invariant mean of $\mathrm{LUC}(S)$. Then there exists a weak* convergent net $(\boldsymbol{\delta}_{s_\alpha})_{\alpha \in \Gamma}$ in $(\mathrm{LUC}(S))^*$ with $s_\alpha \in S$ such that

$$\boldsymbol{\delta}_{s_\alpha} \xrightarrow{\mathrm{w}^*} u. \tag{4.23}$$

Since $(c_{s_\alpha})_{\alpha\in\Gamma}$ is in the compact set C, there exists a convergent subnet of $(c_{s_\alpha})_{\alpha\in\Gamma}$, still denoted by $(c_{s_\alpha})_{\alpha\in\Gamma}$ for convenience, such that

$$c_{s_\alpha} \to c_\infty \in C. \tag{4.24}$$

Let $f \in Y$. We have

$$\langle u, L^c f\rangle = \lim_\alpha \langle \boldsymbol{\delta}_{s_\alpha}, L^c f\rangle = \lim_\alpha (L^c f)(s_\alpha) = \lim_\alpha \langle f,\ c_{s_\alpha}\rangle = \langle f,\ c_\infty\rangle. \tag{4.25}$$

Similar to the proof of (4.19),

$$c_{ss_{\alpha,i}} \in (ss_{\alpha,i})\cdot c = s\cdot(s_{\alpha,i}\cdot c) \subseteq s\cdot c_{s_{\alpha,i}} + F. \tag{4.26}$$

Thus, for every $s \in S$,

$$\begin{aligned}\langle u, l_s(L^c f)\rangle &= \lim_\alpha \left\langle \boldsymbol{\delta}_{s_\alpha}, l_s(L^c f)\right\rangle = \lim_\alpha (L^c f)(ss_\alpha) = \lim_\alpha \langle f, c_{ss_\alpha}\rangle \\ &\le \limsup_\alpha \sup\langle f, s\cdot c_{s_\alpha} + F\rangle \quad \text{(by (4.26))} \\ &= \limsup_\alpha \langle f, s\cdot c_{s_\alpha}\rangle \qquad \text{(since } \langle f, F\rangle \equiv 0) \\ &\le \langle f, s\cdot c_\infty\rangle\end{aligned} \tag{4.27}$$

by the closeness of the action on C, the compactness of C and (4.24). Since u is a left invariant mean of $\mathrm{LUC}(S)$, (4.25) and (4.27) show that, for every $s \in S$)

$$\langle f, c_\infty\rangle \le \langle f, s\cdot c_\infty\rangle, \quad \text{for all } f \in Y.$$

Following the corresponding lines in the proof of Theorem 4.2, (a) $\Rightarrow$ (b), we have

$$c_\infty \in (s\cdot c_\infty + F)\cap C, \quad \text{for all } s \in S.$$

Hence $E_f \neq \emptyset$.

(b) $\Rightarrow$ (c) Follow the proof of Theorem 4.2, (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) Apply the proof of [26, Theorem 1 (F1), $\Rightarrow$ (P1)]. □

Following the proofs of Theorems 4.2 and 4.1, we can obtain the following two results.

THEOREM 4.4. *Let S be a semitopological semigroup. Assume that* $\mathrm{CB}(S)$ *has a left invariant mean. Then the following hold*:

(a) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s\cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is separately closed on C, then $E_f := \{x \in C \mid x \in (s\cdot x) + F,\ \text{for all } s \in S\} \neq \emptyset$.*

(b) *For any linear set-valued action of S on E such that $s\cdot 0 = t\cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is separately closed on C, then $E_f := \{x \in C \mid x \in s\cdot x,$ for all $s \in S\} \neq \emptyset$.*

THEOREM 4.5. *Let S be an algebraic semigroup. The following conditions on S are equivalent*:

(a) *S is left amenable.*
(b) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s \cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is right closed on C, then $E_f := \{x \in C \mid x \in (s \cdot x) + F,\ \text{for all } s \in S\} \neq \emptyset$.*
(c) *For any linear set-valued action of S on E such that $s \cdot 0 = t \cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is right closed on C, then $E_f := \{x \in C \mid x \in s \cdot x,$ for all $s \in S\} \neq \emptyset$.*

5. Invariant means on WLUC(S) and AP(S)

In this section, we present some characterizations on the existence of a left invariant mean on WLUC(S) and AP(S) (see Theorems 5.1 and 5.2), which extend [26, Theorem 4] by Mitchell and [17, Theorem 3.2].

