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POINT-TO-LINE DISTANCES IN THE PLANE OF A TRIANGLE

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ABSTRACT. Many inequalities involving notable points and lines in the plane of a triangle ABC are presented. Most are new; however, they are only conjectured, not proved. Each inequality was detected by computer and then confirmed for 10,740 triangles, selected in such a way that it will be remarkable if any of the conjectures should eventually prove to be false. Ninety-one specific notable points are considered, along with two notable lines, namely, the Euler line, $\mathcal{E}$, and the line perpendicular to $\mathcal{E}$ that passes through the orthocenter of ABC. Typical of the hundreds of inequalities is that the distance from the incenter to $\mathcal{E}$ never exceeds the distance from the symmedian point to $\mathcal{E}$.

1. Introduction. "Computers can solve mathematical problems," writes David Gale [**3**]. "They can also pose them and now, it seems, they may be capable of killing off whole branches of the subject." Gale refers specifically to Euclidean geometry, and even more specifically to a list $\mathcal{L}$ of notable points in the plane of a triangle which exhibit many newly-discovered-by-computer collinearities and other properties. The computer detects *all* cases of collinearity among points in $\mathcal{L}$, so that no further collinear subsets of $\mathcal{L}$ will remain to be found in the future. In that sense, the computer may be capable of killing off a branch, or pruning a twig, of mathematics.

In the same killing spirit we now use the computer to investigate distances among well-known notable points and lines of a triangle. An example of such an inequality is

$$\begin{aligned}&\text{(distance from incenter to Euler line)}\\ &\qquad\qquad \leq \text{(distance from symmedian point to Euler line).}\end{aligned}$$

There appear to be an astonishing number of new inequalities of this sort. In order to list them, it is helpful to use an indexing of notable points, or *centers*, as introduced in [**5**]. Here we consider a total of 91

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centers, listed in the Appendix, beginning with X_1 = incenter, X_2 = centroid, X_3 = circumcenter, and X_4 = orthocenter.

In the plane of a triangle ABC, let L be a line, and let P and P_0 be points such that P_0 does not lie on L. Define

$$D_L(P,P_0) = \delta \frac{(\text{distance between } P \text{ and } L)}{(\text{distance between } P_0 \text{ and } L)}, \tag{1}$$

where

$$\delta = \begin{cases} 1 & \text{if } P \text{ lies on the side of } L \text{ containing } P_0 \text{ or on } L \\ -1 & \text{if } P \text{ lies on the side of } L \text{ opposite the side containing } P_0. \end{cases}$$

Let $D(P) = D_{\mathcal{E}}(P, X_1)$, where $\mathcal{E}$ denotes the Euler line of ABC. Let $\widehat{D}(P) = D_{\mathcal{O}}(P, X_3)$, where $\mathcal{O}$ denotes the line perpendicular to $\mathcal{E}$ and passing through X_4. We call $\mathcal{O}$ the *ortho-Euler line*. In this paper we consider four problems concerning the indexed centers:

Problem 1. For what centers X and Y is $D(X) \leq D(Y)$ for all triangles ABC?

Problem 2. Evaluate inf $D(X)$ and sup $D(X)$ over all triangles ABC.

Problem 3. For what centers X and Y is $\widehat{D}(X) \leq \widehat{D}(Y)$ for all triangles ABC?

Problem 4. Evaluate inf $\widehat{D}(X)$ and sup $\widehat{D}(X)$ over all triangles ABC.

2. Distances in trilinear coordinates. The distance between a point $P = (\alpha', \beta', \gamma')$ given in actual trilinear distances and a line L given by $l\alpha + m\beta + n\gamma = 0$ is

$$\frac{l\alpha' + m\beta' + n\gamma'}{\sqrt{l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C}}. \tag{2}$$

This formula appears in Carr [**1**, article 4624] and elsewhere. (The reader unfamiliar with trilinear coordinates will find [**5**] and references cited therein helpful.) If P is given by trilinears $\alpha : \beta : \gamma$, then the actual trilinear distances for P (i.e., the directed distances from P to the sidelines BC, CA, AB) are $(k\alpha, k\beta, k\gamma)$, where $k = 2\mathcal{A}/(a\alpha + b\beta + c\gamma)$, where $\mathcal{A}$ is the area of the reference triangle ABC. Thus, if $P_0 = \alpha_0 : \beta_0 : \gamma_0$ is a point not on L then the ratio (1) is given by

$$D_L(P, P_0) = \frac{l\alpha + m\beta + n\gamma}{l\alpha_0 + m\beta_0 + n\gamma_0} \frac{a\alpha_0 + b\beta_0 + c\gamma_0}{a\alpha + b\beta + c\gamma}. \tag{3}$$

Note that $D_L(P, P_0) > 0$ if and only if P lies on the side of L that contains P_0.

