ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 1, 2007

AN EIGENVALUE PROBLEM FOR QUASILINEAR SYSTEMS

JOHNNY HENDERSON AND HAIYAN WANG

ABSTRACT. The paper deals with the existence of positive solutions for the n-dimensional quasilinear system $(\mathbf{\Phi}(\mathbf{u}'))' + \lambda\mathbf{h}(t)\mathbf{f}(\mathbf{u}) = 0$, $0 < t < 1$, with the boundary condition $\mathbf{u}(0) = \mathbf{u}(1) = 0$. The vector-valued function $\mathbf{\Phi}$ is defined by $\mathbf{\Phi}(\mathbf{u}) = (\varphi(u_1), \dots, \varphi(u_n))$, where $\mathbf{u} = (u_1, \dots, u_n)$, and φ covers the two important cases $\varphi(u) = u$ and $\varphi(u) = |u|^{p-2}u$, $p > 1$, $\mathbf{h}(t) = \operatorname{diag}[h_1(t), \dots, h_n(t)]$ and $\mathbf{f}(\mathbf{u}) = (f^1(\mathbf{u}), \dots, f^n(\mathbf{u}))$. Assume that f^i and h_i are nonnegative continuous. For $\mathbf{u} = (u_1, \dots, u_n)$, let $f_0^i = \lim_{\|\mathbf{u}\|\to 0} f^i(\mathbf{u})/\varphi(\|\mathbf{u}\|)$, $f_\infty^i = \lim_{\|\mathbf{u}\|\to\infty} f^i(\mathbf{u})/\varphi(\|\mathbf{u}\|)$, $i = 1, \dots, n$, $\mathbf{f}_0 = \max\{f_0^1, \dots, f_0^n\}$ and $\mathbf{f}_\infty = \max\{f_\infty^1, \dots, f_\infty^n\}$. We prove that the boundary value problem has a positive solution, for certain finite intervals of λ, if one of $\mathbf{f}_0$ and $\mathbf{f}_\infty$ is large enough and the other one is small enough. Our methods employ fixed point theorems in a cone.

1. Introduction. In this paper we consider the eigenvalue problem for the system

$$(\mathbf{\Phi}(\mathbf{u}'))' + \lambda\mathbf{h}(t)\,\mathbf{f}(\mathbf{u}) = 0, \quad 0 < t < 1, \tag{1.1}$$

with one of the following three sets of the boundary conditions,

$$\mathbf{u}(0) = \mathbf{u}(1) = 0, \tag{1.2a}$$

$$\mathbf{u}'(0) = \mathbf{u}(1) = 0, \tag{1.2b}$$

$$\mathbf{u}(0) = \mathbf{u}'(1) = 0, \tag{1.2c}$$

where $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{\Phi}(\mathbf{u}) = (\varphi(u_1), \dots, \varphi(u_n))$, $\mathbf{h}(t) = \operatorname{diag} \times [h_1(t), \dots, h_n(t)]$ and $\mathbf{f}(\mathbf{u}) = (f^1(u_1, \dots, u_n), \dots, f^n(u_1, \dots, u_n))$. We understand that $\mathbf{u}$, $\mathbf{\Phi}$ and $\mathbf{f}(\mathbf{u})$ are (column) n-dimensional vector-valued functions. Equation (1.1) means that

$$\begin{cases} (\varphi(u_1'))' + \lambda h_1(t)\, f^1(u_1, \dots, u_n) = 0, & 0 < t < 1 \\ \quad\vdots \\ (\varphi(u_n'))' + \lambda h_n(t)\, f^n(u_1, \dots, u_n) = 0, & 0 < t < 1 \end{cases} \tag{1.3}$$

Received by the editors on July 20, 2003, and in revised form on November 11, 2004.

