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CYCLIC VECTORS IN THE α-BLOCH SPACES

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ABSTRACT. In this paper we try to identify the functions whose polynomial multiples are weak* dense in the β_α spaces. We obtain that if $|f(z)| \geq |g(z)|$ in the open unit disk and g is cyclic in β_α, then f is cyclic in β_α. Especially, for $0 < \alpha < 1$, f is cyclic in β_α if and only if f has no zeros in the closed unit disc.

Introduction. Let D be the open unit disk in the complex plane $\mathbf{C}$. For each $\alpha > 0$, the α-Bloch space of D, denoted by β_α, consists of analytic functions f on D such that

$$\|f\|_{\beta_\alpha} = \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in D\} < \infty.$$

We give a norm in β_α as follows

$$\|f\|_\alpha = |f(0)| + \|f\|_{\beta_\alpha}. \tag{1}$$

With this norm, β_α is a Banach space and $\beta_{\alpha,0}$ a closed subspace. Here $\beta_{\alpha,0}$ denotes the set of those f in β_α for which $(1-|z|^2)^\alpha |f'(z)| \to 0$ as $|z| \uparrow 1$. The space β_α with the norm (1) is isometric to the second dual $\beta_{\alpha,0}^{**}$, see [**9**]. Furthermore, the polynomials are norm dense in $\beta_{\alpha,0}$ and in $\beta_{\alpha,0}^*$, and are weak* dense in β_α. We refer to [**1, 9**] for more information about β_α and $\beta_{\alpha,0}$.

In this paper, we study (weak*) cyclic vectors in β_α. These are the function f in β_α whose polynomial multiples are weak* dense in β_α, i.e., they are cyclic vectors in the weak* topology for the operator of multiplication by z on β_α. If $f \in \beta_\alpha$, let $[f]$ be the weak* closure in β_α of the polynomial multiples of f. Thus, f is cyclic if and only if $[f] = \beta_\alpha$. Note that a duality argument yields the fact that, if f is in $\beta_{\alpha,0}$, then f is (norm) cyclic in $\beta_{\alpha,0}$ if and only if it is weak* cyclic in β_α.

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When we refer to cyclic vectors in β_α, the weak* is always understood. In a special case of $\alpha = 1$, Brown and Shields have shown the following theorems.

Theorem A [**3**, Theorem 2]. *If f, $g \in \beta_1$, $|f(z)| \geq |g(z)|$ in D, and g is cyclic in β_1, then f is cyclic in β_1.*

Theorem B [**3**, Theorem 3]. *If $f \in \beta_1$, f is an outer function, then f is cyclic in β_1.*

The main results in this paper are

Theorem 1. *For $\alpha > 0$, $f, g \in \beta_\alpha$, $|f(z)| \geq |g(z)|$ in D, and g is cyclic in β_α, then f is cyclic in β_α.*

Theorem 2. *For $\alpha \geq 1$, f is an outer function in β_α, then f is cyclic in β_α.*

Theorem 3. *For $0 < \alpha < 1$, $f \in \beta_\alpha$, then f is cyclic in β_α if and only if f has no zeros in the closed unit disc.*

Throughout this paper, C's are positive constants which are not necessarily the same in each appearance.

1. Some sufficient conditions for cyclic. In this section we shall prove Theorems 1 and 2. For this purpose, we need the following lemmas.

Lemma 1. *Let $f \in \beta_\alpha$, $|z| = r$, then*

(a) $|f(z)| \leq (1 + (1/2)\ln(1 + r/1 - r))\|f\|_\alpha$, $\alpha = 1$,

(b) $|f(z)| \leq (1 + (1/(\alpha - 1)(1 - r)^{\alpha-1}))\|f\|_\alpha$, $\alpha > 1$,

(c) $|f(z)| \leq (1 + (1/1 - \alpha))\|f\|_\alpha$, $0 < \alpha < 1$,

(d) $|f(z) - f(tz)| \leq (1/1 - \alpha)\|f\|_{\beta_\alpha}[(1 - tr)^{1-\alpha} - (1 - r)^{1-\alpha}]$, $0 < \alpha < 1$, $0 \leq t < 1$.

The proof follows from a direct calculation; we omit the details.

Lemma 2. *For each* $\alpha > 0$,

(a) *If* $\{f_n\} \subset \beta_\alpha$*, then* $f_n \to 0$ weak* *if and only if* $f_n(z) \to 0$ *for all* z *in* D*, and* $\sup \|f_n\|_\alpha < \infty$,

(b) *If* $\{f_t\} \subset \beta_\alpha$*,* $0 < t \le 1$*, then* $\lim_{t\to 1^-} f_t = 0$ weak* *if and only if* $\lim_{t\to 1^-} f_t(z) = 0$ *for all* z *in* D*, and* $\sup \|f_t\|_\alpha < \infty$.

Proof. For each $z \in D$, the linear functional of evaluation at z is weak* continuous by Lemma 1, β_α is isometric to the $(L_a^1)^*$, L_a^1 (denote the set of analytic functions that are in L^1 with respect to the area measure in D) is a Banach space. Using these facts, the proof can be obtained from [**4**, Proposition 2].