We wish to study distance ratios $D_L(P_1, P_2)$ for various lines and points. Since $D_L(P_1, P_2) = D_L(P_1, P_0)/D_L(P_2, P_0)$ for any three points P_0, P_1, P_2 for which $D_L(P_2, P_0) \neq 0$, it suffices to study $D_L(P, P_0)$ where P_0 remains fixed. Unless the line L contains the incenter, $X_1 = 1 : 1 : 1$, we shall always choose $P_0 = X_1$. We shorten the notation $D_L(P, P_0)$ to $D_L(P)$ and rewrite (3) as

$$D_L(P) = \frac{l\alpha + m\beta + n\gamma}{l + m + n} \frac{a + b + c}{a\alpha + b\beta + c\gamma}. \tag{4}$$

The ratios (3) and (4) serve as a basis for some of what follows (e.g., Columns 2 and 4 of Table 1). Note that inequalities of the form $D_L(X, P_0) \le D_L(Y, P_0)$ simplify via (3) to

$$\frac{l\alpha_X + m\beta_X + n\gamma_X}{a\alpha_X + b\beta_X + c\gamma_X} \le \frac{l\alpha_Y + m\beta_Y + n\gamma_Y}{a\alpha_Y + b\beta_Y + c\gamma_Y}. \tag{5}$$

If $(a\alpha_X + b\beta_X + c\gamma_X)(a\alpha_Y + b\beta_Y + c\gamma_Y) > 0$, then (5) simplifies to

$$\begin{vmatrix} nb - mc & lc - na & ma - lb \\ \alpha_X & \beta_X & \gamma_X \\ \alpha_Y & \beta_Y & \gamma_Y \end{vmatrix} \ge 0. \tag{6}$$

TABLE 1. Distances from centers to the Euler line.

For $i = 1, 2, \ldots, 91$, let $|X_i\mathcal{E}|$ = distance between center X_i and the Euler line $\mathcal{E}$, and let

$\mathcal{E}_i = \delta\frac{|X_i\mathcal{E}|}{|X_1\mathcal{E}|}$, where $\delta = \{ \begin{array}{rl} 1 & \text{if } X_i \text{ lies on the same side of } \mathcal{E} \text{ as } X_1 \text{ or on } \mathcal{E} \\ -1 & \text{if } X_i \text{ lies on the side of } \mathcal{E} \text{ opposite that of } X_1. \end{array}$.

Limiting values (inf and sup) of the ratios $\mathcal{E}_i$ occur in five types:

- E : scalene limit as $(A, B, C) \to (\pi/3, \pi/3, \pi/3)$
- H : scalene limit as $(A, B, C) \to (0, 0, \pi)$
- V : scalene limit as $(A, B, C) \to (0, \pi/2, \pi/2)$
- A : all (A, B, C) satisfying $A+B+C=\pi$, $A > 0$, $B > 0$, $C > 0$, $B \neq C$ $C \neq A$, $A \neq B$
- $\times$: none of the above

The meanings of the heading "next centers $\leq$" will be clear from Figure 1, and similarly for "next centers $\geq$." The notation $L(i, j)$ means the line centers X_i and X_j.

i	$\inf \mathcal{E}_i$	type	$\sup \mathcal{E}_i$	type	next centers $\leq$	next centers $\geq$	remarks
1	1	A	1	A	34,52	7	$L(1,76)\Vert\mathcal{E}$
2					10	28,45,82,83	on $\mathcal{E}$
3					10	28,45,82,83	on $\mathcal{E}$
4					10	28,45,82,83	on $\mathcal{E}$
5					10	28,45,82,83	on $\mathcal{E}$
6	1	V	16/9	E	39,42,54,86	55	
7	1	H	10/9	E	1,56	62,78,79	
8	-2	A	-2	A	73,75	60	
9	$-5/9$	E	$-1/2$	H	68	10	
10	$-1/2$	A	$-1/2$	A	9	2,19	
11	1	H	∞	E	46	14,77,85	$L(11,36)\Vert\mathcal{E}$
12	1/2	E	1	H	45,82,83	17,52	$L(12,35)\Vert\mathcal{E}$
13	8/9	E	3/2	H	17,56	41,58	$L(13,15)\Vert\mathcal{E}$
14	1	V	∞	E	11,59	44	
15	8/9	E	3/2	H	17,56	41,58	$L(15,13)\Vert\mathcal{E}$
16	1	V	∞	E			
17	2/3	E	3/2	H			
18	$-\infty$	$\times$	∞	$\times$			
19	$-1/2$	H	1/9	E	10	28,45,82,83	
20					10	28,45,82,83	on $\mathcal{E}$
21					10	28,45,82,83	on $\mathcal{E}$
22					10	28,45,82,83	on $\mathcal{E}$
23					10	28,45,82,83	on $\mathcal{E}$

TABLE 1. (Continued)

i	$\inf \mathcal{E}_i$	type	$\sup \mathcal{E}_i$	type	next centers $\leq$	next centers $\geq$	remarks
24					10	28,45,82,83	on $\mathcal{E}$
25					10	28,45,82,83	on $\mathcal{E}$
26					10	28,45,82,83	on $\mathcal{E}$
27					10	28,45,82,83	on $\mathcal{E}$
28	0	H	$4/3$	E	2,19	29,62,78,79	
29	0	H	4	E	28		
30	$-\infty$	$\times$	∞	$\times$			
31	1	V	$7/3$	E	55	43,85	
32	1	V	$8/3$	E	55	59	
33	$-\infty$	$\times$	∞	$\times$			
34	-1	E	1	H	81	1	
35	$1/2$	E	1	H	45,82,83	17,52	$L(35,12) \Vert \mathcal{E}$
36	1	H	∞	E	46	14,77,85	$L(36,11) \Vert \mathcal{E}$
37	$11/18$	E	1	V	38, 83	17,52	
38	$1/3$	E	1	V	45	37	
39	1	V	$3/2$	H	58	6,41	
40	$-\infty$	H	-5	E		66, 75	
41	1	V	$5/2$	H	13,39,42,54,62,79	43	
42	1	V	$3/2$	E	62,78	6,41	
43	1	V	3	E	31,41,46,55,59	44	
44	1	V	∞	E	14,43,77		
45	$2/9$	E	1	V	2,19	12,38	
46	1	H	3	E	53	11,43	
47	$-\infty$	$\times$	∞	$\times$			
48	$-\infty$	$\times$	∞	$\times$			
49	$-\infty$	$\times$	∞	$\times$			
50	$-\infty$	$\times$	∞	$\times$			
51	$-\infty$	$\times$	∞	$\times$			
52	$2/3$	E	1	H	37,80,83	1	
53	1	H	2	E	54	46,55	
54	1	H	$5/3$	E	62	6,41,53	
55	1	V	2	E	6,53	31,32,43	
56	$-\infty$	E	1	$\times$		7,13	
57	1	V	2	H	58,62,79	41	
58	1	V	$3/2$	H	13,17,18	39,57	
59	1	V	$8/3$	E	32	14,43,85	
60	-2	H	$-5/3$	E	8	64,72	