By a solution $\mathbf{u}$ to (1.1)–(1.2), we understand a vector-valued function $\mathbf{u} \in C^1([0,1], \mathbf{R}^n)$ with $\mathbf{\Phi}(\mathbf{u}') \in C^1((0,1), \mathbf{R}^n)$, which satisfies (1.1) for $t \in (0,1)$ and one of (1.2). A solution $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))$ is positive if, for each $i = 1, \dots, n$, $u_i(t) \geq 0$ for all $t \in (0,1)$ and there is at least one nontrivial component of $\mathbf{u}$. In fact, we shall show that such a nontrivial component of $\mathbf{u}$ is positive on $(0,1)$.

When $n = 1$, (1.1) reduces to the scalar quasilinear equation

$$(\varphi(u'))' + \lambda h(t)\, f(u) = 0. \tag{1.4}$$

Further, when $\varphi(u) = u$, (1.4) reduces to the classical equation of Emden-Fowler type

$$u'' + \lambda h(t)\, f(u) = 0. \tag{1.5}$$

The existence of positive solutions of boundary value problems for (1.4) and (1.5) originates from a variety of different areas of applied mathematics and physics, and has been intensively studied, see e.g., Agarwal, O'Regan and Wong [**2**] and Wong [**24**].

In connection with the existence of positive radial solutions of partial differential equations in annular regions, Bandle, Coffman and Marcus [**4**] and Lin [**17**] established the existence of positive solutions of boundary value problems for (1.5) under the assumption that f is superlinear, i.e., $f_0 = \lim_{u\to 0} f(u)/u = 0$ and $f_\infty = \lim_{u\to\infty} f(u)/u = \infty$. On the other hand, one of the authors [**20**] obtained the existence of positive solutions boundary value problems for (1.5) under the assumption that f is sublinear, i.e., $f_0 = \infty$ and $f_\infty = 0$.

When $\varphi(u) = |u|^{p-2}u$, $p > 1$, and for even more general functions φ, the problems have been received much attention in the past several decades; see e.g., [**1**–**3, 11, 14, 19** and their references].

If $0 < f_0, f_\infty < \infty$, we [**13**] were able to treat the existence problem, at the expense of a restriction of λ. Roughly, we showed that (1.5) with (1.2) ($n = 1$) has a positive solution for certain finite intervals of λ if one of f_0 and f_∞ is large enough and the other one is small enough. This result was later sharpened by Graef and Yang [**9**] yielding better intervals of λ, but yet for the case when one of f_0 and f_∞ is large enough and the other one is small enough.

In several recent papers [**21, 22**], one of the authors imposed an assumption (see A1) on the function $\varphi(u)$, which covers the two important cases $\varphi(u) = u$ and $\varphi(u) = |u|^{p-2}u$, $p > 1$. Under such an assumption, it is shown that appropriate combinations of superlinearity and sublinearity of $f(u)$ with respect to $\boldsymbol{\varphi}$ at zero and infinity guarantee the existence, multiplicity and nonexistence of positive solutions of (1.1).

The main purpose of this paper is to extend the results in [**13**] to the n-dimensional system (1.1). For this purpose, we use notation in (1.6), $\mathbf{f}_0$ and $\mathbf{f}_\infty$, to characterize superlinearity and sublinearity with respect to φ for (1.1). These are natural extensions of f_0 and f_∞ defined above for the scalar equation (1.5). We are able to show that (1.1) with (1.2) has a positive solution for certain finite intervals of λ if one of $\mathbf{f}_0$ and $\mathbf{f}_\infty$ is large enough and the other one is small enough. We employ a fixed point theorem in a cone due to Krasnoselskii, which is essentially the same as Lemma 2.1.

Let $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_+^n = \Pi_{i=1}^n \mathbf{R}_+$. Also, for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbf{R}_+^n$, let $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$. We make the assumptions:

(A1) φ is an odd, increasing homeomorphism of $\mathbf{R}$ onto $\mathbf{R}$, and there exist two increasing homeomorphisms of $(0,\infty)$ onto $(0,\infty)$ such that

$$\psi_1(\sigma)\,\varphi(x) \le \varphi(\sigma x) \le \psi_2(\sigma)\,\varphi(x), \quad \text{for all} \quad \sigma \quad \text{and} \quad x > 0.$$

(A2) $f^i : \mathbf{R}_+^n \to \mathbf{R}_+$ is continuous, $i = 1, \dots, n$.