Let S be a semitopological semigroup. Let $WLUC(S)$ be the *space of bounded weakly left uniformly continuous functions* on S, i.e. all $f \in CB(S)$ such that the mappings $a \to l_a f$ from S into CB(S) are weakly continuous. Clearly,

$$\mathrm{LUC}(S) \subseteq \mathrm{WLUC}(S) \subseteq \mathrm{CB}(S).$$

Rao showed that WLUC(S) is a closed subspace of CB(S) containing constants and invariant under translations in [31].

Theorem 5.1 is inspired by Mitchell's result: [26, Theorem 4].

THEOREM 5.1. *Let S be a semitopological semigroup. The following conditions on S are equivalent*:

(a) WLUC(S) *has a left invariant mean.*
(b) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s \cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is separately closed on C, then $E_f := \{x \in C \mid x \in (s \cdot x) + F,\ \text{for all } s \in S\} \neq \emptyset$.*
(c) *For any linear set-valued action of S on E such that $s \cdot 0 = t \cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant convex compact set and the action of S on E is separately closed on C, then $E_f := \{x \in C \mid x \in s \cdot x$ for all $s \in S\} \neq \emptyset$.*

PROOF. (a) $\Rightarrow$ (b) Let $Y := \{x^* \in E^* \mid \langle x^*, F\rangle \equiv 0\}$. Fix $c \in C$. Since C is an invariant subset of E, there exists $c_s \in (s \cdot c) \cap C$ for every $s \in S$. By the definition of the space Y, for all $s \in S$ and $f \in Y$, we have $f(s \cdot 0) \equiv 0$ since $s \cdot 0 \subseteq F$.

We define the mapping $L^c\colon Y \to \ell^\infty(S)$ by (for every $f \in Y$)

$$(L^c f)(s) := f(c_s) = f(c_s) + f(s \cdot 0) = f(c_s + s \cdot 0) = f(s \cdot c), \tag{5.1}$$

for all $s \in S$. Let $f \in Y$. Following the corresponding lines in the proof of Theorem 4.2, (a) $\Rightarrow$ (b), we have $L^c f \in CB(S)$. Now we show that $L^c f \in \mathrm{WLUC}(S)$. Let $s_\gamma \to s$ in S. Let μ be a mean of $\mathrm{CB}(S)$. Now we claim

$$\langle \mu,\ l_{s_\gamma}(L^c f)\rangle \to \langle \mu, l_s(L^c f)\rangle. \tag{5.2}$$

Suppose to the contrary that there exists $\lambda > 0$ and a subnet of $(s_\gamma)_\gamma$, still denoted by $(s_\gamma)_\gamma$ for convenience, such that

$$|\langle \mu, l_{s_\gamma}(L^c f)\rangle - \langle \mu, l_s(L^c f)\rangle| > \lambda. \tag{5.3}$$

By Day's result (see [7]), there exists a weak* convergent net $(\mu_\alpha)_{\alpha\in\Gamma}$ in $(\mathrm{CB}(S))^*$ such that $\mu_\alpha \xrightarrow{\mathrm{w}^*} \mu$, where $\mu_\alpha := \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \boldsymbol{\delta}_{s_{\alpha,i}}$ with $\lambda_{\alpha,i} > 0$, $n_\alpha \in \mathbb{N}$, $s_{\alpha,i} \in S$ and $\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} = 1$. Since $\Big(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\Big)_{\alpha\in\Gamma}$ is in the compact set C, there exists a convergent subnet of $\Big(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\Big)_{\alpha\in\Gamma}$, still denoted by $\Big(\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}}\Big)_{\alpha\in\Gamma}$ for convenience, such that

$$\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} c_{s_{\alpha,i}} \to c_\infty \in C. \tag{5.4}$$

Similar to the proof of (4.21), we have

$$\langle \mu, l_t(L^c f)\rangle \le \langle f, t \cdot c_\infty\rangle, \quad \text{for all } t \in S. \tag{5.5}$$

