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NONOSCILLATORY SOLUTIONS OF HIGHER ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Consider the forced higher order nonlinear neutral functional differential equation

$$L_n(x(t) + cx(t-\tau)) + F(t, x(\sigma(t))) = g(t), \quad t \geq t_0.$$

We obtain a global result, with respect to c, which are some sufficient conditions for the existence of a nonoscillatory solution of the above equation. Our results improve essentially and extend a number of existing results.

1. Introduction. Consider the forced higher order nonlinear neutral functional differential equation

$$L_n(x(t) + cx(t-\tau)) + F(t, x(\sigma(t))) = g(t), \quad t \geq t_0, \tag{1}$$

where

$$\begin{aligned} &L_0x(t) = x(t), \\ &L_kx(t) = \frac{1}{r_k(t)}\,(L_{k-1}x(t))', \quad k = 1, 2, \dots, n \ \Big(' = \frac{d}{dt}\Big), \end{aligned}$$

$r_k : [t_0, \infty) \to (0, \infty)$, $k = 1, 2, \dots, n-1$, $r_n \equiv 1$, $\sigma, g : [t_0, \infty) \to \mathbf{R}$, and $F : [t_0, \infty) \times \mathbf{R} \to \mathbf{R}$, $t_0 \geq 0$, are continuous, $\sigma(t) \to \infty$ as $t \to \infty$.

A nontrivial solution x of equation (1) is said to be oscillatory if x has arbitrarily large zeros. Otherwise, x is said to be nonoscillatory. That is, x is nonoscillatory if there exists a $t_1 > t_0$ such that $x(t) \neq 0$ for $t \geq t_1$. In other words, a nonoscillatory solution must be eventually positive or eventually negative, see [**7, 9**].

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Oscillation theory of neutral functional differential equations has developed very rapidly in recent years. It has concerned itself largely with the oscillatory and nonoscillatory properties of solutions, see, e.g., [**1**–**15**] and the references cited therein. The problem of obtaining sufficient conditions to ensure that all solutions of equation (1) and the special forms of equation (1) are oscillatory has been studied by a number of authors, see [**1**–**4, 7, 9, 11, 14**]. Existence of nonoscillatory solutions of first order or higher order neutral differential equations has also been obtained in [**2, 3, 5, 6, 8, 10, 12, 13, 15**].

In [**7**], the authors study the first order functional differential equation

$$\frac{d}{dt}\,(x(t)+cx(t-\tau))+F(t,x(\sigma(t)))=g(t), \quad t\geq t_0. \tag{2}$$

They proved the following result by using the Banach contraction mapping principle:

Theorem A [**7**]. *Assume that*

(C_1) $-1<c<1$;

(C_2) $|F(t,x)|\leq|F(t,y)|$, *as* $|x|\leq|y|$, *and for each closed interval* $[d_1,d_2]$, $0<d_1<d_2$, *there exists* $L(t)$ *such that* $|F(t,x)-F(t,y)|\leq L(t)|x-y|$, $x,y\in[d_1,d_2]$, *and* $\int_{t_0}^{\infty}L(s)\,ds<\infty$.

Further, assume that

$$\int_{t_0}^{\infty}|F(s,d)|\,ds<\infty, \quad \text{for some} \quad d\neq 0,$$

and

$$\int_{t_0}^{\infty}|g(s)|\,ds<\infty.$$

Then (2) *has a bounded nonoscillatory solution.*

In [**12**], Pathi and Rath investigate the existence of nonoscillatory solutions of first order neutral differential equation

$$\frac{d}{dt}\,[x(t)+cx(t-\tau)]+Q(t)f(x(t-\sigma))=g(t), \quad t\geq t_0, \tag{3}$$

where $Q\in C([t_o,\infty),\ \mathbf{R}),\sigma>0$.

Theorem B [**12**]. *Assume that*

(C_3) $c > 0$ *or* $c < -1$;

(C_4) $Q(t) \geq 0$, $f \in C(\mathbf{R}, \mathbf{R})$ *is nondecreasing,* $xf(x) \geq 0$ *for any* $x \neq 0$ *and* f *satisfy the Lipschitz condition on intervals of the type* $[a, b]$, $0 < a < b$.

Further, assume that

$$\int_0^\infty Q(s)\,ds < \infty, \qquad \int_0^\infty |g(s)|\,ds < \infty.$$

Then equation (3) *has a nonoscillatory solution.*

In [**3**], Agarwal, Grace and O'Regan give an existence criteria for nonoscillatory solutions of the second order neutral differential equation

$$L_2(x(t) + cx(t-\tau)) + F(t, x(\sigma(t))) = g(t), \quad t \geq t_0. \tag{4}$$

Theorem C [**3**]. *Assume the following*:

(C_5) $|c| \neq 1$;

(C_6) $xF(t, x) > 0$ *for* $x \neq 0$, $|F(t, x)| \leq |F(t, y)|$ *for* $|x| \leq |y|$, $xy > 0$.