TABLE 1. (Continued)

i	$\inf \mathcal{E}_i$	type	$\sup \mathcal{E}_i$	type	next centers $\leq$	next centers $\geq$	remarks
61	$-\infty$	V	∞	V			$L(61,65)\|\mathcal{E}$
62	1	H	3/2	E	7,28	41,42,54,86	
63	$-\infty$	V	∞	V			
64	$-3/2$	H	∞	E			
65	$-\infty$	V	∞	V			$L(65,61)\|\mathcal{E}$
66	$-32/9$	E	-2	V	40	73	
67	$-\infty$	$\times$	∞	$\times$			
68	$-13/18$	E	$-1/2$	H	81	9	
69	$-\infty$	$\times$	∞	$\times$			
70	$-\infty$	$\times$	∞	$\times$			
71	$-\infty$	$\times$	∞	$\times$			
72	-2	V	$-11/19$	E	60	81	
73	-3	H	-2	V	66	8	
74	$-\infty$	$\times$	∞	$\times$			
75	-5	E	-2	H	40	8	
76	1	A	1	A	34,52	7	$L(76,1)\|\mathcal{E}$
77	1	H	∞	E	11	44	
78	1	V	4/3	E	7,28	42,58,86	
79	1	V	13/9	E	7,28	41,57,86	
80	2/5	V	2/3	E	83	17,52	
81	-1	A	-1	A	72	34,68	
82	1/3	E	1	H	2,19	12	
83	2/5	V	4/9	E	2,19	12,37,80	
84	$-\infty$	V	∞	E			
85	1	V	∞	E	11,31,59		
86	1	V	14/9	E	62,78,79	6	
87	$-\infty$	$\times$	∞	$\times$			
88	$-\infty$	$\times$	∞	$\times$			
89	$-\infty$	$\times$	∞	$\times$			
90	$-\infty$	$\times$	∞	$\times$			
91	$-\infty$	$\times$	∞	E			

3. Problems 1 and 2: Conjectures based on computer output. Three programs were used to find and test possible solutions of Problems 1 and 2. The first, names FIND, finds for each of the 91 centers X_i all those of the other 90 centers X_j that satisfy (5):

$$D(X_i) \leq D(X_j) \tag{7}$$

for 40 thoughtfully chosen triangles.

We call an inequality of the form (7) a *fundamental inequality* if the (single) condition

$D(X_i) < D(X_k) \leq D(X_j)$ or $D(X_i) \leq D(X_k) < D(X_j)$ for all triangles

implies $k = i$ or $k = j$.

The second program, names TEST, rejects inequalities found by FIND that fail for at least one of 10,740 triangles ABC formed systematically as follows:

A ranges from $0.5001°$ to $59.5001°$ in increments of $0.5°$;

B ranges from $(A + .01)°$ to just less than $(90 - A/2)°$ in increments of $0.5°$;

$C = (180 - A - B)°$.

For each i, TEST also prints the least and greatest values of $D(X_i)$ found over the 10,740 triangles. These values indicate that in many cases an infimum or supremum is approached as the triangle, regarded as a variable, approaches one of three limiting configurations:

(i) $(A, B, C) = (\pi/3, \pi/3, \pi/3)$, which we call the E *configuration*;

(ii) $(A, B, C) = (\pi, 0, 0, 0)$, the H *configuration*;

(iii) $(A, B, C) = (0, \pi/2, \pi/2)$, the V *configuration*.

In order to provide data regarding these limiting cases, the third program, named XTREME, evaluates $D(X_i)$ at triangles that are very near a limiting configuration, such as the triangle (59.998°, 59.999°, 60.003°).

Note that each of the 91 centers, when evaluated at an equilateral triangle, either coincides with the incenter or else is undefined, as is the case, for example, with X_{16}. It follows that the Euler line (by which we mean the set of all points $\alpha : \beta : \gamma$ satisfying equation (8) below) is merely a single point in the E configuration. Thus, a limit of $D(X_i)$ as (A, B, C) approaches $(\pi/3, \pi/3, \pi/3)$ certainly cannot be evaluated by substituting $\pi/3$ for the angles, or equivalently, 1 for the sidelengths.

In fact, if ABC is isosceles, then the Euler line contains many of the 91 centers X_i, so that $D(X_i)$ is undefined. Accordingly, when we speak of a limit of a $D(X_i)$ as ABC approaches a fixed configuration, such as E, H, or V, we shall mean a *scalene limit*: the three conditions $B \neq C$, $C \neq A$, $A \neq B$ remain in force during the approach.

The existence of these scalene limits may be difficult to prove in general. However, numerical evidence indicates that the limits do exist. That is, for each finite number listed in Column 2 or Column 4 of Table 1, this is the number approached from a variety of "directions of approach" checked by computer. Many of these numbers have also been obtained analytically using one particular approach, as described in Section 4. It would be of interest to have a general proof of the existence of these limits.

