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APPROXIMATE CONTROLLABILITY FOR ABSTRACT SEMILINEAR IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS BASED ON HAUSDORFF PRODUCT MEASURES

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Abstract. A second order semilinear impulsive functional differential inclusion in a separable Hilbert space is considered. Without imposing hypotheses of the compactness on the cosine families of operators, some sufficient conditions of approximate controllability are formulated in the case where the multivalued nonlinearity of the inclusion is a completely continuous map dominated by a function. By the use of resolvents of controllability Gramian operators and developing appropriate computing techniques for the Hausdorff product measures of noncompactness, the results of approximate controllability for position and velocity are derived. An example is also given to illustrate the application of the obtained results.

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1. Introduction

In this paper we consider the following approximate controllability problem for second order semilinear impulsive functional differential inclusion with delay:

$$\text{(CIP)}\quad \begin{cases} (x'' - Ax)(t) \in F(t, x_t) + Bu(t), & \text{a.e. } t \in I \setminus \{t_1, \ldots, t_m\},\ I = [0, a]; \\ x(t_k^+) - x(t_k^-) = \varphi_k(x(t_k^-)), & k = 1, \ldots, m; \\ x'(t_k^+) - x'(t_k^-) = \psi_k(x(t_k^-)), & k = 1, \ldots, m; \\ x(t) = \phi(t),\ x'(0) = \xi, & t \in J_0,\ J_0 = [-r, 0], \end{cases}$$

where $0 < r, a < +\infty$ and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = a$; the linear operator $A\colon D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}$ in a real separable Hilbert space X; $x\colon [-r, a] \to X$ is a state function; $u(\,\cdot\,) \in L^2(I, U)$ is a control function, U is a Hilbert space and B is a compact linear operator from U into X. The nonlinearity $F\colon I \times \Theta \multimap X$, is a multivalued map, Θ is called a phase space consisting of a class of functions from J_0 into X; $\varphi_k, \psi_k\colon X \to X$ are all single valued mappings, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively; $\phi \in \Theta$, $\xi \in X$. For any function x defined on $[-r, a]$ and any $t \in I$, $x_t \in \Theta$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in J_0 = [-r, 0].$$

Here $x_t(\,\cdot\,)$ represents the history of the state up to the present time t.

In the mathematical control theory, controllability plays an important role (see Klamka [18], [22]). In the past decades, the study of controllability problems described as abstract differential equations or inclusions in infinite-dimensional spaces acquired an growing interest for many researchers, see [1], [2], [11], [19], [23], [26], [28], [29], [38], [41] and the references therein. It should be mentioned that there are many different definitions of controllability. Constrained controllability problems for linear or semilinear systems in infinite-dimensional spaces were discussed (see [20], [21] for examples). Two basic concepts of controllability can be distinguished which are exact and approximate controllability. In general infinite-dimensional spaces, the concept of exact controllability is usually too strong; the one of approximate controllability is more useful in applications (see [9], [27], [32], [37], [31] for examples). Approximate controllability of semilinear functional equations was considered by Dauer and Mahmudov [8]. In addition, Rykaczewski [31] studied approximate controllability for semilinear differential inclusion. Approximate controllability of semilinear impulsive functional differential systems was discussed by Grudzka et al. in [13] and Sakthivel et al. in [33]. Henríquez and Hernández M [15] studied approximate controllability of second-order distributed implicit functional systems.

In many cases, it is advantageous to treat the second order differential systems directly rather than to convert them to first order systems. It is known that the study for second order semilinear systems in abstract spaces relies on the theory of cosine families. Some strict compactness assumptions imposed on the cosine families imply that the underlying space is of finite dimension (see [4]). But there is a lot of difficulty in discuss under some noncompactness assumptions. Recently, Obukhovskiĭ et al. [29] studied the controllability problem for a system governed by a semilinear differential inclusion in a Banach space with a noncompact semigroup; Das et al. [7] studied approximate controllability of a second order neutral differential equation with state dependent delay. In their approach to overcoming the difficulty, computing measures of noncompactness plays a key role.

Our purpose in this paper is to establish criteria for the approximate controllability problem (CIP) without imposing hypotheses of compactness on the cosine family of operators. Up to now, to the best of our knowledge, no results are available for approximate controllability concerning second-order impulsive functional differential inclusion, when the main conditions only depend on the local properties of multivalued map on bounded sets. Motivated by the work in [7], [13], [15], [24], [29], [39], in this paper, we establish some sufficient conditions for approximate controllability in the case where the multivalued nonlinearity of the inclusion is a completely continuous map dominated by a function. By using a fixed point theorem for condensing multivalued maps and developing appropriate computing techniques for the Hausdorff product measures of noncompactness, we derive some results of approximate controllability for position and velocity.

2. Preliminaries and problem formulation

Throughout this paper, we denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}^+$ the sets of real numbers, nonnegative real numbers and positive integers, respectively.

For $a_1, a_2 \in \mathbb{R}$, we denote by $a_1 \vee a_2 = \max\{a_1, a_2\}$ and $a_1 \wedge a_2 = \min\{a_1, a_2\}$. Moreover, $J_0 = [-r, 0]$, $I = [0, a]$, $I_1 = [0, t_1]$, and $I_k = (t_{k-1}, t_k]$, $k = 2, \ldots, m+1$.

Let X be a real separable Hilbert space with the norm $\|\cdot\|$. For $E \subset X$, the notation $\overline{E}$ stands for the closure of E.

Let U be another Hilbert space. By $B(U, X)$ we mean the Banach space of all bounded linear operators from U into X with the norm $|\cdot|_*$, and we abbreviate this notation to $B(X)$ when $U = X$.

For $B \in B(U, X)$, B^* denotes its adjoint. Let J_* be a compact interval in $\mathbb{R}$.

By $C(J_*, X)$ we denote the Banach space consisting of continuous function from J_* into X with the norm

$$\|x\|_C = \sup_{t \in J_*} \|x(t)\|$$

and $C^1(J_*, X)$ the Banach space of continuously differentiable functions endowed with the norm

$$\|x\|_{C^1} = \sup_{t\in J_*} \big[\|x(t)\| + \|x'(t)\|\big].$$

We will consider the space of piece-wise continuous functions

$$\begin{aligned}\mathrm{PC}^1(I) = \big\{x\colon I \to X \mid\ & x'(t) \text{ is continuous at } t \neq t_k,\\ & \text{and } x(t) \text{ is left continuous at } t = t_k,\\ & \text{and } x(t_k^+), x'(t_k^+),\ x'(t_k^-) \text{ exist},\ k = 1, \dots, m\big\}.\end{aligned}$$

When endowed with the norm $\|x\|_\diamond = \sup_{t\in I} \big[\|x(t)\| + \|x'(t)\|\big]$, $\mathrm{PC}^1(I)$ is a Banach space. For $x \in \mathrm{PC}^1(I)$ we have $x'_-(t_k) = x'(t_k^-)$, where $x'_-(t_k)$ is the left derivative of $x(t)$ at $t = t_k$. We will consider the phase space Θ. Suppose that

$$\begin{aligned}\Theta = \big\{x\colon J_0 \to X;\ & x \text{ is continuously differentiable everywhere}\\ & \text{except for a finite number of points at which}\\ & x(s^+),\ x(s^-),\ x'(s^+) \text{ and } x'(s^-) \text{ exist and } x(s) = x(s^-)\big\}.\end{aligned}$$