If

$$\int_{t_0}^\infty r_1(s_1) \int_{s_1}^\infty |F(s, K)|\,ds\,ds_1 < \infty$$

for some constant $K \neq 0$, then (4) has a bounded nonoscillatory solution.

For the second order neutral functional differential equation with positive and negative coefficients

$$\frac{d^2}{dt^2}\,(x(t) + cx(t-\tau)) + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad t \geq t_0, \tag{5}$$

where $c \neq \pm 1$, $\sigma_1 > 0$, $\sigma_2 > 0$, $Q_1(t) \geq 0$ and $Q_2(t) \geq 0$, Kulenovic and Hadziomerspahic [**10**] proved the following results by using Banach contraction mapping principle.

Theorem D [**10**]. *Assume that*

(C_7) $c \neq \pm 1$,

(C_8) $aQ_1(t) - Q_2(t) \geq 0$, *for every* $t \geq T$ *and* $a > 0$.

Further, assume that

$$\int_{t_0}^{\infty} s\, Q_1(s)\, ds < \infty, \qquad \int_{t_0}^{\infty} s\, Q_2(s)\, ds < \infty.$$

Then equation (5) *has a nonoscillatory solution.*

In [**7**], the existence result of nonoscillatory solution for higher order functional differential equation

$$(x(t) + cx(t-\tau))^{(n)} + F(t, x(\sigma(t))) = 0, \quad t \geq t_0, \tag{6}$$

has been obtained.

Theorem E [**7**]. *Assume that*

(C_9) $c < 0$,

(C_{10}) F *is nondecreasing in* x, $F(t,x)x \leq 0$ *for* $(t,x) \in [t_0, \infty) \times \mathbf{R}$.

Further, assume that

$$\int_{t_0}^{\infty} s^{n-1} |F(s, K)|\, ds < \infty, \quad \text{for any} \quad K \neq 0.$$

Then equation (6) *has a nonoscillatory solution.*

Our aim in this paper is to investigate the existence of nonoscillatory solutions of equation (1). By using Krasnoselskii's and Schauder's fixed point theorems and some new techniques, we obtain a global result, with respect to c, which are some sufficient conditions for the existence of a nonoscillatory solution of equation (1). Our results improve and extend Theorems A, B, C, D and E by removing the restrictive conditions (C_1)–(C_{10}).

2. Main results. The following fixed point theorems will be used to prove the main results in this section.

Lemma 1 [**7**] (Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x$$

has a solution in Ω.

Lemma 2 [**7, 9**] (Schauder's fixed point theorem). *Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.*

Theorem 1. *Assume that $c \neq -1$ and that there exists an interval $[a, b] \subset \mathbf{R}^+$ such that*

$$\begin{aligned}(7)\quad &\int_{t_0}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} \sup_{w\in[a,b]} |F(s, w)|\, ds\, ds_{n-1} \cdots ds_1 < \infty,\end{aligned}$$

and

$$\begin{aligned}(8)\quad &\int_{t_0}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} |g(s)|\, ds\, ds_{n-1} \cdots ds_1 < \infty.\end{aligned}$$

Then (1) has a bounded nonoscillatory solution.

Proof. The proof of this theorem will be divided into five cases in terms of c. Let $C([t_0, \infty), \mathbf{R})$ be the set of all continuous functions with the norm $||x|| = \sup_{t\geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbf{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbf{R})$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), \mathbf{R}) : a \leq x(t) \leq b,\ t \geq t_0\}.$$

Case 1. For the case $-1 < c \le 0$, by (7) and (8), we choose a $T > t_0$ sufficiently large such that

$$\int_T^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^\infty \Big(\sup_{w \in [a,b]} |F(s,w)| + |g(s)| \Big) \, ds \, ds_{n-1} \cdots ds_1 < \frac{(c+1)(b-a)}{2}.$$

Define two maps S_1 and $S_2 : \Omega \to C([t_0,\infty), \mathbf{R})$ as follows:

$$(S_1 x)(t) = \begin{cases} \dfrac{(c+1)(a+b)}{2} - cx(t-\tau) & t \ge T, \\ (S_1 x)(T) & t_0 \le t \le T. \end{cases}$$

$$(S_2 x)(t) = \begin{cases} (-1)^{n+1} \displaystyle\int_t^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\ \quad \displaystyle\int_{s_{n-1}}^\infty \Big(F(s, x(\sigma(s))) - g(s) \Big) \, ds \, ds_{n-1} \cdots ds_1 & t \ge T, \\ (S_2 x)(T) & t_0 \le t \le T. \end{cases}$$