Because $D(X)$ failes to exist when (A, B, C) is one of $(0, 0, \pi)$, $(\pi/3, \pi/3, \pi/3)$, $(0, \pi/2, \pi/2)$, many of the inequalities $D(X_i) \leq D(X_j)$ can be strengthened to $D(X_i) < D(X_j)$; that is, the inf and sup are not attained for any (A, B, C). It would appear that the only exceptions are the obvious ones: $i \in \{8, 10, 76, 81\}$, since in these cases $D(X_i)$ is constant over all triangles.

Consider the symmedian point, X_6, as an example. Columns 2 and 4 assert that X_6 always lies on the side of the Euler line that X_1 lies on, and that

$$|X_1\mathcal{E}| \leq |X_6\mathcal{E}| \leq (16/9)|X_1\mathcal{E}|$$

for all triangles. Column 3 asserts that $|X_6\mathcal{E}|$ comes arbitrarily near $|X_1\mathcal{E}|$ as ABC scalene-approaches the V configuration; and Column 5, that $|X_6\mathcal{E}|$ comes arbitrarily near $(16/9)|X_1\mathcal{E}|$ as ABC scalene-approaches the E configuration.

Column 6 asserts that for all triangles ABC, we have $D(X_i) \leq D(X_6)$ for i in the set $S = \{39, 42, 54, 86\}$, and, further, that if $D(X_j) \leq D(X_6)$ for some $j \notin S \cup \{6\}$, then $D(X_j) \leq D(X_i)$ for some $i \in S$. Similarly, for Column 7.

Now, applying Table 1 to the set $S \cup \{55\}$, we can build chains of inequalities:

$$D(X_{58}) \leq D(X_{39}) \leq D(X_6) \leq D(X_{55}) \leq D(X_{32})$$

and

$$D(X_{62}) \leq D(X_{42}) \leq D(X_6) \leq D(X_{55}) \leq D(X_{43}).$$

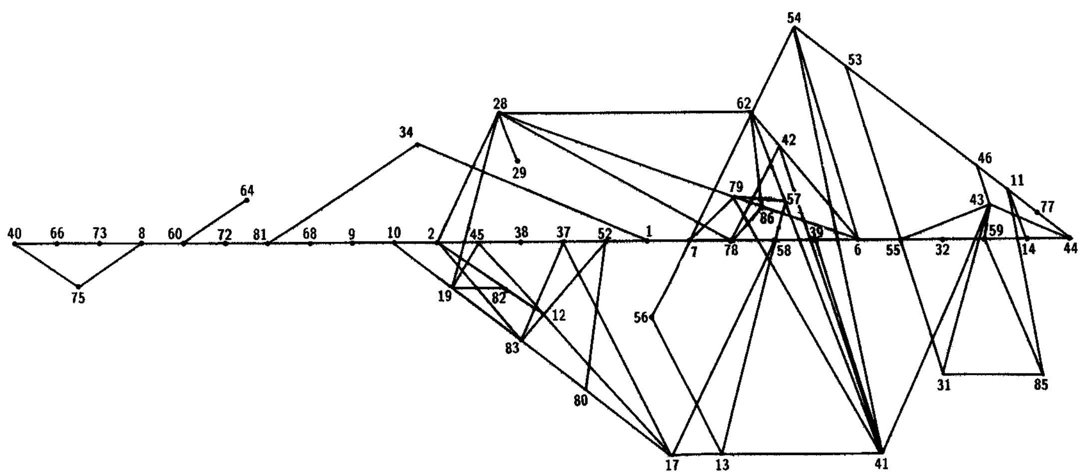

FIGURE 1. $D(X_i) \leq D(X_j) \Leftrightarrow j$ is path-connected to i, with i to the left of j. ("Path-connected" means through labeled nodes; e.g., 42 is path-connected to 31, but not to 41).

The graph in Figure 1 shows all such chains. In this graph, two vertices i and j are connected by an edge if and only if the vertex at the left-most end of the edge, say i, satisfies $D(X_i) \leq D(X_j)$ for all triangles (i.e., the 10,740 sampled and, we conjecture, all others). Thus, for example, four edges lead into vertex 6 and one edge leads out, in agreement with the discussion above of the set S. The longest chain is indicated by the horizontal line from 40 on the far left to 44 on the right. The 25 edges in this chain represent 25 fundamental inequalities.

In Figure 1 the digit 2 represents X_2, the centroid, whose distance from $\mathcal{E}$ is zero. Loosely speaking, numbers i to the left of 2 represent centers always on the side of $\mathcal{E}$ opposite that of X_1, and numbers i to the right of 2, centers always on the same side of $\mathcal{E}$ as X_1. More precisely, we must take "to the left of 2" to mean "is connected to 2 by a path having 2 as its right-most vertex," and similarly for "to the right of 2."Note that neither vertex 19 nor vertex 34 fits either description, indicating that, of the centers represented in Figure 1, only these two cross the Euler line; the others, except for X_2 of course, stay on one side of $\mathcal{E}$ or the other as ABC ranges through the set of all triangles.