Endowed with the integral norm

$$\|x\|_\Theta = \frac{1}{r} \int_{-r}^{0} \big[\|x(t)\| + \|x'(t)\|\big]\, dt,$$

it is clear that Θ is a linear normed space. We do not consider the space Θ with the sup norm, because it creates problems: the function x_t is not necessarily measurable (see Example 3.1 in [14]). Notice that the same construction can be applied also in [6], [13]. The phase space Θ is a one of initial trajectories, from which solutions of System (CIP) start their lives. Set $J = J_0 \cup I = [-r, a]$. For a function $x\colon J \to X$, $x\lfloor_I$ denotes the *restriction* of x to I. Let

$$\mathrm{PC}^1(J) = \big\{x\colon J \to X \mid x\lfloor_I \in \mathrm{PC}^1(I) \text{ and } x\lfloor_{J_0} \in \Theta\big\}.$$

For $x \in \mathrm{PC}^1(J)$, the norm of x is defined by $\|x\|_* = \max\{\|x\|_\Theta, \|x\|_\diamond\}$. Clearly, $\mathrm{PC}^1(J)$ is a linear normed space. It is evident that $x_n \to x$ in $\mathrm{PC}^1(J)$ if and only if $x_n \to x$ in Θ and in $\mathrm{PC}^1(I)$.

We denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and put

$$\begin{aligned}\mathcal{P}_{\mathrm{cl}}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is closed}\},\\ \mathcal{P}_{\mathrm{bd}}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is bounded}\},\\ \mathcal{P}_{\mathrm{cp}}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is compact}\},\\ \mathcal{P}_{\mathrm{cv,cp}}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is compact and convex}\}.\end{aligned}$$

Let Y be a metric space. For $Z \subset Y$ and $y \in Y$, we denote by $d(y, Z)$ the distance from y to Z. Let $T\colon Y \multimap X$ be a multivalued map. A point $z \in Y \subset X$ is called a *fixed point* of T if $z \in T(z)$. The fixed point set of T will be denoted by $\mathrm{Fix}(T)$. T is a *closed graph* map if the graph of T is closed in $Y \times X$. For $V \subset X$, suppose that $T^+(V) = \{y \in Y : T(y) \cap V \neq \emptyset\}$. T is called *upper semicontinuous* (u.s.c. for short) if for each nonempty closed set $V \subset X$, $T^+(V)$ is closed in Y. T is said to be *precompact* if $T(D)$ is relatively compact in X for each bounded subset D of Y. T is said to be *completely continuous* (see [5]) if it is u.s.c. and precompact. T is said to be *quasicompact* if $T(D)$ is relatively compact in X for each relatively compact set $D \subset Y$.

Lemma 2.1 (see [10]). *If $T\colon Y \to \mathcal{P}_{\mathrm{cp}}(X)$ is quasicompact and has closed graph, then T is an u.s.c. map.*

Let β_H be the Hausdorff measure of noncompactness. A multivalued map $T\colon Y \multimap X$ is said to be *β_H-condensing* if, for each bounded non-relatively-compact subset E of Y, $T(E)$ is bounded and satisfies $\beta_H(T(E)) < \beta_H(E)$.

The following fixed point theorem for condensing multivalued maps will be needed.

Lemma 2.2 (see [25], [30]). *Let X be a Banach space and $T\colon X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ an u.s.c. β_H-condensing multivalued map. Then either*

(a) $\mathrm{Fix}(T) \neq \emptyset$, *or*
(b) *the set* $\Omega = \{u \in X : \exists\, \lambda > 1,\ \lambda u \in T(u)\}$ *is unbounded.*

Let $\mathcal{L}(I)$ be the Lebesgue σ-algebra of I. A multivalued map $T\colon I \multimap X$ is said to be *measurable* if for each closed set $V \subset X$, $T^+(V) \in \mathcal{L}(I)$. If T is measurable and has nonempty closed values, then T admits a measurable selector (see [3]). By $L^1(I,X)$ we denote the Banach space of all Bochner integrable mapping from I into X with the norm

$$\|x\|_{L^1} = \int_0^a \|x(t)\|\,dt$$

and $L^2(I,U)$ the Hilbert space of all Bochner square integrable mapping from I into U with the norm

$$\|u\|_{L^2} = \bigg(\int_0^a \|u(t)\|^2\,dt\bigg)^{1/2}.$$

A multivalued map $T\colon I \multimap X$ is said to be *integrable* if it admits a integrable selector; and to be *integrably bounded* if there exists a function $p_0 \in L^1(I,\mathbb{R}^+)$ such that $\sup_{z \in T(t)} \|z\| \leq p_0(t)$ for almost every $t \in I$. If $T\colon I \multimap X$ is an integrable, integrably bounded multivalued map such that $\beta_H(T(t)) \leq \alpha(t)$ for almost every $t \in I$, where $\alpha \in L^1(I,\mathbb{R}^+)$, then (see [16], [29])

$$\beta_H\bigg(\int_0^t T(s)\,ds\bigg) \leq \int_0^t \alpha(s)\,ds, \quad \text{for all } t \in I.$$

For the Hausdorff product measures of noncompactness, the following inequalities will be needed.

Lemma 2.3 (see [40]). *Let X_1, X_2 be two Banach spaces, and $P \in B(X_1, X_2)$. The norm in $X_1 \times X_2$ is defined by*

$$\|(x_1, x_2)\| = \|x_1\| + \|x_2\|, \quad \text{for } (x_1, x_2) \in X_1 \times X_2.$$

(a) *If $D \subset X_1$ is bounded, then $\beta_H(PD) \le |P|_* \beta_H(D)$.*
(b) *If $D_1 \subset X_1$ and $D_2 \subset X_2$ are bounded, then*

$$\beta_H(D_1 \times D_2) \le \beta_H(D_1) + \beta_H(D_2).$$

Lemma 2.4 (see [40]). *Let J_* be a compact interval in $\mathbb{R}$, $Q \subset C^1(J_*, X)$ and $t \in J_*$. Let $Q(t)$, $Q(J_*)$, $Q'(t)$, $Q'(J_*)$ be subsets of X defined respectively by*

$$Q(t) = \{x(t) : x \in Q\}, \qquad Q(J_*) = \{x(t) : x \in Q,\ t \in J_*\},$$
$$Q'(t) = \{x'(t) : x \in Q\}, \qquad Q'(J_*) = \{x'(t) : x \in Q,\ t \in J_*\}.$$

If Q is bounded in $C^1(J_*, X)$ and Q' is equicontinuous in $C(J_*, X)$, then

(a) $\beta_H(Q(J_*)) = \max_{t \in J_*} \beta_H(Q(t))$ *and* $\beta_H(Q'(J_*)) = \max_{t \in J_*} \beta_H(Q'(t))$.
(b) $\beta_H(Q(J_*)) \vee \beta_H(Q'(J_*)) \le \beta_H(Q) \le \beta_H(Q(J_*)) + \beta_H(Q'(J_*))$.