Note that $L^c(-f) = -L^c f$ by (5.1) and then $l_t(L^c(-f)) = -l_t(L^c f)$ for all $t \in S$. Hence, by (5.5),

$$-\langle \mu, l_t(L^c f)\rangle = \langle \mu, l_t(L^c(-f))\rangle \le \langle -f, t \cdot c_\infty\rangle, \quad \text{for all } t \in S.$$

Then, combining with (5.5), we have

$$\langle \mu, l_t(L^c f)\rangle = \langle f, t \cdot c_\infty\rangle, \quad \text{for all } t \in S. \tag{5.6}$$

Thus (5.3) and (5.6) show that

$$|\langle f, s_\gamma \cdot c_\infty\rangle - \langle f, s \cdot c_\infty\rangle| > \lambda. \tag{5.7}$$

Since C is an invariant set, (5.4) implies that there exists $v_\gamma \in (s_\gamma \cdot c_\infty) \cap C$. By the compactness of C, there exists a subnet of $(v_\gamma)_\gamma$, still denote by $(v_\gamma)_\gamma$ for convenience, such that

$$v_\gamma \to v \in C. \tag{5.8}$$

Since the action of S on E is separately closed on C and $s_\gamma \to s$, (5.8) implies that $v \in s \cdot c_\infty$. Thus, by the continuity of f, (5.8) and (5.1) show that

$$\langle f, s_\gamma \cdot c_\infty\rangle = \langle f, v_\gamma\rangle \to \langle f, v\rangle = \langle f, s \cdot c_\infty\rangle,$$

which contradicts (5.7). Hence (5.2) holds.

Since each $y^* \in (\mathrm{CB}(S))^*$ can be expressed as a linear combination of two means on $\mathrm{CB}(S)$, (5.2) shows that

$$\langle y^*, l_{s_\gamma}(L^c f)\rangle \to \langle y^*, l_s(L^c f)\rangle, \quad \text{for all } y^* \in (\mathrm{CB}(S))^*.$$

Hence $L^c f \in \mathrm{WLUC}(S)$.

Following the corresponding parts in the proof of Theorem 4.2, (a) $\Rightarrow$ (b), we have (b) holds.

(b) $\Rightarrow$ (c) Follow the proof of Theorem 4.2, (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) Apply the same proof of [26, Theorem 4, (F4) $\Rightarrow$ (P4)]. $\square$

Given a set-valued action of S on E, let $C \subseteq E$ be an invariant compact set. We say that the action of S on C is *equicontinuous* if, for each $x \in C$ and each open neighbourhood U of 0 in E, there exists an open neighbourhood V of x in E such that, for every $y \in V \cap C$ and every $s \in S$, there exist $v \in (s \cdot y) \cap C$ and $w \in (s \cdot x) \cap C$ with $v - w \in U$.

Theorem 5.2 is inspired by [17, Lemma 3.1 and Theorem 3.2].

THEOREM 5.2. *Let S be a semitopological semigroup. The following conditions on S are equivalent*:

(a) $\mathrm{AP}(S)$ *has a left invariant mean.*

(b) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s \cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant convex compact set such that the action of S on E is separately closed on C and the action of S on C is equicontinuous, then $E_f := \{x \in C \mid x \in (s \cdot x) + F,\ \text{for all } s \in S\} \neq \emptyset$.*

(c) *For any linear set-valued action of S on E such that $s \cdot 0 = t \cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant convex compact set such that the action of S on E is separately closed on C and the action of S on C is equicontinuous, then $E_f := \{x \in C \mid x \in s \cdot x,\ \text{for all } s \in S\} \neq \emptyset$.*

PROOF. (a) $\Rightarrow$ (b) Let the space Y be defined as $Y := \{x^* \in E^* \mid \langle x^*, F\rangle \equiv 0\}$. Fix $c \in C$. Since C is an invariant subset of E, there exists $c_s \in (s \cdot c) \cap C$ for every $s \in S$. By the definition of the space Y, for all $s \in S$ and $f \in Y$, we have $f(s \cdot 0) \equiv 0$ since $s \cdot 0 \subseteq F$.