i) We shall show that, for any $x, y \in \Omega$, $S_1 x + S_2 y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $t \ge T$, we get

$$\begin{aligned} &(S_1 x)(t) + (S_2 y)(t) \\ &\le \frac{(c+1)(a+b)}{2} - cx(t-\tau) + \int_t^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \\ &\qquad \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^\infty \Big(|F(s, y(\sigma(s)))| + |g(s)| \Big) \, ds \, ds_{n-1} \cdots ds_1 \end{aligned}$$

$$\begin{aligned}
&\le \frac{(c+1)(a+b)}{2} - cb + \int_T^\infty r_1(s_1)\int_{s_1}^\infty r_2(s_2)\cdots \\
&\qquad\qquad \cdots\int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times\int_{s_{n-1}}^\infty \Big(\sup_{w\in[a,b]}|F(s,w)|+|g(s)|\Big)\,ds\,ds_{n-1}\cdots ds_1 \\
&\le \frac{(c+1)(a+b)}{2} - cb + \frac{(c+1)(b-a)}{2} = b.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&(S_1x)(t)+(S_2y)(t) \\
&\ge \frac{(c+1)(a+b)}{2} - cx(t-\tau) - \int_t^\infty r_1(s_1)\int_{s_1}^\infty r_2(s_2)\cdots \\
&\qquad\qquad \cdots\int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times\int_{s_{n-1}}^\infty \Big(|F(s,y(\sigma(s)))|+|g(s)|\Big)\,ds\,ds_{n-1}\cdots ds_1 \\
&\ge \frac{(c+1)(a+b)}{2} - ca - \int_T^\infty r_1(s_1)\int_{s_1}^\infty r_2(s_2)\cdots \\
&\qquad\qquad \cdots\int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times\int_{s_{n-1}}^\infty \Big(\sup_{w\in[a,b]}|F(s,w)|+|g(s)|\Big)\,ds\,ds_{n-1}\cdots ds_1 \\
&\ge \frac{(c+1)(a+b)}{2} - ca - \frac{(c+1)(b-a)}{2} = a.
\end{aligned}$$

Hence,

$$a \le (S_1x)(t)+(S_2y)(t) \le b, \quad \text{for} \quad t\ge t_0.$$

Thus we have proved that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

ii) We shall show that S_1 is a contraction mapping on Ω.

In fact, for $x,y\in\Omega$ and $t\ge T$, we have

$$|(S_1x)(t)-(S_1y)(t)| \le -c|x(t-\tau)-y(t-\tau)| \le -c||x-y||.$$

This implies that

$$||S_1x - S_1y|| \le -c\,||x - y||.$$

Since $0 < -c < 1$, we conclude that S_1 is a contraction mapping on Ω.

iii) We now show that S_2 is completely continuous.

First, we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \ge T$, we have

$$\begin{aligned}
&|(S_2x_k)(t) - (S_2x)(t)| \\
&\qquad \le \int_t^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^\infty \Big(|F(s, x_k(\sigma(s))) - F(s, x(\sigma(s)))| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad \le \int_T^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^\infty \Big(|F(s, x_k(\sigma(s))) - F(s, x(\sigma(s)))| \Big)\, ds\, ds_{n-1} \cdots ds_1
\end{aligned}$$

Since $|F(t, x_k(\sigma(t))) - F(t, x(\sigma(t)))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{k\to\infty} ||(S_2x_k)(t) - (S_2x)(t)|| = 0$. This means that S_2 is continuous.

Next, we show that $S_2\Omega$ is relatively compact. It suffices to show that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result, we only need to show that, for any given $\varepsilon > 0$, $[T, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. By (7) and (8), for any $\varepsilon > 0$, take $T^* \ge T$ large enough so that

$$\begin{aligned}
&\int_{T^*}^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\
&\qquad\qquad \int_{s_{n-1}}^\infty \Big(\sup_{w\in[a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 < \frac{\varepsilon}{2}.
\end{aligned}$$

Then for $x \in \Omega$, $t_2 > t_1 \geq T^*$,

$$
\begin{aligned}
&|(S_2x)(t_2) - (S_2x)(t_1)| \\
&\qquad \leq \int_{t_2}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad\quad + \int_{t_1}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad \leq \int_{t_2}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad\quad + \int_{t_1}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

For $x \in \Omega$ and $T \leq t_1 < t_2 \leq T^*$,

$$
\begin{aligned}
&|(S_2x)(t_2) - (S_2x)(t_1)| \\
&\qquad \leq \int_{t_1}^{t_2} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad \leq \int_{t_1}^{t_2} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1.
\end{aligned}
$$

Then there exists a $\delta > 0$ such that

$$|(S_2x)(t_2) - (S_2x)(t_1)| < \varepsilon, \quad \text{if} \quad 0 < t_2 - t_1 < \delta.$$

For any $x \in \Omega$, $t_0 \le t_1 < t_2 \le T$, it is easy to see that

$$|(S_2x)(t_2) - (S_2x)(t_1)| = 0 < \varepsilon.$$

Therefore $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact. By Lemma 1 (Krasnoselskii's fixed point theorem), there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. It is easy to see that $x_0(t)$ is a bounded nonoscillatory solution of equation (1). This completes the proof in this case.