Let $F\colon I \times \Theta \multimap X$ be a multivalued map. F is said to be *locally integrably bounded* (or *α_λ-locally integrably bounded*) if for each $\lambda > 0$, there exists $\alpha_\lambda \in L^1(I, \mathbb{R}^+)$ such that

$$\|u\|_\Theta \le \lambda \ \Rightarrow\ \sup\{\|z\| : z \in F(t, u)\} \le \alpha_\lambda(t) \quad \text{for a.e. } t \in I.$$

For $x \in \mathrm{PC}^1(J)$, we use the notation $S^1_F(x)$ to denote the set of integrable selectors (possibly empty), i.e.

$$S^1_F(x) = \big\{f \in L^1(I, X) : f(t) \in F(t, x_t) \text{ for a.e. } t \in I\big\}. \tag{2.1}$$

The following lemma is a consequence of Lemma 3.8 in [40].

Lemma 2.5 (see [40]). *Let $F\colon I \times \Theta \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be a map such that $t \mapsto F(t, x_t)$ is measurable and $u \mapsto F(t, u)$ is u.s.c. and locally integrably bounded. Then*:

(a) *the map $S^1_F\colon \mathrm{PC}^1(J) \multimap L^1(I, X)$ has nonempty, closed, convex values*;
(b) *if $\Gamma\colon L^1(I, X) \to \mathrm{PC}^1(J)$ is a continuous linear operator, then $\Gamma \circ S^1_F\colon \mathrm{PC}^1(J) \multimap \mathrm{PC}^1(J)$ is a closed graph map.*

A one-parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators is called a *strongly continuous cosine family* in X if

(C1) $C(0) = I_X$ (I_X is the identity operator);
(C2) $C(t + \tau) + C(t - \tau) = 2C(t)C(\tau)$ for all $t, \tau \in \mathbb{R}$;
(C3) for each $x \in X$, the mapping $C(t)x$ is continuous in $t \in \mathbb{R}$.

If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family, then $\{S(t) : t \in \mathbb{R}\}$ is the associated *sine family* defined by

$$S(t)x = \int_0^t C(\tau)x\, d\tau, \quad x \in X,\ t \in \mathbb{R}.$$

The *infinitesimal generator* $A\colon D(A) \subset X \to X$ of a cosine family is a linear operator defined by

$$Ax = \frac{d^2}{dt^2}\, C(t)x\bigg|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$. Also, E denotes the space $E = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$. We refer to Travis and Webb [35], [36] for the detailed study of the family of cosine and sine operators. It is knows that A is a closed linear operator in $D(A)$, $D(A) \subset E$ and $\overline{D(A)} = X$. Moreover, $\{S(t) : t \in \mathbb{R}\}$ is uniformly continuous, and there exist $M_0 \geq 1$ and $\omega > 0$ such that

$$|C(t)|_* \leq M_0 e^{\omega|t|} \quad \text{and} \quad |S(t) - S(r)|_* \leq M_0 \bigg|\int_r^t e^{\omega|\tau|}\, d\tau\bigg|, \quad \text{for } t, r \in \mathbb{R}.$$

Let $M = (1 \vee a)M_0 e^{\omega a}$. For $t, r \in I$ and $\tau \in [0, t \wedge r]$, we have the following inequalities: $|C(t)|_* \leq M$, $|S(t)|_* \leq M$ and

$$|S(t-\tau) - S(r-\tau)|_* \leq 2M|S((t-r)/2))|_*. \tag{2.2}$$

It is known from Kisińsky [17], that E endowed with the norm $\|x\|_E = \|x\| + \sup_{t\in I}\|AS(t)x\|$ is a Banach space. From this definition it follows that $M_A = \sup_{t\in I}|AS(t)|_* \leq 1$ in $B(E, X)$. If an operator $W \in B(X)$, then we have $W \in B(E, X)$ and $|W|_{*B(E,X)} \leq |W|_{*B(X)}$ since $\|\cdot\| \leq \|\cdot\|_E$. Therefore, for $t, s \in I$, $\tau \in [0, t \wedge r]$ we have

$$|C(t-\tau) - C(s-\tau)|_* \leq 2M_A|S((t-s)/2)|_* \quad \text{in } B(E, X). \tag{2.3}$$

For the sake of explicit, the space E mentioned in the sequel means $(E, \|\cdot\|_E)$.

In order to address the problem, we shall introduce some operators as preparation.

Definition 2.6. A function $x \in \mathrm{PC}^1(J)$ is said to be a mild solution of class C^1 (see [34]) or C^1-solution of the system (CIP) if there exists $f \in L^1(I, X)$ such that $f(t) \in F(t, x_t)$ almost everywhere on I, and $x(t)$ is given by

$$x(t) = \begin{cases} \phi(t), & t \in J_0, \\ z(t), & t \in I, \end{cases}$$

where

$$
\begin{aligned}
z(t) = {} & C(t)\phi(0) + S(t)\xi \\
& + \sum_{0<t_k<t} \big[C(t-t_k)\varphi_k(x(t_k)) + S(t-t_k)\psi_k(x(t_k))\big] \\
& + \int_0^t S(t-\tau)f(\tau)\,d\tau + \int_0^t S(t-\tau)Bu(\tau)\,d\tau;
\end{aligned}
\tag{2.4}
$$

its derivative is given by

$$
x'(t) = \begin{cases} \phi'(t), & t \in J_0, \\ z'(t), & t \in I, \end{cases}
$$

where

$$
\begin{aligned}
z'(t) = {} & AS(t)\phi(0) + C(t)\xi \\
& + \sum_{0<t_k<t} \big[AS(t-t_k)\varphi_k(x(t_k)) + C(t-t_k)\psi_k(x(t_k))\big] \\
& + \int_0^t C(t-\tau)f(\tau)d\tau + \int_0^t C(t-\tau)Bu(\tau)\,d\tau.
\end{aligned}
\tag{2.5}
$$

Definition 2.7. For every initial state $\phi \in \Theta$ with $\phi(0) \in E$ and $\xi \in E$, system (CIP) is said to be position approximate controllable on J if

$$
\overline{\mathcal{R}(a;\phi,\xi)} = X, \quad \text{where } \mathcal{R}(a;\phi,\xi) = \{x(a;\phi,\xi;u) \mid u \in L^2(I,U)\};
$$

and is said to be velocity approximate controllable on J if

$$
\overline{\mathcal{R}'(a;\phi,\xi)} = X, \quad \text{where } \mathcal{R}'(a;\phi,\xi) = \{x'(a;\phi,\xi;u) \mid u \in L^2(I,U)\}.
$$

Next, we introduce the following two resolvent operators. Let

$$
\begin{aligned}
\Upsilon_0^a &= \int_0^a S(a-\tau)BB^*S^*(a-\tau)\,d\tau, \\
\Psi_0^a &= \int_0^a C(a-\tau)BB^*C^*(a-\tau)\,d\tau
\end{aligned}
\tag{2.6}
$$

be two controllability Gramian operators. Then the resolvent operators

$$
R(\delta,-\Upsilon_0^a), R(\delta,-\Psi_0^a) \in B(X) \quad \text{for } \delta > 0
$$

are given by

$$
R(\delta,-\Upsilon_0^a) = (\delta I_X + \Upsilon_0^a)^{-1} \quad \text{and} \quad R(\delta,-\Psi_0^a) = (\delta I_X + \Psi_0^a)^{-1},
\tag{2.7}
$$

where I_X is the identity operator on X. Since the operators Υ_0^a and Ψ_0^a are clearly positive, $R(\delta,-\Upsilon_0^a)$ and $R(\delta,-\Psi_0^a)$ are well defined.