Lemma 3. *If* $f \in \beta_\alpha$*, then* $\|f_t\|_{\beta_\alpha} \le \|f\|_{\beta_\alpha}$*,* $0 < t \le 1$*, where* $f_t(z) = f(tz)$ *for all* $z \in D$.

The proof follows from a direct calculation; we omit the details.

Lemma 4. *For* $\alpha > 1$*, if* $f \in \beta_\alpha$*, then* $\sup(1-|z|^2)^{\alpha-1}|f(z)| \le C < \infty$.

Proof. For $f \in \beta_\alpha$, then $f' \in L^1(D, (1-|z|^2)^\alpha\, dA(z))$. By [**10**, Section 4.2.1], we have

$$f'(z) = (\alpha+1)\int_D \frac{(1-|w|^2)^\alpha f'(w)}{(1-z\overline{w})^{2+\alpha}}\, dA(w).$$

Taking the line integral from 0 to z,

$$f(z) - f(0) = \int_D \frac{(1-|w|^2)^\alpha f'(w)}{\overline{w}} \left[\frac{1}{(1-z\overline{w})^{1+\alpha}} - 1\right] dA(w).$$

Using Taylor expansion, we get

$$\int_D \frac{(1-|w|^2)^\alpha f'(w)}{\overline{w}}\, dA(w) = 0.$$

So

$$
\begin{aligned}
|f(z)-f(0)| &\le \|f\|_{\beta_\alpha}\int_D \frac{dA(w)}{|w|(1-z\overline{w})^{1+\alpha}}\\
&= \|f\|_{\beta_\alpha}\left[\int_{D/2}\frac{dA(w)}{|w|(1-z\overline{w})^{1+\alpha}}+\int_{D\setminus D/2}\frac{dA(w)}{|w|(1-z\overline{w})^{1+\alpha}}\right]\\
&\le \|f\|_{\beta_\alpha}\left[2\int_0^{1/2}\frac{dr}{(1-r)^{1+\alpha}}+\int_{D\setminus D/2}\frac{2dA(w)}{(1-z\overline{w})^{1+\alpha}}\right]\\
&\le \|f\|_{\beta_\alpha}\left[(2^\alpha-1)\,\frac{2}{\alpha}+2\int_D\frac{dA(w)}{(1-z\overline{w})^{1+\alpha}}\right].
\end{aligned}
$$

However, by [**10** Lemma 4.2.2],

$$\int_D \frac{dA(w)}{(1-z\overline{w})^{1+\alpha}} \le \frac{C}{(1-|z|^2)^{\alpha-1}}.$$

Thus,

$$\sup(1-|z|^2)^{\alpha-1}|f(z)| \le C < \infty.$$

Lemma 5. *If $g\in H^\infty$, $\alpha>0$, $f\in\beta_\alpha$ and $fg\in\beta_\alpha$, then $fg\in[f]$.*

Proof. For $0<t<1$, $g_t(z)=g(tz)$, we can easily show that if P_n is the partial sum of the power series for g_t, then $P_nf\to g_tf$ (norm) as $n\to\infty$. Thus, we have g_tf is in the weak* closure of polynomial of f, which implies $g_tf\in[f]$. For $z\in D$, $\lim_{t\to1^-} g_t(z)f(z)=g(z)f(z)$, if $\sup\|g_tf\|_\alpha<\infty$, then $g_tf\to gf$ weak* by Lemma 2, thus $fg\in[f]$. Now we are going to show that $\sup\|g_tf\|_\alpha<\infty$.

For $\alpha>1$, using Lemma 4, we see that

$$
\begin{aligned}
(1-|z|^2)^\alpha|(g_tf)'| &\le (1-|z|^2)^\alpha|f'||g_t| + (1-|z|^2)^\alpha|g_t'||f|\\
&\le \|g\|_\infty\|f\|_{\beta_\alpha} + (1-|z|^2)^{\alpha-1}|f|(1-|z|^2)|g_t'|\\
&\le \|g\|_\infty\|f\|_{\beta_\alpha} + C(1-|z|^2)\,\frac{1}{(1-|tz|)}\|g\|_\infty\\
&< \infty.
\end{aligned}
$$

Hence,

$$\sup\|g_tf\|_\alpha<\infty.$$

For $0 < \alpha < 1$, we have

$$(1-|z|^2)^\alpha |(g_t f)'| \le \|g\|_\infty \|f\|_{\beta_\alpha} + (1-|z|^2)^\alpha |f g_t'|.$$

Then

$$\begin{aligned}(1-|z|^2)^\alpha |f g_t'| &\le (1-|z|^2)^\alpha |f - f_t||g_t'| + (1-|z|^2)^\alpha |f_t g_t'| \\ &= \varphi_1 + \varphi_2.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}\varphi_1 &\le \frac{1}{1-\alpha}(1-r^2)^\alpha \|f\|_{\beta_\alpha}[(1-tr)^{1-\alpha} - (1-r)^{1-\alpha}]\,\frac{1}{(1-tr)}\|g\|_\infty \\ &\le C(\alpha)\|f\|_{\beta_\alpha}\left[\left(\frac{1-r}{1-tr}\right)^\alpha - \frac{1-r}{1-tr}\right]\|g\|_\infty \\ &\le 2C(\alpha)\|f\|_{\beta_\alpha}\|g\|_\infty < \infty.\end{aligned}$$