(A3) $h_i(t) : [0,1] \to \mathbf{R}_+$ is continuous and $h_i(t) \not\equiv 0$ on any subinterval of $[0,1]$, $i = 1, \dots, n$.

Let

$$\gamma_i(t) = \frac{1}{8}\left[\int_{1/4}^{t} \psi_2^{-1}\left(\int_s^t h_i(\tau)\,d\tau\right) ds + \int_t^{3/4} \psi_2^{-1}\left(\int_t^s h_i(\tau)\,d\tau\right) ds\right],$$
$$t \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad i = 1, \dots, n.$$

It follows from (A1)–(A3) that

$$\Gamma = \min\left\{\gamma_i(t) : \frac{1}{4} \le t \le \frac{3}{4},\ i = 1, \dots, n\right\} > 0,$$
$$\chi = \sum_{i=1}^n \psi_1^{-1}\left(\int_0^1 h_i(\tau)\,d\tau\right) > 0.$$

In order to state our results we introduce the notation

$$
\begin{aligned}
&f_0^i = \lim_{\|\mathbf{u}\|\to 0} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)}, \qquad f_\infty^i = \lim_{\|\mathbf{u}\|\to \infty} \frac{f^i(\mathbf{u})}{\varphi(\|\mathbf{u}\|)},\\
&\qquad\qquad \mathbf{u} \in \mathbf{R}_+^n, \quad i = 1, \dots, n,\\
&\mathbf{f}_0 = \max\{f_0^1, \dots, f_0^n\}, \quad \mathbf{f}_\infty = \max\{f_\infty^1, \dots, f_\infty^n\}.
\end{aligned}
\tag{1.6}
$$

Although we will not provide its proof until Section 3, we state at this point our main result of the paper:

Theorem 1.1. *Let* (A1)–(A3) *hold. Assume* $0 < \mathbf{f}_0 < \infty$ *and* $0 < \mathbf{f}_\infty < \infty$.

(a) *If*

$$\psi_2\left(\frac{1}{\Gamma\,\psi_2^{-1}(\mathbf{f}_0)}\right) < \lambda < \psi_1\left(\frac{1}{\chi\,\psi_1^{-1}(\mathbf{f}_\infty)}\right),$$

then (1.1)–(1.2) *has a positive solution.*

(b) *If*

$$\psi_2\left(\frac{1}{\Gamma\,\psi_2^{-1}(\mathbf{f}_\infty)}\right) < \lambda < \psi_1\left(\frac{1}{\chi\,\psi_1^{-1}(\mathbf{f}_0)}\right),$$

then (1.1)–(1.2) *has a positive solution.*

2. Preliminaries. The following well-known result from the fixed point index theory is crucial in our arguments.

Lemma 2.1 ([**6, 10, 15**]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (1.1)–(1.2), let X be the Banach space $\Pi_{i=1}^n C[0,1]$ and, for $\mathbf{u} = (u_1, \dots, u_n) \in X$,

$$\|\mathbf{u}\| = \sum_{i=1}^n \sup_{t\in[0,1]} |u_i(t)|.$$

For $\mathbf{u} \in X$ or $\mathbf{R}_+^n$, $\|\mathbf{u}\|$ denotes the norm of $\mathbf{u}$ in X or $\mathbf{R}_+^n$, respectively.

Define K to be a cone in X by

$$K = \Big\{\mathbf{u} = (u_1, \dots, u_n) \in X : u_i(t) \geq 0,\ t \in [0,1],\ i = 1, \dots, n, \\ \text{and} \min_{1/4\leq t\leq 3/4} \sum_{i=1}^n u_i(t) \geq \frac{1}{4}\|\mathbf{u}\|\Big\}.$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| < r\}.$$

Note that $\partial\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\| = r\}$.