Definition 2.8. Let $F\colon I \times \Theta \multimap X$ be a multivalued map and $S_F^1(x) \neq \emptyset$ for all $x \in \mathrm{PC}^1(J)$, where $S_F^1(x)$ is defined by (2.1). For every $x_a, x_a^1 \in X$, $x \in \mathrm{PC}^1(J)$ and $f \in S_F^1(x)$, define the control maps

$$u_x(t) = B^* S^*(a-t) R(\delta, -\Upsilon_0^a)\big\{p(x) : f \in S_F^1(x)\big\}, \quad \text{for } t \in I, \tag{2.8}$$

where

$$p(x) = \bigg[x_a - C(a)\phi(0) - S(a)\xi - \int_0^a S(a-\tau) f(\tau)\, d\tau \\ - \sum_{k=1}^m C(a-t_k)\varphi_k(x(t_k)) + S(a-t_k)\psi_k(x(t_k))\bigg].$$

And $u_x^1(t) = B^* C^*(a-t) R(\delta, -\Psi_0^a)\big\{p_1(x) : f \in S_F^1(x)\big\}$, for $t \in I$, where

$$p_1(x) = \bigg[x_a^1 - AS(a)\phi(0) - C(a)\xi - \int_0^a C(a-\tau) f(\tau)\, d\tau \\ - \sum_{k=1}^m AS(a-t_k)\varphi_k(x(t_k)) + C(a-t_k)\psi_k(x(t_k))\bigg].$$

It is evident that u_x and u_x^1 are all multivalued maps, and $u_x, u_x^1 \subset L^2(I, U)$.

Definition 2.9. Let $F\colon I \times \Theta \multimap X$ be a multivalued map and $S_F^1(x) \neq \emptyset$ for all $x \in \mathrm{PC}^1(J)$, where $S_F^1(x)$ is defined by (2.1). Let $\Gamma\colon L^1(I, X) \to \mathrm{PC}^1(I)$ and $\Gamma_\diamond\colon L^2(I, U) \to \mathrm{PC}^1(I)$ be linear operators defined by

$$(\Gamma f)(t) = \int_0^t S(t-\tau) f(\tau)\, d\tau, \qquad t \in I; \tag{2.9}$$

$$(\Gamma_\diamond u_x)(t) = \int_0^t S(t-\tau) B u_x(\tau)\, d\tau, \quad t \in I. \tag{2.10}$$

Let $\Lambda_0, \Lambda\colon \mathrm{PC}^1(I) \to \mathrm{PC}^1(I)$ be single valued mappings ($t \in I$) defined by

$$(\Lambda_0 x)(t) = C(t)\phi(0) + S(t)\xi; \tag{2.11}$$

$$(\Lambda x)(t) = \sum_{0<t_k<t} [C(t-t_k)\varphi_k(x(t_k)) + S(t-t_k)\psi_k(x(t_k))]. \tag{2.12}$$

For $x \in \mathrm{PC}^1(I)$, we define a multivalued map $T\colon \mathrm{PC}^1(I) \multimap \mathrm{PC}^1(I)$ by

$$Tx = \big\{y \in \mathrm{PC}^1(I) : y(t) = (\Lambda_0 x)(t) + (\Lambda x)(t) \\ + (\Gamma_\diamond u_x)(t) + (\Gamma f)(t),\ f \in S_F^1(x)\big\}, \tag{2.13}$$

i.e. $Tx = \Lambda_0 x + \Lambda x + \Gamma_\diamond u_x + \big(\Gamma \circ S_F^1\big)(x)$. Replacing u_x by u_x^1 in the definition of T, we define a multivalued map $T_1\colon \mathrm{PC}^1(I) \multimap \mathrm{PC}^1(I)$ by

$$T_1 x = \{y \in \mathrm{PC}^1(I) : y(t) = (\Lambda_0 x)(t) + (\Lambda x)(t) \\ + (\Gamma_\diamond u_x^1)(t) + (\Gamma f)(t),\ f \in S_F^1(x)\}, \tag{2.14}$$

i.e. $T_1x = \Lambda_0 x + \Lambda x + \Gamma_\diamond u_x^1 + \big(\Gamma \circ S_F^1\big)(x)$.

Let $\widehat{\Gamma}\colon L^1(I,X) \to \mathrm{PC}^1(J)$ and $\widehat{\Gamma}_\diamond\colon L^2(I,U) \to \mathrm{PC}^1(J)$ be linear operators defined by

$$\big(\widehat{\Gamma} f\big)(t) = \begin{cases} 0, & t \in J_0, \\ (\Gamma f)(t), & t \in I; \end{cases} \quad \text{and} \quad \big(\widehat{\Gamma}_\diamond u_x\big)(t) = \begin{cases} 0, & t \in J_0, \\ (\Gamma_\diamond u_x)(t), & t \in I. \end{cases}$$

Let $\widehat{\Lambda}_0, \widehat{\Lambda}\colon \mathrm{PC}^1(J) \to \mathrm{PC}^1(J)$ be single valued mappings defined by

$$(\widehat{\Lambda}_0 x)(t) = \begin{cases} \phi(t), & t \in J_0, \\ (\Lambda_0 x)(t), & t \in I; \end{cases} \quad \text{and} \quad (\widehat{\Lambda} x)(t) = \begin{cases} 0, & t \in J_0, \\ (\Lambda x)(t), & t \in I. \end{cases}$$

For $x \in \mathrm{PC}^1(J)$, we define multivalued maps $\widehat{T}, \widehat{T}_1\colon \mathrm{PC}^1(J) \multimap \mathrm{PC}^1(J)$ by

$$\widehat{T} x = \widehat{\Lambda}_0 x + \widehat{\Lambda} x + \widehat{\Gamma}_\diamond u_x + (\widehat{\Gamma} \circ S_F^1)(x);$$
$$\widehat{T}_1 x = \widehat{\Lambda}_0 x + \widehat{\Lambda} x + \widehat{\Gamma}_\diamond u_x^1 + (\widehat{\Gamma} \circ S_F^1)(x).$$

Note that $\widehat{T}\lfloor_\Theta$ and $\widehat{T}_1\lfloor_\Theta$ are constant functors, $\widehat{T}\lfloor_{\mathrm{PC}^1(I)} = T$ and $\widehat{T}_1\lfloor_{\mathrm{PC}^1(I)} = T_1$. It is clear that, if there exists $x \in \mathrm{Fix}(T)$ or $x \in \mathrm{Fix}(T_1)$, and let

$$\widehat{x}(t) = \begin{cases} \phi(t), & t \in J_0, \\ x(t), & t \in I, \end{cases}$$

then $\widehat{x} \in \mathrm{Fix}(\widehat{T})$ or $\widehat{x} \in \mathrm{Fix}(\widehat{T}_1)$. Hence, from (2.4) and (2.5) we see that $\widehat{x}$ is a C^1-solution of the system (CIP).