Case 2. For the case $c < -1$, by (7) and (8), we choose a $T > t_0$ sufficiently large such that

$$\begin{aligned} &-\frac{1}{c}\int_{T+\tau}^{\infty} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\cdots\int_{s_2}^{\infty} r_{n-1}(s_{n-1}) \\ &\quad \int_{s_{n-1}}^{\infty}\Big(\sup_{w\in[a,b]}|F(s,w)| + |g(s)|\Big)\,ds\,ds_{n-1}\cdots ds_1 < \frac{(c+1)(b-a)}{2c}. \end{aligned}$$

Define two maps S_1 and $S_2 : \Omega \to C([t_0, \infty), \mathbf{R})$ as follows:

$$(S_1x)(t) = \begin{cases} \dfrac{(c+1)(a+b)}{2c} - \dfrac{x(t+\tau)}{c} & t \ge T, \\ (S_1x)(T) & t_0 \le t \le T. \end{cases}$$

$$(S_2x)(t) = \begin{cases} \dfrac{(-1)^{n+1}}{c}\displaystyle\int_{t+\tau}^{\infty} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\cdots\int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) & \\ \quad \displaystyle\int_{s_{n-1}}^{\infty}\Big(F(s,x(\sigma(s))) - g(s)\Big)\,ds\,ds_{n-1}\cdots ds_1 & t \ge T, \\ (S_2x)(T) & t_0 \le t \le T. \end{cases}$$

i) We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $t \geq T$, we get

$$
\begin{aligned}
&(S_1x)(t) + (S_2y)(t) \\
&\quad \leq \frac{(c+1)(a+b)}{2c} - \frac{x(t+\tau)}{c} \\
&\qquad - \frac{1}{c}\int_{t+\tau}^{\infty} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\cdots\int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty}\Big(|F(s, y(\sigma(s)))| + |g(s)|\Big)\, ds\, ds_{n-1}\cdots ds_1 \\
&\quad \leq \frac{(c+1)(a+b)}{2c} - \frac{b}{c} - \frac{1}{c}\int_{T}^{\infty} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\cdots \\
&\qquad\quad \cdots\int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\quad \times \int_{s_{n-1}}^{\infty}\Big(\sup_{w\in[a,b]}|F(s,w)| + |g(s)|\Big)\, ds\, ds_{n-1}\cdots ds_1 \\
&\quad \leq \frac{(c+1)(a+b)}{2c} - \frac{b}{c} + \frac{(c+1)(b-a)}{2c} = b.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&(S_1x)(t) + (S_2y)(t) \\
&\quad \geq \frac{(c+1)(a+b)}{2c} - \frac{x(t+\tau)}{c} + \frac{1}{c}\int_{t+\tau}^{\infty} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\cdots \\
&\qquad\qquad \cdots\int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty}\Big(|F(s, y(\sigma(s)))| + |g(s)|\Big)\, ds\, ds_{n-1}\cdots ds_1 \\
&\quad \geq \frac{(c+1)(a+b)}{2c} - \frac{a}{c} + \frac{1}{c}\int_{T}^{\infty} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\cdots \\
&\qquad\qquad \cdots\int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty}\Big(\sup_{w\in[a,b]}|F(s,w)| + |g(s)|\Big)\, ds\, ds_{n-1}\cdots ds_1 \\
&\quad \geq \frac{(c+1)(a+b)}{2c} - \frac{a}{c} - \frac{(c+1)(b-a)}{2c} = a.
\end{aligned}
$$

Hence,

$$a \le (S_1x)(t) + (S_2y)(t) \le b, \quad \text{for} \quad t \ge t_0.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

ii) We shall show that S_1 is a contraction mapping on Ω.

In fact, for $x, y \in \Omega$ and $t \ge T$, we have

$$|(S_1x)(t) - (S_1y)(t)| \le -\frac{1}{c}\,|x(t-\tau) - y(t-\tau)| \le -\frac{1}{c}\,||x-y||.$$

This implies that

$$||S_1x - S_1y|| \le -\frac{1}{c}\,||x-y||.$$

Since $0 < -1/c < 1$, we conclude that S_1 is a contraction mapping on Ω.

Proceeding similarly as in the proof of case 1, we obtain that the mapping S_2 is completely continuous. By Lemma 1, there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of equation (1). This completes the proof in this case.