Let $\mathbf{T}_\lambda : K \to X$ be a map with components $(T_\lambda^1, \dots, T_\lambda^n)$. We define T_λ^i, $i = 1, \dots, n$, by

$$\text{(2.7)} \quad T_\lambda^i \mathbf{u}(t) = \begin{cases} \displaystyle\int_0^t \varphi^{-1}\bigg(\int_s^{\sigma_i} \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\, d\tau\bigg)\, ds & 0 \leq t \leq \sigma_i, \\ \displaystyle\int_t^1 \varphi^{-1}\bigg(\int_{\sigma_i}^{s} \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\, d\tau\bigg)\, ds & \sigma_i \leq t \leq 1, \end{cases}$$

where $\sigma_i = 0$ for (1.1)–(1.2b) and $\sigma_i = 1$ for (1.1)–(1.2c). For (1.1)–(1.2a), $\sigma_i \in (0,1)$ is a solution of the equation

$$\text{(2.8)} \qquad \Theta^i \mathbf{u}(t) = 0, \quad 0 \leq t \leq 1,$$

where the map $\Theta^i : K \to C[0,1]$ is defined by

$$\text{(2.9)} \quad \begin{aligned} \Theta^i \mathbf{u}(t) = &\int_0^t \varphi^{-1}\bigg(\int_s^t \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\, d\tau\bigg)\, ds \\ &- \int_t^1 \varphi^{-1}\bigg(\int_t^s \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\, d\tau\bigg)\, ds, \quad 0 \leq t \leq 1. \end{aligned}$$

By virtue of Lemma 2.2, the operator $\mathbf{T}_\lambda$ is well defined.

Lemma 2.2. *Assume* (A1)–(A3) *hold. Then, for any* $\mathbf{u} \in K$ *and* $i = 1, \dots, n$, $\Theta^i \mathbf{u}(t) = 0$ *has at least one solution in* $(0,1)$. *In addition, if* $\sigma_i^1 < \sigma_i^2 \in (0,1)$, $i = 1, \dots n$, *are two solutions of* $\Theta^i \mathbf{u}(t) = 0$, *then* $h_i(t) f^i(\mathbf{u}(t)) \equiv 0$ *for* $t \in [\sigma_i^1, \sigma_i^2]$ *and any* $\sigma_i \in [\sigma_i^1, \sigma_i^2]$ *is also a solution of* $\Theta^i \mathbf{u}(t) = 0$. *Furthermore,* $\mathbf{T}_\lambda^i \mathbf{u}(t)$, $i = 1, \dots, n$, *is independent of the choice of* $\sigma_i \in [\sigma_i^1, \sigma_i^2]$.

Proof. Let $\alpha^i(\tau) = \lambda h_i(\tau) f^i(\mathbf{u}(\tau))$. If $\int_0^1 \alpha^i(\tau)\, dt = 0$, we may choose any $\sigma_i \in (0,1)$. Let's assume $\int_0^1 \alpha^i(\tau)\, dt > 0$. Therefore, $\Theta^i \mathbf{u}(0) < 0$ and $\Theta^i \mathbf{u}(1) > 0$. It follows from the continuity of $\Theta^i \mathbf{u}(t)$ that $\Theta^i \mathbf{u}(t) = 0$ has at least one solution on (0,1). In addition, $\Theta^i \mathbf{u}(t)$ is a nondecreasing function on $[0,1]$. If $\sigma_i^1 < \sigma_i^2 \in (0,1)$ are two solutions of $\Theta^i \mathbf{u}(t) = 0$, it is not hard to show that $\int_{\sigma_i^1}^{\sigma_i^2} \varphi^{-1}(\int_s^{\sigma_i^2} \alpha^i(\tau)\, d\tau)\, ds = 0$. Therefore, $\alpha^i(\tau) \equiv 0$ on $[\sigma_i^1, \sigma_i^2]$. Let $\sigma_i \in [\sigma_i^1, \sigma_i^2]$. Then it is easy to verify that σ_i is a solution of $\Theta^i \mathbf{u}(t) = 0$. Hence, (2.7) implies