Concerning the operators A and B, we assume the following hypothesis (where Υ_0^a, Ψ_0^a, $R(\delta, -\Upsilon_0^a)$ and $R(\delta, -\Psi_0^a)$ are defined by (2.6) and (2.7)):

- (H_A) A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$; $\{S(t) : t \in \mathbb{R}\}$ is a sine family; and $M_A = \sup_{t \in I} |AS(t)|_*$ in $B(E,X)$.
- (H_B) $\delta R(\delta, -\Upsilon_0^a) \to 0$ as $\delta \to 0^+$, in operator strong topology (pointwise).
- (H_B^1) $\delta R(\delta, -\Psi_0^a) \to 0$ as $\delta \to 0^+$, in operator strong topology (pointwise).

3. Criteria for approximate controllability

Theorem 3.1. *Let $\phi \in \Theta$ with $\phi(0) \in E$ and $\xi \in E$. Let* (H_A), (H_B) *and the following conditions be satisfied*:

- (H_I) *For the mappings $\varphi_k, \psi_k\colon X \to E$, there exist constants $a_k, b_k > 0$ ($k = 1, \ldots, m$) such that for all $x, y \in X$,*

$$\|\varphi_k(x) - \varphi_k(y)\|_E \le a_k \|x - y\|, \qquad \|\psi_k(x) - \psi_k(y)\|_E \le b_k \|x - y\|.$$

(H_F) *$F\colon I \times \Theta \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ is a completely continuous map such that $t \mapsto F(t, x_t)$ is measurable for $x \in \mathrm{PC}^1(J)$ and almost every $t \in I$ and there exist a function $\alpha \in L^1(I, \mathbb{R}^+)$ and a continuous nondecreasing function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ such that for all $w \in \Theta$ and almost every $t \in I$,*

$$\sup\{\|z\|_E : z \in F(t, w)\} \le \alpha(t)\Phi(\|w\|_\Theta).$$

If $\gamma = 2(M \vee M_A) \sum_{k=1}^{m} (a_k + b_k) < 1$ and

$$\int_1^{+\infty} \frac{d\tau}{\Phi(\tau)} = +\infty, \tag{3.1}$$

then the system (CIP) *is position approximate controllable on J.*

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Let T be a multivalued map defined by* (2.13). *If $S^1_F(x) \neq \emptyset$ for all $x \in \mathrm{PC}^1(J)$, then $Tx \subset \mathrm{PC}^1(I)$ for all $x \in \mathrm{PC}^1(I)$, i.e. T is well defined.*

Proof. Let Γ, $\Gamma_\diamond$, Λ_0, Λ be defined by (2.9)–(2.12), respectively. Suppose that $x \in \mathrm{PC}^1(I)$, $f \in S^1_F(x)$ and $t \in I$. From the strong continuity of $S(t)$ and $C(t)$ we see that $\Gamma f, \Gamma_\diamond u_x, \Lambda_0 x \in C(I, X)$. Suppose that

$$\overline{x}(t) = \begin{cases} x(t), & t \in I_k; \\ x(t^+_{k-1}), & t = t_{k-1}, \end{cases} \quad \text{and} \quad \overline{x}'(t) = \begin{cases} x'(t), & t \in I_k; \\ x'(t^+_{k-1}), & t = t_{k-1}. \end{cases}$$

It is clear that $x \in \mathrm{PC}^1(I)$ if and only if $\overline{x} \in C^1(\overline{I_k}, X)$ for $k = 1, \ldots, m+1$. Hence, from the strong continuity of $S(t)$ and $C(t)$ it is easy to check that Λx is continuous in each I_k and $(\Lambda x)(t^+_{k-1})$ exists. Moreover, for $t \in I$ we have

$$(\Gamma f)'(t) = \int_0^t C(t-\tau) f(\tau)\, d\tau; \tag{3.2}$$

$$(\Gamma_\diamond u_x)'(t) = \int_0^t C(t-\tau) B u_x(\tau)\, d\tau; \tag{3.3}$$

$$(\Lambda_0 x)'(t) = AS(t)\phi(0) + C(t)\xi; \tag{3.4}$$

$$(\Lambda x)'(t) = \sum_{0<t_k<t} [AS(t-t_k)\varphi_k(x(t_k)) + C(t-t_k)\psi_k(x(t_k))]. \tag{3.5}$$

From (3.2)–(3.5) it is easy to see that $(\Gamma f)', (\Gamma_\diamond u_x)', (\Lambda_0 x)' \in C(I, X)$ and $(\Lambda x)'(t)$ is continuous in each I_k and $(\Lambda x)'(t^+_{k-1})$ exists. Hence, Γf, $\Gamma_\diamond u_x$, $\Lambda_0 x$, Λx in $\mathrm{PC}^1(I)$, and so we obtain $Tx \subset \mathrm{PC}^1(I)$. $\square$

Lemma 3.3. *Suppose that* (H_A) *is satisfied. Then $\Gamma\colon L^1(I, X) \to \mathrm{PC}^1(I)$ and $\Gamma_\diamond\colon L^2(I, U) \to \mathrm{PC}^1(I)$ are all continuous linear operators, and*

$$\|\Gamma f\|_\diamond \le 2M\|f\|_{L^1}; \tag{3.6}$$

$$\|\Gamma_\diamond u\|_\diamond \le 2M\sqrt{a}\ |B|_* \|u\|_{L^2}. \tag{3.7}$$

Proof. It is enough to check that Γ and $\Gamma_\diamond$ are bounded. In fact, for $f \in L^1(I,X)$ and $u \in L^2(I,U)$, from (2.9), (3.2), (2.10), (3.3) and (2.2) we have

$$\begin{aligned}
\|\Gamma f\|_\diamond &= \sup_{t\in I}[\|(\Gamma f)(t)\| + \|(\Gamma f)'(t)\|] \\
&\le \sup_{t\in I}\bigg[\int_0^t \|S(t-\tau)f(\tau)\|\,d\tau + \int_0^t \|C(t-\tau)f(\tau)\|\,d\tau\bigg] \le 2M\|f\|_{L^1}; \\
\|\Gamma_\diamond u\|_\diamond &= \sup_{t\in I}[\|(\Gamma_\diamond u)(t)\| + \|(\Gamma_\diamond u)'(t)\|] \\
&\le 2M|B|_* \int_0^a \|u(\tau)\|\,d\tau \le 2M\sqrt{a}\,|B|_*\|u\|_{L^2},
\end{aligned}$$

which shows that (3.6) and (3.7) hold, and so Γ and $\Gamma_\diamond$ are bounded. □

Lemma 3.4. *Let S^1_F be defined by* (2.1) *and* (H_F) *be satisfied. Then $S^1_F(x)\neq\emptyset$ and $S^1_F(x)$ is convex and closed for all $x \in \mathrm{PC}^1(I)$.*

Proof. If $\lambda > 0$ is given and $\|u\|_\Theta \le \lambda$, then from (H_F) we have

$$\sup\{\|z\|_E : z \in F(t,u)\} \le \alpha(t)\Phi(\lambda), \quad \text{for a.e. } t \in I.$$

This shows that F is $\alpha(\,\cdot\,)\Phi(\lambda)$-locally integrably bounded. Since $t \mapsto F(t,x_t)$ is measurable, by Lemma 2.5 (a) we see that $S^1_F(x) \neq \emptyset$ and $S^1_F(x)$ is convex and closed for all $x \in \mathrm{PC}^1(J)$. This implies that $S^1_F(x) \neq \emptyset$ and $S^1_F(x)$ is convex and closed for all $x \in \mathrm{PC}^1(I)$. □

Lemma 3.5. $\Gamma \circ S^1_F$ *is precompact in* $\mathrm{PC}^1(I)$.