We define the mapping $L^c \colon Y \to \ell^\infty(S)$, for every $f \in Y$, by

$$(L^c f)(s) := f(c_s) = f(c_s) + f(s \cdot 0) = f(c_s + s \cdot 0) = f(s \cdot c), \tag{5.9}$$

for all $s \in S$. Let $f \in Y$. Following the corresponding lines in the proof of Theorem 4.2, (a) $\Rightarrow$ (b), we have $L^c f \in \mathrm{CB}(S)$. Now we show that $L^c f \in \mathrm{AP}(S)$. It suffices to show that

$$\mathcal{RO}(L^c f) = \{r_a(L^c f) \mid a \in S\} \quad \text{is relatively compact in } \ell^\infty(S).$$

We have, for every $a \in S$,

$$(r_a(L^c f))(t) = (L^c f)(ta) = f(ta \cdot c) = f(t \cdot (a \cdot c)) = f(t \cdot c_a) = (L^{c_a} f)(t),$$

for all $t \in S$ (by (5.9)). Thus $r_a(L^c f) = L^{c_a} f$ for all $a \in S$. Then

$$\mathcal{RO}(L^c f) = \{L^{c_a} f \mid a \in S\}. \tag{5.10}$$

On the other hand, let $\varepsilon > 0$ and set

$$U_\varepsilon := f^{-1}\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right) := \left\{x \in E \;\middle|\; |f(x)| < \frac{\varepsilon}{2}\right\}.$$

Then U_ε is an open neighbourhood of 0 in E by the continuity of f. Let $x \in C$. Since the action of S on C is equicontinuous, there exists an open neighbourhood V of x in E such that, for each $y \in V \cap C$ and $s \in S$, there exist $v_{y,s} \in (s \cdot y) \cap C$ and $w_{y,s} \in (s \cdot x) \cap C$ with $v_{y,s} - w_{y,s} \in U$. Then

$$|f(v_{y,s}) - f(w_{y,s})| = |f(v_{y,s} - w_{y,s})| < \frac{\varepsilon}{2}, \quad \text{for all } y \in V \cap C.$$

Hence (5.9) shows that

$$\|L^y f - L^x f\|_\infty = \sup_{s \in S} |f(s \cdot y) - f(s \cdot x)| = \sup_{s \in S} |f(v_{y,s}) - f(w_{y,s})| \le \frac{\varepsilon}{2}, \tag{5.11}$$

for all $y \in V \cap C$. Since $\{c_a \mid a \in S\} \subseteq C$ and C is compact, (5.10) and (5.11) imply that $\mathcal{RO}(L^c f)$ is relatively compact in $\ell^\infty(S)$. Hence $L^c f \in AP(S)$.

Following the corresponding parts in the proof of Theorem 4.2, (a) $\Rightarrow$ (b), we have (b) holds.

(b) $\Rightarrow$ (c) Follow the proof of Theorem 4.2, (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) Apply exactly the proof of the second statement in [17, Theorem 3.2]. $\square$

6. Application to left amenability of F-algebras

Let X be a *Banach A-bimodule* and A be a *Banach algebra*. A linear mapping $D\colon A \to X$ is called a *derivation* if it satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b, \quad \text{for all } a, b \in X.$$

Derivations in the form $D(a) := a \cdot x_0 - x_0 \cdot a$ ($a \in A$) for some fixed $x_0 \in X$ are called *inner derivations*.

A Banach algebra A is an *F-algebra* [21] (also known as *Lau algebras* [30]) if it is the (unique) predual of a W^*-algebra $\mathfrak{M}$ and the identity e of $\mathfrak{M}$ is a multiplicative linear functional on A. Since $A^{**} = \mathfrak{M}^*$, we denote by $P_1(A^{**})$ the set of all normalized positive linear functionals on $\mathfrak{M}$, i.e.

$$P_1(A^{**}) := \{m \in A^{**} \mid m \ge 0,\ m(e) = 1\}.$$

In this case, $P_1(A^{**})$ is a semigroup with the first (or second) Arens multiplication. A mean $m \in P_1(A^{**})$ on A^* is called a *topological left invariant mean*, abbreviated as TLIM, if $a \cdot m = m$ for all $a \in P_1(A) = P_1(A^{**}) \cap A$; in other words, $m \in P_1(A^{**})$ is a TLIM if $m(x \cdot a) = m(x)$ for all $a \in P_1(A)$ and $x \in A^*$.