Some of the 91 centers listed in the appendix do not appear in Figure 1. For example, centers numbered 3, 4, 5, 20, 21, 22, 23, 24, 25, 26, 27 are all on the Euler line and so are represented by center 2, the centroid, in Figure 1. Center 16 is represented by 14, since the line of centers 14 and 16 is parallel to the Euler line. (The only known proof of this parallelism is tedious and analytic; a geometric proof is editorially solicited at the end of [**4**].) Other such parallelisms are indicated in Column 8 of Table 1. Finally, for each of the centers X numbered 18, 30, 33, 47, 48, 49, 50, 51, 61, 63, 64, 65, 67, 69, 70, 71, 74, 84, 87, 88, 89, 90, 91, there is a triangle for which $D(X) < D(40)$ and also a triangle for which $D(X) > D(44)$; moreover, if X and Y are any two of these, then $D(X) < D(Y)$ in some triangle but $D(X) > D(Y)$ in some other triangle. These assertions were all confirmed by a subroutine added to the program TEST.

4. Problems 3 and 4. Table 1 and Figure 1 furnish many (conjectured) solutions to Problems 1 and 2. In a similar way, Problems 3 and 4 lead to Table 2. We restrict Table 2 to only 12 of the 91 centers:

TABLE 2. Distances from centers to the ortho-Euler line.

The ortho-Euler line is the line through X_4 perpendicular to the Euler line. An equation $l\alpha + m\beta + n\gamma = 0$ for this line is given by $l = l(A,B,C) = \cos A(2\tan A - \tan B - \tan C)$, $m = l(B,C,A)$, and $n = l(C,A,B)$. For $i = 1,2,\ldots,91$, let $|X_i\mathcal{O}|$ = distance between center X_i and the ortho-Euler line $\mathcal{O}$, and let

$O_i = \delta\frac{|X_i\mathcal{O}|}{|X_3\mathcal{O}|}$, where $\delta = \begin{cases} 1 & \text{if } X_i \text{ lies on the same side of } \mathcal{O} \text{ as } X_3 \\ -1 & \text{if } X_i \text{ lies on the side of } \mathcal{O} \text{ opposite that of } X_3 \end{cases}$.

Limiting values (inf and sup) of the ratios O_i occur in four types:

E: scalene limit as $(A,B,C) \to (\pi/3,\pi/3,\pi/3)$

H: scalene limit as $(A,B,C) \to (0,0,\pi)$

V: scalene limit as $(A,B,C) \to (0,\pi/2,\pi/2)$

A: all (A,B,C) satisfying $A+B+C=\pi$, $A>0$, $B>0$, $C>0$, $B\neq C$, $C\neq A$, $A\neq B$

i	$\inf O_i$	type	$\sup O_i$	type	next centers $\leq$	next centers $\geq$	remarks
1	0	V	2/3	H	6,7,13	2	
2	2/3	A	2/3	A	1,5,13	9,10	
3	1	A	1	A	9,10		
4					40	5,6,7,13	on $\mathcal{O}$
5	1/2	A	1/2	A	4	2	
6	0	V	2/3	H	4	1	
7	0	V	2/3	H	4	1	
8	2/3	H	2	V	9,10		
9	2/3	H	1	V	2	3,8	
10	2/3	H	1	V	2	3,8,9	
13	0	V	2/3	H	4	1,2	
40	$-\infty$	H	$-1/2$	V		4	

5. Limits of $D_L(X)$ in Table 1. Recall that the scalene limits of $D_L(X)$ in (4), appearing in Columns 2 and 4 of Table 1, have not been proved to exist. Assuming that they do exist, we can evaluate some of them along well-chosen directions of approach. In order to carry out such evaluations, we use an equation of the form $l\alpha + m\beta + n\gamma = 0$ for the Euler line (e.g., Carr [**1**, article 4644]):

$$\alpha \sin 2A \sin(B-C) + \beta \sin 2B \sin(C-A) + \gamma \sin 2C \sin(A-B) = 0,$$

which can be rewritten as

(8)
$$\alpha a(b^2+c^2-a^2)(b^2-c^2)+\beta b(c^2+a^2-b^2)(c^2-a^2)+\gamma c(a^2+b^2-c^2)(a^2-b^2)=0.$$