Case 3. For the case $0 \le c < 1$, by (7) and (8), we choose a $T > t_0$ sufficiently large such that

$$\begin{aligned}&\int_T^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\ &\quad\times \int_{s_{n-1}}^\infty \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)|\Big)\, ds\, ds_{n-1} \cdots ds_1 < \frac{(1-c)(b-a)}{2}.\end{aligned}$$

Define two maps S_1 and $S_2 : \Omega \to C([t_0,\infty), \mathbf{R})$ as follows:

$$(S_1x)(t) = \begin{cases} \dfrac{(c+1)(a+b)}{2} - cx(t-\tau) & t \ge T, \\ (S_1x)(T) & t_0 \le t \le T.\end{cases}$$

$$(S_2x)(t) = \begin{cases} (-1)^{n+1} \int_t^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\ \qquad \int_{s_{n-1}}^\infty \Big(F(s, x(\sigma(s))) - g(s) \Big) \, ds \, ds_{n-1} \cdots ds_1 & t \ge T, \\ (S_2x)(T) & t_0 \le t \le T. \end{cases}$$

The rest of the proof is similar to that of Case 1 and, thus, is omitted.

Case 4. For the case $c > 1$, by (7) and (8), we choose a $T > t_0$ sufficiently large such that

$$\frac{1}{c} \int_{T+\tau}^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\ \times \int_{s_{n-1}}^\infty \Big(\sup_{w \in [a,b]} |F(s,w)| + |g(s)| \Big) \, ds \, ds_{n-1} \cdots ds_1 < \frac{(c-1)(b-a)}{2c}.$$

Define two maps S_1 and $S_2 : \Omega \to C([t_0, \infty), \mathbf{R})$ as follows:

$$(S_1x)(t) = \begin{cases} \dfrac{(c+1)(a+b)}{2c} - \dfrac{x(t+\tau)}{c} & t \ge T, \\ (S_1x)(T) & t_0 \le t \le T. \end{cases}$$

$$(S_2x)(t) = \begin{cases} \dfrac{(-1)^{n+1}}{c} \int_{t+\tau}^\infty r_1(s_1) \int_{s_1}^\infty r_2(s_2) \cdots \int_{s_{n-2}}^\infty r_{n-1}(s_{n-1}) \\ \quad \int_{s_{n-1}}^\infty \Big(F(s, x(\sigma(s))) - g(s) \Big) \, ds \, ds_{n-1} \cdots ds_1 \\ & t \ge T, \\ (S_2x)(T) & t_0 \le t \le T. \end{cases}$$

The rest of the proof is similar to that of Case 2 and, thus, is omitted.

Case 5. For the case $c = 1$, by (7) and (8), we choose a $T > t_0$ sufficiently large such that

$$\int_{T+\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 < \frac{(b-a)}{2}.$$

Define a map $S : \Omega \to C([t_0,\infty), \mathbf{R})$ as follows:

$$(Sx)(t) = \begin{cases} \dfrac{a+b}{2} + (-1)^{n+1} \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ \quad \displaystyle\int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} \Big(F(s, x(\sigma(s)) - g(s) \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq T, \\ (Sx)(T) \qquad\qquad\qquad\qquad\qquad\qquad\qquad t_0 \leq t \leq T. \end{cases}$$

i) We shall show that, for any $x, y \in \Omega$, $S\Omega \in \Omega$.

In fact, for every $x \in \Omega$ and $t \geq T$, we get

$$\begin{aligned} (Sx)(t) &\leq \frac{(a+b)}{2} + \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ &\leq \frac{(a+b)}{2} + \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ &\leq \frac{(a+b)}{2} + \frac{(b-a)}{2} = b. \end{aligned}$$

Furthermore, we have

$$\begin{aligned}(Sx)(t) \geq \frac{(a+b)}{2} &- \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big) \, ds \, ds_{n-1} \cdots ds_1 \\ \geq \frac{(a+b)}{2} &- \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big) \, ds \, ds_{n-1} \cdots ds_1 \\ \geq \frac{(a+b)}{2} &- \frac{(b-a)}{2} = a\end{aligned}$$

Hence, $S\Omega \subset \Omega$.

Proceeding similarly as in the proof of Case 1 we obtain that the mapping S is completely continuous. By Lemma 2, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$, that is,

$$x_0(t) = \begin{cases} \frac{a+b}{2} + (-1)^{n+1} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) & \\ \quad \int_{s_{n-1}}^{\infty} \Big(F(s, x_0(\sigma(s))) - g(s) \Big) \, ds \, ds_{n-1} \cdots ds_1 & t \geq T, \\ x_0(T) & t_0 \leq t \leq T. \end{cases}$$

It follows that

$$\begin{aligned}x_0(t) &+ x_0(t-\tau) \\ &= a + b + (-1)^{n+1} \int_t^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} \Big(F(s, x_0(\sigma(s))) - g(s) \Big) \, ds \, ds_{n-1} \cdots ds_1.\end{aligned}$$

Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of equation (1). This completes the proof of Theorem 1.