Examples of F-algebras include the predual algebras of a Hopf von Neumann algebra (in particular, quantum group algebras), the group algebra $L^1(G)$ of a locally compact group G, the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of a topological group G (see [5], [21], [22]). They also include the measure algebra $M(S)$ of a locally compact semigroup S. Moreover, the hypergroup algebra $L^1(H)$ and the measure algebra $M(H)$ of a locally compact hypergroup H with a left Haar measure are F-algebras. In this case, it was shown in [37, Theorem 5.2.2] (see also [38, Remark 5.3]) that $(L^1(H))^* = L^\infty(H)$ is not a Hopf von Neumann algebra unless H is a locally compact group.

An F-algebra A is called *left amenable* if, for each Banach A-bimodule X with the left module action specified by $a \cdot x := \langle a, e\rangle\, x$, for all $a \in A$, $x \in X$, every continuous derivation from A into X^* is inner. The following result was shown in [21, Theorems 4.1 and 4.6].

Lemma 6.1. *Let A be an F-algebra. Then the following are equivalent.*

(a) *There is a* TLIM *for A^*.*
(b) *The algebra A is left amenable.*
(c) *There exists a net $(m_\alpha) \subseteq P_1(A)$ such that $am_\alpha - m_\alpha \to 0$ in the norm topology for each $a \in P_1(A)$.*

We note here that, being F-algebras, the group algebra $L^1(G)$ and the measure algebra $M(G)$ of a locally compact group G are left amenable if and only if G is an amenable group, while the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ are always left (and right) amenable [21]. The hypergroup algebra $L^1(H)$ of a locally compact hypergroup H with a left Haar measure is left amenable if and only if H is an amenable hypergroup [35]. Also the left amenability of the predual algebra of a Hopf von Neumann algebra, as an F-algebra, coincides with that studied in [32], [36] (see also [2] and references therein).

Mitchell showed in [26] that a semitopological semigroup S is *extremely left amenable* (i.e. LUC(S) has a multiplicative left invariant mean) if and only if S has the following fixed point property:

(F_E) Every jointly continuous representation of S on a compact Hausdorff space C has a common fixed point in C.

For an F-algebra A, the left amenability of A is equivalent to the extreme left amenability of $P_1(A)$.

Lemma 6.2 (See [25, Theorem 3.2]). *Let A be an F-algebra. Then A is left amenable if and only if $P_1(A)$ has the fixed point property (F_E).*

The following theorem is now a consequence of Theorem 4.3.

THEOREM 6.3. *Let A be an F-algebra. Then A is left amenable if and only if the following fixed point property holds for the topological semigroup $S := P_1(A)$ with the norm topology and multiplication of the F-algebra A:*

(a) *For any linear set-valued action of S on E such that there exists a closed subspace F of E with $F \supseteq s \cdot 0$ (for all $s \in S$), if $C \subseteq E$ is an invariant compact set and the action of S on E is jointly closed on C, then $E_f := \{x \in C \mid x \in (s \cdot x) + F,\ \text{for all}\ s \in S\} \neq \emptyset$.*

(b) *For any linear set-valued action of S on E such that $s \cdot 0 = t \cdot 0$ is closed (for all $s, t \in S$), if $C \subseteq E$ is an invariant compact set and the action of S on E is jointly closed on C, then $E_f := \{x \in C \mid x \in s \cdot x,\ \text{for all}\ s \in S\} \neq \emptyset$.*

7. Remarks and open problems

REMARK 7.1. Theorems 2.2 and 3.1 extend the first author's results: [18, Theorem 1] and [20, Theorem 1], respectively.

REMARK 7.2. An extension of Markov–Kakutani fixed point theorem for a family of set-valued mappings was presented in [24] recently.

PROBLEM 7.3. Can we improve our results in Sections 4 and 5 by weakening the constraints on the linear set-valued action of S on E?

PROBLEM 7.4. Can we extend the fixed point properties in [19] for set-valued mappings?

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