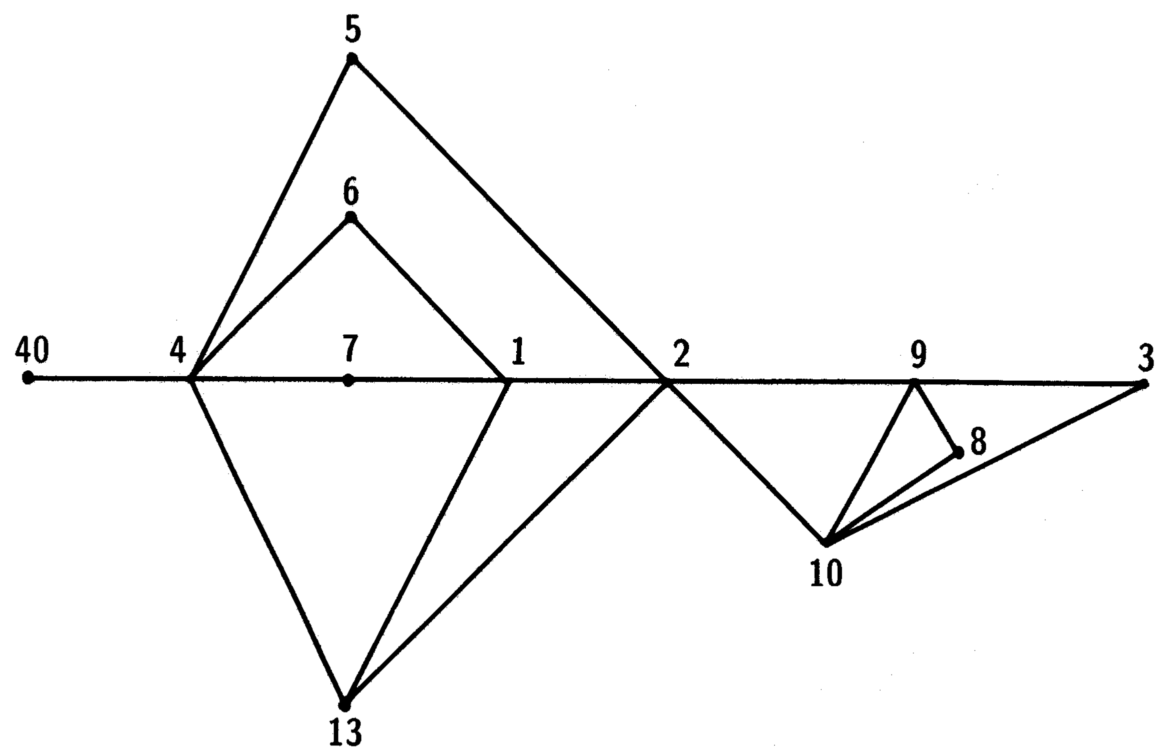

FIGURE 2. $\hat{D}(X_i) \leq \hat{D}(X_j) \Leftrightarrow j$ is path-connected to i, with i to the left of j.

Recall [**5, 6, 7**] that a center $X = \alpha : \beta : \gamma$ is defined from a center-function f:

$$\alpha = f(a,b,c), \qquad \beta = f(b,c,a), \qquad \gamma = f(c,a,b) \tag{9}$$

where $f(a,b,c) = f(a,c,b)$ for all triples (a,b,c) of sidelengths of triangles.

Case 1. V configuration. Let $a = x$, $b = 1-x$, $c = 1+x$. Then (8) gives, for the line $\mathcal{E}$,

$$\begin{aligned} l &= a(b^2+c^2-a^2)(b^2-c^2) = -4x^2(x^2+2) \\ m &= b(c^2+a^2-b^2)(c^2-a^2) = x(1-x)(2x^2+9x+4) \\ n &= c(a^2+b^2-c^2)(a^2-b^2) = -m(-x) \end{aligned}$$

and (9) gives, for the center X,

$$\begin{aligned} \alpha_0(x) &= f(x, 1-x, 1+x) \\ \beta_0(x) &= f(1-x, 1+x, x) \\ \gamma_0(x) &= f(1+x, x, 1-x). \end{aligned}$$

Substitute into (4) and cancel x from the numerator and denominator to see that

$$\text{if } \beta_0(0)+\gamma_0(0) \neq 0, \quad \text{then } \lim_{x\to 0} D_L(X) = 1.$$

Matching this limit in Table 1 are $\inf \mathcal{E}_6$, $\inf \mathcal{E}_{14}$, and others, but *not* $\inf \mathcal{E}_{61}$, $\inf \mathcal{E}_{72}$, and others.

Case 2. H configuration. Let $a \neq 1$, $b = 1-x$, $c = 1+x$. Then

$$\begin{aligned} l &= -ax(2x^2+2-a^2) \\ m &= (x-1)(a^4 - a^2(x-1)^2 - 4x^3 - 8x^2 - 4x) \\ n &= -m(-x). \end{aligned}$$

Write

$$\begin{aligned} \alpha_a(x) &= f(a, 1-x, 1+x) \\ \beta_a(x) &= f(1-x, 1+x, a) \\ \gamma_a(x) &= f(1+x, a, 1-x). \end{aligned}$$

Substitute into (4) and note that the quotient has indeterminate form $0/0$. If $a\alpha_a(0) + \beta_a(0) + \gamma_a(0) \neq 0$, then L'Hospital's rule yields

(10)
$$\lim_{x\to 0} D_L(X) = \frac{4a(a^2-2)\alpha_a(0)+(a^4-3a^2+4)(\beta_a(0)+\gamma_a(0))+a^2(a^2-1)(\gamma_a'(0)-\beta_a'(0))}{2(a-1)^2(a+2)(a\alpha_a(0)+b\beta_a(0)+c\gamma_a(0))}.$$

We have the H configuration when $a = 2$. Since $\gamma_2(0) = \beta_2(0)$ and $\gamma_2'(0) = -\beta_2'(0)$, we conclude that

(11) if $\alpha_0(0) + \beta_0(0) \neq 0$, then $\displaystyle\lim_{x\to 0} D_L(X) = 1 - \frac{3\beta_2'(0)}{2(\alpha_2(0) + \beta_2(0))}$.

For example, in Table 1, corresponding to (11) are $\inf \mathcal{E}_7$ and $\inf \mathcal{E}_9$, but *not* $\inf \mathcal{E}_{40}$.

Case 3. E configuration. Let $a = 1$, $b = 1 - x$, $c = 1 + x$. Then l, m, n and $\alpha_1(x), \beta_1(x), \gamma_1(x)$ are given by substituting 1 for a in the formulas for Case 2. In (4), cancel x and apply L'Hospital's rule twice to find that if $\alpha_1(0) + \beta_1(0) + \gamma_1(0) \neq 0$, then

(12)
$$\lim_{x\to 0} D_L(X) = \frac{8\alpha_1(0)+5(\beta_1(0)+\gamma_1(0))+7(\gamma_1'(0)-\beta_1'(0))+2\alpha_1''(0)-\beta_1''(0)-\gamma_1''(0)}{6(\alpha_1(0)+\beta_1(0)+\gamma_1(0))}.$$

Since $\gamma_1(0) = \beta_1(0)$, $\gamma_1'(0) = -\beta_1'(0)$, and $\gamma_1''(0) = \beta_1''(0)$, we conclude that

(13)
if $\alpha_1(0)+2\beta_1(0) \neq 0$, then $\displaystyle\lim_{x\to 0} D_L(X) = 1+\frac{\alpha_1''(0) - \beta_1''(0) - 7\beta_1'(0)}{9\alpha_1(0)}$.