$$
T_\lambda^i \mathbf{u}(t) = \begin{cases} \displaystyle\int_0^t \varphi^{-1}\left(\int_s^{\sigma_i^1} \alpha^i(\tau)\, d\tau\right) ds & 0 \le t \le \sigma_i^1, \\ \displaystyle\int_0^{\sigma_i^1} \varphi^{-1}\left(\int_s^{\sigma_i^1} \alpha^i(\tau)\, d\tau\right) ds & \sigma_i^1 \le t \le \sigma_i, \\ \displaystyle\int_{\sigma_i^2}^1 \varphi^{-1}\left(\int_{\sigma_i^2}^s \alpha^i(\tau)\, d\tau\right) ds & \sigma_i \le t \le \sigma_i^2, \\ \displaystyle\int_t^1 \varphi^{-1}\left(\int_{\sigma_i^2}^s \alpha^i(\tau)\, d\tau\right) ds & \sigma_i^2 \le t \le 1, \end{cases} \tag{2.10}
$$

which is independent of $\sigma_i \in [\sigma_i^1, \sigma_i^2]$. $\square$

The following lemma is a standard result due to the concavity of a real-valued function $u(t)$ on $[0,1]$, see e.g., [**21**–**23**].

Lemma 2.3. *Assume* φ *is an odd, increasing homeomorphism of* $\mathbf{R}$ *onto* $\mathbf{R}$. *Let* $0 \le u(t) \in C^1[0,1]$ *and* $\varphi(u'(t))$ *be nonincreasing on* $[0,1]$.

Then

$$u(t) \geq \min\{t, 1-t\} \sup_{t\in[0,1]} u(t) \quad \text{for} \quad t \in [0,1].$$

In particular, $\min_{1/4\leq t\leq 3/4} u(t) \geq 1/4 \sup_{t\in[0,1]} u(t)$.

We remark that, according to Lemma 2.3, any nontrivial component of nonnegative solutions of (1.1)–(1.2) is positive on $(0,1)$.

Lemma 2.4. *Assume* (A1)–(A3) *hold. Then* $\mathbf{T}_\lambda(K) \subset K$ *and* $\mathbf{T}_\lambda : K \to K$ *is compact and continuous.*

Proof. Lemma 2.3 implies that $\mathbf{T}_\lambda(K) \subset K$. It is not hard to show that $\mathbf{T}_\lambda : K \to K$ is compact and continuous. □

Lemma 2.5 [21, 22]. *Assume* (A1) *holds. Then for all* $\sigma, x \in (0,\infty)$

$$\psi_2^{-1}(\sigma)x \leq \varphi^{-1}(\sigma\,\varphi(x)) \leq \psi_1^{-1}(\sigma)x.$$

Proof. Since $\sigma = \psi_1(\psi_1^{-1}(\sigma)) = \psi_2(\psi_2^{-1}(\sigma))$ and $\varphi(\varphi^{-1}(\sigma\,\varphi(x))) = \sigma\,\varphi(x)$, it follows that

$$\psi_2(\psi_2^{-1}(\sigma))\,\varphi(x) = \varphi(\varphi^{-1}(\sigma\,\varphi(x))) = \psi_1(\psi_1^{-1}(\sigma))\,\varphi(x).$$

On the other hand, we have by (A1)

$$\psi_1(\psi_1^{-1}(\sigma))\,\varphi(x) \leq \varphi(\psi_1^{-1}(\sigma)x) \quad \text{and} \quad \psi_2(\psi_2^{-1}(\sigma))\,\varphi(x) \geq \varphi(\psi_2^{-1}(\sigma)x).$$

Hence, $\varphi(\psi_2^{-1}(\sigma)x) \leq \varphi(\varphi^{-1}(\sigma\,\varphi(x))) \leq \varphi(\psi_1^{-1}(\sigma)x)$.