Theorem 2. *Assume that $c = -1$ and that there exists an interval $[a, b] \subset \mathbf{R}^+$ such that*

$$\int_{t_0}^{\infty} s_1 r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} \sup_{w\in[a,b]} |F(s,w)|\, ds\, ds_{n-1} \cdots ds_1 < \infty, \tag{9}$$

and

$$\int_{t_0}^{\infty} s_1 r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} |g(s)|\, ds\, ds_{n-1} \cdots ds_1 < \infty. \tag{10}$$

Then (1) *has a bounded nonoscillatory solution.*

Proof. By a known result [**7**, Theorem 3.2.6], (9) and (10) are equivalent to

$$\sum_{j=0}^{\infty} \int_{t_0+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} \sup_{w\in[a,b]} |F(s,w)|\, ds\, ds_{n-1} \cdots ds_1 < \infty, \tag{11}$$

and

$$\sum_{j=0}^{\infty} \int_{t_0+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} |g(s)|\, ds\, ds_{n-1} \cdots ds_1 < \infty, \tag{12}$$

respectively. We choose a sufficiently large $T > t_0$ such that

$$\sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 < \frac{b-a}{2}.$$

Define a mapping $S : \Omega \to C([t_0, \infty), \mathbf{R})$ as follows:

$$(Sx)(t)$$
$$= \begin{cases} \dfrac{a+b}{2} + (-1)^n \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) & \\ \quad \displaystyle\int_{s_{n-1}}^{\infty} \Big(F(s, x(\sigma(s))) - g(s) \Big)\, ds\, ds_{n-1} \cdots ds_1 & t \geq T, \\ (Sx)(T) & t_0 \leq t \leq T. \end{cases}$$

We shall show that $S\Omega \subset \Omega$.

In fact, for every $x \in \Omega$ and $t \geq T$, we get

$$\begin{aligned} (Sx)(t) \leq \frac{a+b}{2} &+ \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ \leq \frac{a+b}{2} &+ \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ \leq \frac{a+b}{2} &+ \frac{b-a}{2} = b. \end{aligned}$$

Furthermore, we have

$$\begin{aligned}(Sx)(t) &\geq \frac{a+b}{2} - \sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ &\geq \frac{a+b}{2} - \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\ &\geq \frac{a+b}{2} - \frac{b-a}{2} = a.\end{aligned}$$

Hence, $S\Omega \subset \Omega$.

We now show that S is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq T$, we have

$$\begin{aligned}|(Sx_k)(t) - (Sx)(t)| &\leq \sum_{j=1}^{\infty} \int_{T+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ &\qquad \int_{s_{n-1}}^{\infty} |F(s, x_k(\sigma(s))) - F(s, x(\sigma(s)))|\, ds\, ds_{n-1} \cdots ds_1\end{aligned}$$

Since $|F(t, x_k(\sigma(t))) - F(x(\sigma(t)))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{k\to\infty} ||(Sx_k)(t) - (Sx)(t)|| = 0$. This means that S is continuous.

In the following, we show that $S\Omega$ is relatively compact. By (11) and (12), for any $\varepsilon > 0$, take $T^* \geq T$ large enough so that

$$\sum_{j=1}^{\infty} \int_{T^*+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 < \frac{\varepsilon}{2}.$$

Then for $x \in \Omega$, $t_2 > t_1 \geq T^*$,

$$\begin{aligned}
&|(Sx)(t_2) - (Sx)(t_1)| \\
&\quad\leq \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s,x(\sigma(s)))| + |g(s)|\Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad + \sum_{j=1}^{\infty} \int_{t_1+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s,x(\sigma(s)))| + |g(s)|\Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad\leq \sum_{j=1}^{\infty} \int_{t_2+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)|\Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad + \sum_{j=1}^{\infty} \int_{t_1+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)|\Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For $T \leq t_1 < t_2 \leq T^*$, we choose a sufficiently large $J \in \mathbf{N}^+$ such that

$T + j\tau \geq T^*$ as $j \geq J$. For $x \in \Omega$

$$\begin{aligned}
&|(Sx)(t_2) - (Sx)(t_1)| \\
&\quad \leq \sum_{j=1}^{\infty} \int_{t_1+j\tau}^{t_2+j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad \leq \sum_{j=1}^{J} \int_{t_1+j\tau}^{t_2+j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad + \sum_{j=J+1}^{\infty} \int_{t_1+j\tau}^{t_2+j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(|F(s, x(\sigma(s)))| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\quad \leq \sum_{j=1}^{J} \int_{t_1+j\tau}^{t_2+j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 \\
&\qquad + \sum_{j=1}^{\infty} \int_{T^*+j\tau}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\
&\qquad\qquad \times \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1.
\end{aligned}$$