Corresponding to (13) in Table 1 are $\sup \mathcal{E}_6$, $\sup \mathcal{E}_7$, and $\inf \mathcal{E}_{12}$, but not $\sup \mathcal{E}_{11}$ and $\inf \mathcal{E}_{56}$.

6. Conclusion. Instead of using the Euler line $\mathcal{E}$ and the ortho-Euler line $\mathcal{O}$ as reference lines with respect to which to form distance ratios, one could use other central lines. (A *central line* is defined in [**6**] as a line that passes through two distinct centers.) Tables like Table 1 could be used to record the results, including scalene limits. As mentioned earlier, we would like to see a proof of the existence of these limits.

Methods discussed in Chapter III, "Homogeneous Symmetric Polynomial Geometric Inequalities," of [**8**] do not appear to apply directly to inequalities conjectured in our Section 3. In this regard, suppose an inequality (5) holds for all a, b, c where $X_i = \alpha_i : \beta_i : \gamma_i$ and $X_j = \alpha_j : \beta_j : \gamma_j$, where $\alpha_i, \beta_i, \gamma_i$ are homogeneous polynomials of degree m in a, b, c and $\alpha_j, \beta_j, \gamma_j$ are homogeneous polynomials of degree n in a, b, c. Then (4) yields an inequality of the form (6), except that in some cases, $\geq$ must be changed to $\leq$. The left-hand side of (6) represents a homogeneous polynomial of formal degree $6 + m + n$. However, the polynomial (11) is generally *not* symmetric in a, b, c.

As an example of (11), one can write out the inequality mentioned in the second sentence of Section 1: $D(X_1) \leq D(X_6)$. Writing as usual s_k for the sum $a^k + b^k + c^k$, the result is

$$s_1 s_2^3 - 3s_2^2 s_3 + 2s_2 s_5 + 2s_1 s_2 s_4 - 2s_1 s_6 - 2s_2^2 \sum ab^2 \\ + 4s_2 \sum a^3 b^2 - 4s_1 \sum a^4 b^2 \geq 0,$$

where $\sum a^i b^2$ abbreviates $a^i b^2 + b^i c^2 + c^i a^2$ for $i = 1, 2, 4$.

In [**6**], we distinguish between 1-lines and 0-lines. The distinction need not be repeated here, but we do mention that the Euler line is a 1-line, whereas the ortho-Euler line is a 0-line. For any 0-line L, the left-hand side of (6) is symmetric in a, b, c. Methods of Chapter III of [**7**], as well as methods introduced in [**2**] and [**9**], and references cited in those papers, then perhaps apply in some cases when the degree of the polynomials is ≤ 6.

As a final thought on Section 3, we note that the orthogonal lines $\mathcal{E}$ and $\mathcal{O}$ meet at X_4, which we may regard as the origin of an ordinary cartesian coordinate system, where 1 unit along the x axis, which we choose to be $\mathcal{E}$, is the distance $|X_3X_4|$, and 1 unit along the y axis is the distance $|X_1\mathcal{E}|$. Then there is for each center X a rectangle (or point, segment, unbounded strip, etc.) $R(X)$ to which X is confined. For example, $R(X_6) = [1, 16/9] \times [0, 2/3]$, and $R(X_9) = [-1/2, -5/9] \times [2/3, 1]$.

Appendix

The specific triangle centers referred to above are listed here. This list is reduced from [**4**], where many details, such as Euclidean constructions, collinearities, and references are given. Each center is represented by a name of the form $\mathbf{X_i}$, followed by an equation of the form $\alpha = f(a,b,c)$ or $\alpha = g(A,B,C)$. For each α, one can write out trilinear coordinates for $\mathbf{X_i}$, using

$$\alpha = f(a,b,c), \beta = f(b,c,a), \gamma = f(c,a,b) \text{ or } \alpha = g(A,B,C),$$
$$\beta = g(B,C,A), \gamma = g(C,A,B).$$

$\mathbf{X_1}$ $\alpha = 1$ (incenter)

$\mathbf{X_2}$ $\alpha = 1/a$ (centroid)

$\mathbf{X_3}$ $\alpha = \cos A$, or $\alpha = a(b^2 + c^2 - a^2)$ (circumcenter)

$\mathbf{X_4}$ $\alpha = \sec A$ (orthocenter)

$\mathbf{X_5}$ $\alpha = \cos(B - C)$, or $\alpha = \cos A + 2\cos B \cos C$ (center of the nine-point circle)

$\mathbf{X_6}$ $\alpha = a$ (symmedian point, Lemoine point)

$\mathbf{X_7}$ $\alpha = \dfrac{1}{a(b+c-a)}$, or $\sec^2 A/2$ (Gergonne point)

$\mathbf{X_8}$ $\alpha = (b + c - a)/a$, or $\alpha = \csc^2 A/2$ (Nagel point)

$\mathbf{X_9}$ $\alpha = b + c - a$, or $\alpha = \cot A/2$ (Mittenpunkt)

$\mathbf{X_{10}}$ $\alpha = (b + c)/a$ (Spieker center)