Proof. If D is a bounded subset of $\mathrm{PC}^1(J)$, then there exists $\lambda_1 > 0$ such that $\|x\|_* \le \lambda_1$ for all $x \in D$. This implies that $\|x\|_\Theta \le \lambda_1$. Thus, by ($\mathrm{H}_F$) there exists $\alpha \in L^1(I,\mathbb{R}^+)$ such that $\|f(t)\|_E \le \alpha(t)\Phi(\lambda_1)$ for all $f \in S^1_F(D)$ and almost every $t \in I$. Since for each $f \in S^1_F(D)$,

$$\|S(t-\tau)f(\tau)\| \le M\Phi(\lambda_1)\alpha(\tau) \quad \text{and} \quad \|C(t-\tau)f(\tau)\| \le M\Phi(\lambda_1)\alpha(\tau),$$

we see that $\{S(t-\tau)f(\tau) : f \in S^1_F(D)\}$ and $\{C(t-\tau)f(\tau) : f \in S^1_F(D)\}$ are all integrably bounded multivalued maps at $\tau \in I$. By (H_F), $\{f(\tau) : f \in S^1_F(D)\} \subset \{F(\tau,x_\tau) : x \in D\}$ is relatively compact. Thus, according to Lemma 2.3 (a), we obtain

$$\beta_H\{S(t-\tau)f(\tau) : f \in S^1_F(D)\} = 0 \quad \text{and} \quad \beta_H\{C(t-\tau)f(\tau) : f \in S^1_F(D)\} = 0.$$

Hence, for each fixed $t \in I$ we have

$$\begin{aligned}
\beta_H\bigg(\int_0^t \big\{S(t-\tau)f(\tau) : f \in S^1_F(D)\big\}\,d\tau\bigg) &= 0, \\
\beta_H\bigg(\int_0^t \big\{C(t-\tau)f(\tau) : f \in S^1_F(D)\big\}\,d\tau\bigg) &= 0.
\end{aligned}$$

This shows that, for each fixed $t \in I$,

$$\bigg\{ \int_0^t S(t-\tau)f(\tau)d\tau : f \in S_F^1(D) \bigg\} \quad \text{and} \quad \bigg\{ \int_0^t C(t-\tau)f(\tau)\,d\tau : f \in S_F^1(D) \bigg\}$$

are all relatively compact in X. Next we show that $\widehat{\Gamma} \circ S_F^1(D)$ is an equicontinuous family of $\mathrm{PC}^1(J)$. Let $f \in S_F^1(D)$, $t, r \in I$ and $0 \le r < t \le a$. Then, using (2.2) and (2.3), from (2.9) and (3.2) we have

$$\begin{aligned}
&\|(\Gamma f)(t) - (\Gamma f)(r)\| \\
&\quad \le \bigg\| \int_0^t [S(t-\tau) - S(r-\tau)]f(\tau)\,d\tau \bigg\| + \bigg\| \int_r^t S(r-\tau)f(\tau)\,d\tau \bigg\| \\
&\quad \le 2M\Phi(\lambda_1)\|\alpha\|_{L^1} |S((t-r)/2)|_* + M\Phi(\lambda_1) \int_r^t \alpha(\tau)\,d\tau; \\
&\|(\Gamma f)'(t) - (\Gamma f)'(r)\| \\
&\quad \le \bigg\| \int_0^t [C(t-\tau) - C(r-\tau)]f(\tau)\,d\tau \bigg\| + \bigg\| \int_r^t C(r-\tau)f(\tau)\,d\tau \bigg\| \\
&\quad \le 2\bigg\| AS((t-r)/2) \int_0^t S((t+r)/2-\tau)f(\tau)\,d\tau \bigg\| + M\Phi(\lambda_1) \int_r^t \alpha(\tau)d\tau.
\end{aligned}$$

Hence, from the uniform continuity of $S(t)$, the absolutely integral continuity of α, the relative compactness of $\big\{ \int_0^t S((t+r)/2-\tau)f(\tau)\,d\tau : f \in S_F^1(D) \big\}$ we see that $\|(\Gamma f)(t) - (\Gamma f)(r)\| \to 0$ and $\|(\Gamma f)'(t) - (\Gamma f)'(r)\| \to 0$ hold uniformly as $t - r \to 0$. This shows that $\widehat{\Gamma} \circ S_F^1(D)$ is an equicontinuous subset of $\mathrm{PC}^1(J)$. As a consequence of the Arzela–Ascoli theorem, $\widehat{\Gamma} \circ S_F^1(D)$ is relatively compact in $\mathrm{PC}^1(J)$. Hence $\widehat{\Gamma} \circ S_F^1$ is precompact in $\mathrm{PC}^1(J)$. This follows that $\Gamma \circ S_F^1$ is precompact in $\mathrm{PC}^1(I)$. □

Lemma 3.6. *$\Gamma \circ S_F^1 : \mathrm{PC}^1(I) \to \mathcal{P}_{\mathrm{cp,cv}}(\mathrm{PC}^1(I))$ is a closed graph map.*

Proof. By Lemmas 3.3–3.5, $\Gamma \circ S_F^1(x)$ is convex and compact for all $x \in \mathrm{PC}^1(I)$. From the proof of Lemma 3.4 we see that F is $\alpha(\cdot)\Phi(\lambda)$-locally integrably bounded. Since F is an u.s.c. map, the assertion is valid by Lemma 2.5 (b). □

Lemma 3.7. *Λ is γ-Lipschitz, where* $\gamma = 2(M \vee M_A) \sum_{k=1}^m (a_k + b_k)$.

Proof. Let $x, y \in \mathrm{PC}^1(I)$ be fixed and $t \in I$. By (2.12), (3.5) and (H_I) we have

$$\begin{aligned}
\|\Lambda x - \Lambda y\|_\diamond &= \sup_{t \in I} [\|(\Lambda x)(t) - (\Lambda y)(t)\| + \|(\Lambda x)'(t) - (\Lambda y)'(t)\|] \\
&\le \|x - y\|_\diamond \bigg[2(M \vee M_A) \sum_{k=1}^m a_k + 2M \sum_{k=1}^m b_k \bigg] \le \gamma \|x - y\|_\diamond,
\end{aligned}$$

which shows that Λ is γ-Lipschitz. □

Lemma 3.8. *$\Gamma_\diamond u_x \colon \mathrm{PC}^1(I) \to \mathcal{P}_{\mathrm{cp,cv}}(\mathrm{PC}^1(I))$ is a closed graph map.*

Proof. Since $\Gamma_\diamond \colon L^2(I,U) \to \mathrm{PC}^1(I)$ is continuous by Lemma 3.3, it is enough to prove that $u_x \colon \mathrm{PC}^1(I) \multimap L^2(I,U)$ is a closed graph map with compact and convex values. From (2.8) we see that

$$u_x(t) = B^* S^*(a-t) R(\delta, -\Upsilon_0^a) p(x), \qquad \text{where } p(x) = x_a - (\Lambda_0 x)(a) - (\Lambda x)(a) - (\Gamma \circ S_F^1(x))(a). \tag{3.8}$$