Then there exists a $\delta > 0$ such that

$$\begin{aligned}
&\sum_{j=1}^{J} \int_{t_1+j\tau}^{t_2+j\tau} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \\
&\cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \Big)\, ds\, ds_{n-1} \cdots ds_1 < \frac{\varepsilon}{2}, \\
&\qquad\qquad\qquad \text{if} \quad 0 < t_2 - t_1 < \delta.
\end{aligned}$$

Hence,

$$|(Sx)(t_2) - (Sx)(t_1)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{if} \quad 0 < t_2 - t_1 < \delta.$$

For any $x \in \Omega$, $t_0 \le t_1 < t_2 \le T$, it is easy to see that

$$|(Sx)(t_2) - (Sx)(t_1)| = 0 < \varepsilon.$$

Therefore $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S\Omega$ is relatively compact. By Lemma 2 (Schauder's fixed point theorem), there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. That is,

$$x_0(t) = \begin{cases} \dfrac{a+b}{2} + (-1)^n \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{\infty} r_1(s_{n-1}) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) & \\ \quad \displaystyle\int_{s_{n-1}}^{\infty} \Big(F(s, x_0(\sigma(s)) - g(s)\Big)\, ds\, ds_{n-1} \cdots ds_1 & t \ge T, \\ x_0(T) & t_0 \le t \le T. \end{cases}$$

It follows that

$$\begin{aligned} x_0(t) - x_0(t-\tau) = (-1)^{n+1} \int_t^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \\ \times \int_{s_{n-1}}^{\infty} \Big(F(s, x_0(\sigma(s)) - g(s)\Big)\, ds\, ds_{n-1} \cdots ds_1, \\ t \ge T. \end{aligned}$$

Clearly, $x_0 = x_0(t)$ is a bounded nonoscillatory solution of equation (1). This completes the proof of Theorem 2.

When $r_k(t) \equiv 1$, $k = 1, 2, \dots, n$, equation (1) reduces to

$$(x(t) + cx(t-\tau))^{(n)} + F(t, x(\sigma(t)) = g(t), \quad t \ge t_0. \tag{13}$$

By using Theorems 1 and 2, we obtain the following results.

Corollary 1. *Assume that $c \ne -1$ and that there exists an interval $[a, b] \subset \mathbf{R}^+$ such that*

$$\int_{t_0}^{\infty} s^{n-1} \sup_{w \in [a,b]} |F(s, w)|\, ds < \infty, \tag{14}$$

and

$$\int_{t_0}^{\infty} s^{n-1}|g(s)|\,ds < \infty. \tag{15}$$

Then (13) *has a bounded nonoscillatory solution.*

Proof. We note that (14) and (15) are equivalent to

$$\int_{t_0}^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{n-2}}^{\infty}\int_{s_{n-1}}^{\infty}\sup_{w\in[a,b]}|F(s,w)|\,ds\,ds_{n-1}\cdots ds_1 < \infty, \tag{16}$$

and

$$\int_{t_0}^{\infty}\int_{s_1}^{\infty}\cdots\int_{s_{n-2}}^{\infty}\int_{s_{n-1}}^{\infty}|g(s)|\,ds\,ds_{n-1}\cdots ds_1 < \infty, \tag{17}$$

respectively. This implies that (7) and (8) hold, so the proof is complete.

Corollary 2. *Assume that $c = -1$ and that there exists an interval $[a,b] \subset \mathbf{R}^+$ such that*

$$\int_{t_0}^{\infty} s^n \sup_{w\in[a,b]}|F(s,w)|\,ds < \infty, \tag{18}$$

and

$$\int_{t_0}^{\infty} s^n|g(s)|\,ds < \infty. \tag{19}$$

Then (13) *has a bounded nonoscillatory solution.*

The proof is similar to that of Corollary 1.

Example 1. Consider the higher order neutral differential equation

$$(x(t)+cx(t-\tau))^{(n)} + \frac{1}{t^{\alpha}}\,x^{\beta}(t-\sigma) = 0, \quad t \geq t_0 \tag{20}$$

where n is a positive integer, $c \in \mathbf{R}, \tau$, $\sigma \geq 0$, $\beta > 0$.

When $c \neq -1$, $\alpha > n$, for any real number $b > a > 0$,

$$\int_{t_0}^{\infty} s^{n-1} \sup_{w\in[a,b]} \left\{ \frac{w^\beta}{s^\alpha} \right\} ds \leq b^\beta \int_{t_0}^{\infty} \frac{1}{s^{\alpha+1-n}}\, ds < \infty,$$

by Corollary 1, equation (20) has a bounded nonoscillatory solution.