$\mathbf{X_{11}}$ $\alpha = 1 - \cos(B - C)$ (Feuerbach point)

$\mathbf{X_{12}}$ $\alpha = 1 + \cos(B - C)$ (harmonic conjugate of X_{11} with respect to X_1 and X_5

$\mathbf{X_{13}}$ $\alpha = \csc(A + \pi/3)$ (1^{st} isogonic center, Fermat point)

$\mathbf{X_{14}}$ $\alpha = \csc(A - \pi/3)$ (2^{nd} isogonic center)

$\mathbf{X_{15}}$ $\alpha = \sin(A + \pi/3)$ (1^{st} isodynamic point)

$\mathbf{X_{16}}$ $\alpha = \sin(A - \pi/3)$ (2^{nd} isodynamic point)

$\mathbf{X_{17}}$ $\alpha = \csc(A + \pi/6)$ (1^{st} Napoleon point)

$\mathbf{X_{18}}$ $\alpha = \csc(A - \pi/6)$ (2^{nd} Napoleon point)

$\mathbf{X_{19}}$ $\alpha = \tan A$, or $\alpha = \sin 2B + \sin 2C - \sin 2A$ (crucial point)

$\mathbf{X_{20}}$ $\alpha = \cos A - \cos B \cos C$ (De Longchamps point)

$\mathbf{X_{21}}$ $\alpha = 1/(\cos B + \cos C)$ (Schiffler point)

$\mathbf{X_{22}}$ $\alpha = a(b^4 + c^4 - a^4)$ (Exeter point)

$\mathbf{X_{23}}$ $\alpha = a(b^4 + c^4 - a^4 - b^2c^2)$ (far-out point)

$\mathbf{X_{24}}$ $\alpha = \sec A \cos 2A$ (center of perspective of ABC and orthic-of-orthic triangle)

$\mathbf{X_{25}}$ $\alpha = \sin A \tan A$ (homothetic center of orthic and tangential triangles)

$\mathbf{X_{26}}$ $\alpha = a(b^2 \cos 2B + c^2 \cos 2C - a^2 \cos 2A)$ (circumcenter of the tangential triangle)

$\mathbf{X_{27}}$ $\alpha = (\sec A)/(b + c)$

$\mathbf{X_{28}}$ $\alpha = a^2 \cos(B - C)$ (centroid of the orthic triangle)

$\mathbf{X_{29}}$ $\alpha = \cos 2A \cos(B - C)$ (orthocenter of the orthic triangle)

$\mathbf{X_{30}}$ $\alpha = \sin 2A \cos(B - C)$

$\mathbf{X_{31}}$ $\alpha = a^2$ (2^{nd} power point)

$\mathbf{X_{32}}$ $\alpha = a^3$ (3^{rd} power point)

$\mathbf{X_{33}}$ $\alpha = 1 + \sec A$

$\mathbf{X_{34}}$ $\alpha = 1 - \sec A$

$\mathbf{X_{35}}$ $\alpha = 1 + 2\cos A$

$\mathbf{X_{36}}$ $\alpha = 1 - 2\cos A$

$\mathbf{X_{37}}$ $\alpha = b + c$

$\mathbf{X_{38}}$ $\alpha = b^2 + c^2$

$\mathbf{X_{39}}$ $\alpha = a(b^2 + c^2)$ (Brocard midpoint)

$\mathbf{X_{40}}$ $\alpha = 1/(\cos B + \cos C - \cos A - 1)$

$\mathbf{X_{41}}$ $\alpha = a^2(b + c - a)$

$\mathbf{X_{42}}$ $\alpha = a(b + c)$

$\mathbf{X_{43}}$ $\alpha = ca + ab - bc$

$\mathbf{X_{44}}$ $\alpha = b + c - 2a$

$\mathbf{X_{45}}$ $\alpha = 2b + 2c - a$

$\mathbf{X_{46}}$ $\alpha = \cos B + \cos C - \cos A$

$\mathbf{X_{47}}$ $\alpha = \cos 2A$

$\mathbf{X_{48}}$ $\alpha = \sin 2A$

$\mathbf{X_{49}}$ $\alpha = \cos 3a$

$\mathbf{X_{50}}$ $\alpha = \sin 3A$

$$\begin{aligned}
&\mathbf{X_{51}} &&= \mathbf{x_5^{-1}} \quad \alpha = \sec(B-C)\\
&\mathbf{X_{52}} &&= \mathbf{x_7^{-1}} \quad \alpha = a(b+c-a),\ \text{or } \alpha = \cos^2 A/2,\ \text{or}\\
& &&\alpha = 1+\cos A\\
&\mathbf{X_{53}} &&= \mathbf{x_8^{-1}} \quad \alpha = \sin^2 A/2,\ \text{or } \alpha = -1+\cos A\\
&\mathbf{X_{54}} &&= \mathbf{x_9^{-1}} \quad \alpha = 1/(b+c-a),\ \text{or } \alpha = \tan A/2\\
&\mathbf{X_{55}} &&= \mathbf{x_{10}^{-1}} \quad \alpha = a/(b+c)\\
&\mathbf{X_{56}} &&= \mathbf{x_{11}^{-1}} \quad \alpha = 1/(1-\cos(B-C))\\
&\mathbf{X_{57}} &&= \mathbf{x_{12}^{-1}} \quad \alpha = 1/(1+\cos(B-C))\\
&\mathbf{X_{57+n}} &&= \mathbf{x_{12+n}^{-1}} \quad\quad \text{for } n = 1,2,3,\dots,34.
\end{aligned}$$

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