It is evident that $\Lambda_0 \colon \mathrm{PC}^1(I) \to \mathrm{PC}^1(I)$ is continuous. In view of Lemma 3.7, $\Lambda \colon \mathrm{PC}^1(I) \to \mathrm{PC}^1(I)$ is also continuous. Hence, from (3.8) and Lemma 3.6 it follows that u_x has closed graph. Also, according to (3.8) and Lemmas 3.4 and 3.5, u_x has compact and convex values. □

Proof of Theorem 3.1. Let T be a multivalued map defined by (2.13). By Lemmas 3.4 and 3.2, $T(x) \subset \mathrm{PC}^1(I)$ for all $x \in \mathrm{PC}^1(I)$. Next, we show that T is a β_H-condensing multivalued map. Suppose that D is a bounded subset of $\mathrm{PC}^1(I)$. It is evident that $\beta_H(\Lambda_0(D)) = 0$. From Lemma 3.5 we see that $\beta_H(\Gamma \circ S_F^1(D)) = 0$. We claim that

$$\beta_H\{\Gamma_\diamond u_x : x \in D\} = 0. \tag{3.9}$$

In fact, using the same manner as the proof of Lemm 3.5 we can check that $\{(\Gamma_\diamond u_x)' : x \in D\}$ is equacontinuous in I. It is easy to see that $\{p(x) : x \in D\}$ is bounded. Since B is compact, B^* is compact, and so is $B^* S^*(a-\tau) R(\delta, -\Upsilon_0^a)$. From (3.8) we have $\beta_H\{u_x(\tau) : x \in D\} = 0$. In view of Lemmas 2.3 (a) and 2.4 we obtain

$$\begin{aligned}
\beta_H\{\Gamma_\diamond u_x : x \in D\} &\le \max_{t\in I} \beta_H \bigg\{ \int_0^t S(t-\tau) B u_x(\tau)\, d\tau : x \in D \bigg\} \\
&\quad + \max_{t\in I} \beta_H \bigg\{ \int_0^t C(t-\tau) B u_x(\tau)\, d\tau : x \in D \bigg\} \\
&\le 2 \int_0^a M |B|_* \beta_H\{u_x(\tau) : x \in D\}\, d\tau = 0,
\end{aligned}$$

i.e. (3.9) holds. Therefore, from (3.9) and Lemma 3.7 we have

$$\begin{aligned}
\beta_H(T(D)) \le \beta_H(\Lambda_0(D)) + \beta_H(\Lambda(D)) + \beta_H\{\Gamma_\diamond u_x : x \in D\} \\
+ \beta_H(\Gamma \circ S_F^1(D)) = \beta_H(\Lambda(D)) \le \gamma \beta_H(D),
\end{aligned}$$

which shows that T is a β_H-condensing map due to $\gamma < 1$. This implies that T is quasicompact. From Lemmas 3.6 and 3.8 it is easy to see that $T \colon \mathrm{PC}^1(I) \to \mathcal{P}_{\mathrm{cp,cv}}(\mathrm{PC}^1(I))$ has closed graph. Therefore, we conclude that T is u.s.c. with compact and convex values due to Lemma 2.1. It remains to show that the set $\Omega = \{y \in \mathrm{PC}^1(I) : \exists\, \lambda > 1,\ \lambda y \in T(y)\}$ is bounded. Let $y \in \Omega$ be any element.

Then there exist $\lambda > 1$ and $f_y \in S^1_F(y)$ such that $\lambda y = \Lambda_0 y + \Lambda y + \Gamma f_y + \Gamma_\diamond u_y$. Thus, for each $t \in I$, from $\lambda^{-1} < 1$ we have

$$\begin{aligned}(3.10)\quad \|y(t)\| + \|y'(t)\| \leq \|(\Lambda_0 y)(t)\| + \|(\Lambda_0 y)'(t)\| + \|\Lambda y)(t)\| \\ + \|\Lambda y)'(t)\| + \|\Gamma f_y(t)\| + \|(\Gamma f_y)'(t)\| + \|\Gamma_\diamond u_y(t)\| + \|(\Gamma_\diamond u_y)'(t)\|.\end{aligned}$$

Suppose that C_1 and C_2 are constants given by

$$\begin{aligned}C_1 &= (M + M_A)\|\phi(0)\|_E + 2M\|\xi\|_E;\\ C_2 &= 2(M \vee M_A)\sum_{k=1}^{m}[\|\varphi_k(0)\|_E + \|\psi_k(0)\|_E].\end{aligned}$$

Then, for $t \in I$, from (H_F) and (H_I) we have the following estimates:

$$(3.11)\quad \|(\Lambda_0 y)(t)\| + \|(\Lambda_0 y)'(t)\| \leq (M + M_A)\|\phi(0)\|_E + 2M\|\xi\|_E = C_1;$$

$$(3.12)\quad \|(\Gamma f_y)(t)\| + \|(\Gamma f_y)'(t)\| \leq 2M\int_0^t \alpha(\tau)\Phi(\|y_\tau\|_\Theta)\,d\tau;$$

$$(3.13)\quad \|(\Lambda y)(t)\| + \|(\Lambda y)'(t)\| \leq 2(M \vee M_A)\sum_{0<t_k<t}(a_k + b_k)\|y(t_k)\| + C_2.$$

By (H_B) and (2.8) we have $\|\delta R(\delta, -\Upsilon_0^a)p(y)\| \to 0$ as $\delta \to 0^+$. This implies that, for all $t \in I$,

$$\delta[\|\Gamma_\diamond u_y(t)\| + \|(\Gamma_\diamond u_y)'(t)\|] \to 0 \quad \text{as } \delta \to 0^+.$$

Hence, we can suppose that

$$(3.14)\quad \|\Gamma_\diamond u_y(t)\| + \|(\Gamma_\diamond u_y)'(t)\| \leq \delta^{-1}, \quad t \in I.$$

Consider the function $\mu(t)$ defined by

$$\mu(t) = \sup\{\|y(\tau)\| + \|y'(\tau)\| : \tau \in [0, t]\}, \quad t \in I.$$

Let $\nu(t)$, $\sigma(t)$ be functions defined by $\nu(t) = \|\phi\|_\Theta + \mu(t)$ and

$$\sigma(t) = \frac{\|\phi\|_\Theta + C_1 + C_2 + \delta^{-1}}{1-\gamma} + \frac{2M}{1-\gamma}\int_0^t \alpha(\tau)\Phi(\nu(\tau))\,d\tau.$$

Then $\|y(t_k)\| \leq \mu(t)$ if $0 < t_k < t$. From $\lambda y \in T(y)$ we see that

$$\|y\|_\Theta \leq \lambda^{-1}\|\phi\|_\Theta \leq \|\phi\|_\Theta.$$

Thus, for each $t \in I$, we have $\|y_t\|_\Theta \leq \max\{\|\phi\|_\Theta, \mu(t)\} \leq \nu(t)$. Since Φ is nondecreasing, we have $\Phi(\|y_t\|_\Theta) \leq \Phi(\nu(t))$. From (3.11)–(3.14) and (3.10) it follows that

$$(3.15)\quad \mu(t) \leq C_1 + C_2 + \delta^{-1} + \gamma\nu(t) + 2M\int_0^t \alpha(\tau)\Phi(\nu(\tau))\,d\tau.$$