When $c = -1$, $\alpha > n+1$, for any real number $b > a > 0$,

$$\int_{t_0}^{\infty} s^{n} \sup_{w\in[a,b]} \left\{ \frac{w^\beta}{s^\alpha} \right\} ds \leq b^\beta \int_{t_0}^{\infty} \frac{1}{s^{\alpha-n}}\, ds < \infty,$$

by Corollary 2, equation (20) has a bounded nonoscillatory solution.

Remark 1. Minor adjustments are only necessary to discuss the neutral functional differential equation

$$(21)\quad L_n(x(t)+cx(t-\tau))+F(t,x(\sigma_1(t)),\dots,x(\sigma_m(t))) = g(t), \quad t \geq t_0,$$

where $m \geq 1$ is an integer.

Theorem 3. *Assume that $c \neq -1$ and that there exist some interval $[a_i, b_i] \subset \mathbf{R}^+$, $i = 1, 2, \dots, m$, such that*

$$\int_{t_0}^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} \sup_{(w_1,w_2,\dots,w_m)\in[a_1,b_1]\times[a_2,b_2]\cdots\times[a_m,b_m]} |F(s,w_1,w_2,\dots,w_m)|\, ds\, ds_{n-1}\cdots ds_1 < \infty,$$

and (8) *holds. Then* (21) *has a bounded nonoscillatory solution.*

Theorem 4. *Assume that $c = -1$ and that there exist some interval $[a_i, b_i] \subset \mathbf{R}^+$, $i = 1, 2, \dots, m$ such that*

$$\int_{t_0}^{\infty} s_1 r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} \sup_{(w_1,w_2,\dots,w_m)\in[a_1,b_1]\times[a_2,b_2]\cdots\times[a_m,b_m]} |F(s,w_1,w_2,\dots,w_m)|\, ds\, ds_{n-1}\cdots ds_1 < \infty,$$

and (10) *holds. Then* (21) *has a bounded nonoscillatory solution.*

Remark 2. Theorems 1 and 2 improve and extend Theorem C by removing the conditions (C_5) and (C_6). Corollaries 1 and 2 improve and extend Theorems A, B and E by removing the conditions (C_1), (C_2), (C_3), (C_4), (C_9) and (C_{10}). Theorems 3 and 4 improve and extend Theorem D by removing the conditions (C_7) and (C_8).

REFERENCES

1. R.P. Agarwal and S.R. Grace, *The oscillation of higher-order differential equations with deviating arguments*, Comput. Math. Appl. **38** (1999), 185–190.

2. R.P.Agarwal, S.R. Grace and D. O'Regan, *Oscillation theory for difference and functional differential equations*, Kluwer Acad. Publ., Dordrecht, 2000.

3. ———, *Oscillation theory for second order dynamic equations*, Taylor and Francis, London, 2003.

4. ———, *Oscillation criteria for certain nth order differential equations with deviating arguments*, J. Math. Anal. Appl. **262** (2001), 601–622.

5. M.P. Chen, J.S. Yu and Z.C. Wang, *Nonoscillatory solutions of neutral delay differential equations*, Bull. Austral. Math. Soc. **48**(1993), 475–483.

6. H.A. El-Morshedy and K. Gopalsamy, *Nonoscillation, oscillation and convergence of a class of neutral equations*, Nonlinear Anal. **40** (2000), 173–183.

7. L.H. Erbe, Q.K. Kong and B.G. Zhang, *Oscillation theory for functional differential equations*, Marcel Dekker, New York, 1995.

8. J.R. Graef, B. Yang and B.G. Zhang, *Existence of nonoscillatory and oscillatory solutions of neutral differential equations with positive and negative coefficients*, Math. Bohem. **124** (1999), 87–102.

9. I. Gyori and G. Ladas, *Oscillation theory for delay differential equations with applications*, Oxford Univ. Press, London, 1991.

10. M.R.S. Kulenovic and S. Hadziomerspahic, *Existence of nonoscillatory solution of second order linear neutral delay equation*, J. Math. Anal. Appl. **228** (1998), 436–448.

11. C.H. Ou and J.S.W. Wong, *Forced oscillation of nth-order functional differential equations*, J. Math. Anal. Appl. **262** (2001), 722–732.

12. N. Parhi and R.N. Rath, *Oscillation criteria for forced first order neutral differential equations with variable coefficients*, J. Math. Anal. Appl. **256** (2001), 525–241.

13. S. Tanaka, *Existence of positive solutions for a class of higher order neutral differential equations*, Czechoslovak Math. J. **51** (2001), 573–583.

14. Yong Zhou, *Oscillation of neutral functional differential equations*, Acta Math. Hungar. **86** (2000), 205–212.

15. Yong Zhou and B.G. Zhang, *Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients*, Appl. Math. Lett. **15** (2002), 867–874.

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