From Lemma 3,

$$\begin{aligned}\varphi_2 &= (1-|z|^2)^\alpha |(f_t g_t)' - f_t' g_t| \\ &\le (1-|z|^2)^\alpha |(fg)_t'| + (1-|z|^2)^\alpha |f_t'||g_t| \\ &\le \|(fg)_t\|_{\beta_\alpha} + \|g\|_\infty \|f_t\|_{\beta_\alpha} \\ &\le \|fg\|_{\beta_\alpha} + \|g\|_\infty \|f\|_{\beta_\alpha} < \infty.\end{aligned}$$

Thus,

$$\sup \|g_t f\|_\alpha < \infty.$$

For $\alpha = 1$, one can obtain the proof from [**3**, Lemma 4]. The proof is completed. □

Proof of Theorem 1. We have $g/f \in H^\infty$ and $(g/f)f = g \in [f]$, which implies that f is cyclic.

Corollary 1. *For $\alpha > 0$, $f \in \beta_\alpha$ and $|f(z)| \ge C > 0$ in D, then f is cyclic in β_α.*

To give the proof Theorem 2, we need the following proposition.

Proposition 1. (a) *f is cyclic in H^∞ (with the* weak* *topology) if and only if f is an outer function.*

(b) *For $\alpha > 1$, f is cyclic in H^∞, then f is cyclic in β_α.*

Proof. (a) is in [**8**, Theorem 5.5], we only need to prove (b).

Let f be cyclic in H^∞. Then there exists a sequence of polynomials P_n such that $P_n f \to 1$ weak*; this implies that $P_n(z)f(z) \to 1$ for all z in D and $\sup \|P_n f\|_{H^\infty} < \infty$.

Moreover, for $\alpha \geq 1$, $H^\infty \subset \beta_\alpha$, gives that identity i: $H^\infty \to \beta_\alpha$ is bounded. So $\sup \|P_n f\|_\alpha < \infty$. Hence, by Lemma 2, f is cyclic in β_α.

Proof of Theorem 2. Let

$$g(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|g^*(t)|\,dt\right\},$$

where $f^*(t) = \lim_{r\to 1} f(re^{it})$ almost everywhere and

$$g^*(t) = \begin{cases} 1 & |f^*(t)| \geq 1, \\ |f^*(t)| & |f^*(t)| < 1. \end{cases}$$

We see that $g \in H^\infty \subset \beta_\alpha$ is an outer function and therefore cyclic in H^∞; by Proposition 1, g is cyclic in β_α. Furthermore,

$$\begin{aligned} |g(z)| &= \exp\left\{\int_0^{2\pi} P_z(t)\log|g^*(t)|\,dt\right\} \\ &\leq \exp\left\{\int_0^{2\pi} P_z(t)\log|f^*(t)|\,dt\right\} \\ &= |f(z)|, \quad z \in D, \end{aligned}$$

where P_z is the Poisson kernel for the point z, so f is cyclic by Theorem 1.

2. A sufficient and necessary condition for $0 < \alpha < 1$. As usual, by the disc algebra A we mean the space of functions continuous on the closed unit disc and analytic in D, with the supremum norm.

Lemma 6. *For every* $0 < \alpha < 1$, $\beta_\alpha \subset A$.

Proof. See [**5**, p. 74].

Lemma 7. *For* $f \in A$, *then* f *is cyclic in* A, *with the norm topology, if and only if* $f(z)$ *has no zeros in the closed unit disc.*

Proof. For A a Banach algebra, it can be shown that maximal ideal space is the closed unit disc, see [**6**, p. 189]. One shows that f is cyclic if and only if it lies in no proper closed ideal. Thus the cyclic vectors are precisely the invertible elements in the Banach algebra A, so f is cyclic in A if and only if $f(z)$ has no zeros in the closed unit disc.

Proof of Theorem 3. For $0 < \alpha < 1$, $f \in \beta_\alpha$, f is cyclic in β_α. Let $t = (1+\alpha)/2$; then $\alpha < t < 1$. From Lemma 2, we can easily show that f is cyclic in β_t. However, $f \in \beta_{t,o}$, shows f is cyclic in $\beta_{t,o}$. Thus, there exists a sequence of polynomials $P_n(z)$ such that $P_n f \to 1$ (norm) as $n \to \infty$. By Lemma 1, we have

$$\|P_n f - 1\|_A \leq \left(1 + \frac{1}{1-t}\right)\|P_n f - 1\|_t,$$

thus f is cyclic in A. Hence, by Lemma 7, f has no zeros in the closed unit disc.

On the other hand, since f is continuous in the closed unit disc, there exists $C > 0$ such that $|f(z)| \geq C$ for all z in D. Thus, f is cyclic by Corollary 1.

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