Thus, by (3.15) and $\gamma < 1$ we have

$$(3.16)\quad \nu(t) \leq \frac{\|\phi\|_\Theta + C_1 + C_2 + \delta^{-1}}{1-\gamma} + \frac{2M}{1-\gamma}\int_0^t \alpha(\tau)\Phi(\nu(\tau))\,d\tau = \sigma(t).$$

Note that $\sigma(0) = (\|\phi\|_\Theta + C_1 + C_2 + \delta^{-1})/(1-\gamma) < +\infty$. Using the nondecreasing character of the function Φ we obtain

$$\sigma'(t) = \frac{2M}{1-\gamma}\,\alpha(t)\Phi(\nu(t)) \le \frac{2M}{1-\gamma}\,\alpha(t)\Phi(\sigma(t)).$$

This implies that

$$\int_{\sigma(0)}^{\sigma(t)} \frac{d\tau}{\Phi(\tau)} \le \frac{2M}{1-\gamma}\int_0^a \alpha(\tau)\,d\tau < +\infty. \tag{3.17}$$

From (3.1) and (3.17) it follows that $\sigma(t)$ is bounded in I. Hence, there exists a constant $\rho_\diamond$ such that $\sigma(t) \le \rho_\diamond$ for all $t \in I$; and so, for each $y \in \Omega$ we have $\|y\|_\diamond = \mu(a) \le \nu(a) \le \sigma(a) \le \rho_\diamond$, which shows that Ω is bounded.

Notice that $\mathrm{PC}^1(I)$ is a Banach space. As a consequence of Lemma 2.2 we deduce that T has at least one fixed point, and so $\widehat{T}$ has at least one fixed point. Let x be a fixed point of $\widehat{T}$, then $x(\,\cdot\,;\phi,\xi;u)$ is the mild solution of System (CIP) under the control u given by (2.4). Thus, there is $f \in S^1_F(x)$ such that

$$\begin{aligned} x(a) &= x(a;\phi,\xi;u) = (\Lambda_0 x)(a) + (\Lambda x)(a) + (\Gamma_\diamond u_x)(a) + (\Gamma f)(a)\\ &= (\Lambda_0 x)(a) + (\Lambda x)(a) + (\Gamma f)(a)\\ &\quad + \int_0^a S(a-\tau)BB^*S^*(a-\tau)R(\delta,-\Upsilon_0^a)p(x)\,d\tau, \end{aligned}$$

where $p(x) = x_a - (\Lambda_0 x)(a) - (\Lambda x)(a) - (\Gamma f)(a)$. By the definition of the operators Υ_0^a and $R(\delta,-\Upsilon_0^a)$, we have

$$x(a) - x_a = -\delta R(\delta,-\Upsilon_0^a)[x_a - (\Lambda_0 x)(a) - (\Lambda x)(a) - (\Gamma f)(a)]. \tag{3.18}$$

From (3.18) and assumption (H_A) we know that the system (CIP) is position approximate controllable on the interval J. $\square$

In the same manner as the proof of Theorem 3.1, by means of the resolvent operator $R(\delta,-\Psi_0^a)$ and the operator T_1 defined by (2.14), we have the following consequence.

Theorem 3.9. *Let $\phi \in \Theta$ with $\phi(0) \in E$ and $\xi \in E$. Let* (H_A), (H_B^1) *hold and conditions* (H_I), $\mathrm{H}_F)$ *in Theorem* 3.1 *be satisfied. If*

$$\gamma = 2(M \vee M_A)\sum_{k=1}^m (a_k + b_k) < 1 \quad \textit{and} \quad \int_1^{+\infty} \frac{d\tau}{\Phi(\tau)} = +\infty,$$

then the system (CIP) *is velocity approximate controllable on J.*

4. An example

In this section, as an application of our results we consider the following controllability problem for second order partial delay differential inclusions:

$$
\text{(PCIP)}\qquad \begin{cases} y_{tt}(t,\omega) - y_{\omega\omega}(t,\omega) \in F(t, y(t-s,\omega)) + Bu(t), & \\ & t \in I \setminus \{t_k\}_{k=1}^m, \\ & \omega \in \Omega,\ s \in J_0; \\ y(t,0) = y(t,\pi) = 0, & t \in I; \\ y(t_k^+,\omega) - y(t_k,\omega) = \varphi_k(y(t_k,\,\cdot\,),\omega), & k = 1,\dots,m; \\ y_t(t_k^+,\omega) - y_t(t_k^-,\omega) = \psi_k(y(t_k,\,\cdot\,),\omega), & k = 1,\dots,m; \\ y(s,\omega) = \phi(s,\omega),\ y_t(0,\omega) = \xi(\omega), & s \in J_0, \end{cases}
$$

where $I = [0,a]$, $J_0 = [-r,0]$, $\Omega = [0,\pi]$; $u(\,\cdot\,) \in L^2(I,\mathbb{R})$; $t_k \in (0,a)$, $\xi(\,\cdot\,) \in L^2(\Omega,\mathbb{R})$; B is a compact linear operator; $F\colon I\times\mathbb{R} \multimap \mathbb{R}$ is a multivalued function and $\phi\colon J_0 \times \Omega \to \mathbb{R}$ is a single function such that $\phi(\,\cdot\,,\omega)\colon J_0 \to L^2(\Omega,\mathbb{R})$ is continuously differentiable.

We put $y(t,\,\cdot\,) = x(t)$ and choose the space $X = L^2(\Omega,\mathbb{R})$. Define $A\colon D(A) \subset X \to X$ by $Ax = x''$ with domain

$$
\begin{aligned} D(A) = \{x \in X : x \text{ and } x' \text{ are absolutely continuous,} \\ x'' \in X \text{ and } x(0) = x(\pi) = 0\}, \end{aligned}
$$

then $y_{\omega\omega}(t,\omega) = Ax(t)$, and it is well known that (see [12], [17], [35] for more details)

$$
E = \{x \in X : x \text{ are absolutely continuous, } x' \in X \text{ and } x(0) = x(\pi) = 0\};
$$

A has the spectral representation

$$
Ax = \sum_{n=1}^{\infty} -n^2\langle x, e_n\rangle e_n, \quad x \in D(A),
$$

where $\{-n^2 : n \in \mathbb{Z}^+\}$ is the discrete spectrum of A and

$$
\left\{e_n : e_n(\omega) = \sqrt{\frac{2}{\pi}}\sin n\omega,\ \omega \in \Omega,\ n \in \mathbb{Z}^+\right\}
$$

is the orthogonal set of normalized eigenfunctions. Also, it can be shown that A is the infinitesimal generator of the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ given by

$$
C(t)x = \sum_{n=1}^{\infty} \cos nt\,\langle x, e_n\rangle\, e_n, \quad x \in X,
$$

and the associated sine family is given by

$$
S(t)x = \sum_{n=1}^{\infty} \frac{\sin nt}{n}\,\langle x, e_n\rangle\, e_n, \quad x \in X,
$$

We endow E with the norm $\|\cdot\|_E$ and take the spaces Θ and PC^1 as above. Then the control system (PCIP) is converted into (CIP). Let conditions (H_A), (H_B), (H_B^1), (H_F) and (H_I) in Theorems 3.1 and 3.9 are satisfied. If $\gamma < 1$ and $\int_1^{+\infty} d\tau/\Phi(\tau) = +\infty$, then the system (PCIP) is approximately controllable for position and velocity.

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