Thus, we obtain $\psi_2^{-1}(\sigma)x \leq \varphi^{-1}(\sigma\,\varphi(x)) \leq \psi_1^{-1}(\sigma)x$. □

Lemma 2.6. *Assume* (A1)–(A3) *hold. Let* $\mathbf{u} = (u_1, \dots, u_n) \in K$ *and* $\eta > 0$. *If there exists a component* f^i *of* $\mathbf{f}$ *such that*

$$f^i(\mathbf{u}(t)) \geq \varphi\left(\eta \sum_{i=1}^n u_i(t)\right) \quad \text{for} \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \psi_2^{-1}(\lambda)\,\Gamma\,\eta\|\mathbf{u}\|.$$

Proof. Note, from the definition of $\mathbf{T}_\lambda \mathbf{u}$, that $T_\lambda^i \mathbf{u}(\sigma_i)$ is the maximum value of $T_\lambda^i \mathbf{u}$ on [0,1]. If $\sigma_i \in [1/4, 3/4]$, we have

$$\begin{aligned}\|\mathbf{T}_\lambda \mathbf{u}\| &\geq \sup_{t\in[0,1]} |T_\lambda^i \mathbf{u}(t)| \\ &\geq \frac{1}{2}\bigg[\int_{1/4}^{\sigma_i} \varphi^{-1}\bigg(\int_s^{\sigma_i} \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\,d\tau\bigg)\,ds \\ &\qquad + \int_{\sigma_i}^{3/4} \varphi^{-1}\bigg(\int_{\sigma_i}^{s} \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\,d\tau\bigg)\,ds\bigg] \\ &\geq \frac{1}{2}\bigg[\int_{1/4}^{\sigma_i} \varphi^{-1}\bigg(\int_s^{\sigma_i} \lambda h_i(\tau)\, \varphi(\eta\sum_{j=1}^n u_j(\tau))\,d\tau\bigg)\,ds \\ &\qquad + \int_{\sigma_i}^{3/4} \varphi^{-1}\bigg(\int_{\sigma_i}^{s} \lambda h_i(\tau)\, \varphi(\eta\sum_{j=1}^n u_j(\tau))\,d\tau\bigg)\,ds\bigg],\end{aligned}$$

and in view of Lemma 2.3 and condition (A1), we find

$$\begin{aligned}\|\mathbf{T}_\lambda \mathbf{u}\| &\geq \frac{1}{2}\bigg[\int_{1/4}^{\sigma_i} \varphi^{-1}\bigg(\int_s^{\sigma_i} \psi_2(\psi_2^{-1}(\lambda))h_i(\tau)\,\varphi\Big(\frac{\eta}{4}\,\|\mathbf{u}\|\Big)\,d\tau\bigg)\,ds \\ &\qquad + \int_{\sigma_i}^{3/4} \varphi^{-1}\bigg(\int_{\sigma_i}^{s} \psi_2(\psi_2^{-1}(\lambda))h_i(\tau)\,\varphi\Big(\frac{\eta}{4}\,\|\mathbf{u}\|\Big)\,d\tau\bigg)\,ds\bigg] \\ &\geq \frac{1}{2}\bigg[\int_{1/4}^{\sigma_i} \varphi^{-1}\bigg(\int_s^{\sigma_i} h_i(\tau)\,d\tau\,\varphi\Big(\psi_2^{-1}(\lambda)\,\frac{\eta}{4}\,\|\mathbf{u}\|\Big)\bigg)\,ds \\ &\qquad + \int_{\sigma_i}^{3/4} \varphi^{-1}\bigg(\int_{\sigma_i}^{s} h_i(\tau)\,d\tau\,\varphi\Big(\psi_2^{-1}(\lambda)\,\frac{\eta}{4}\,\|\mathbf{u}\|\Big)\bigg)\,ds\bigg].\end{aligned}$$

Now, because of Lemma 2.5, we have

$$\begin{aligned}&\|\mathbf{T}_\lambda \mathbf{u}\| \\ &\geq \frac{\psi_2^{-1}(\lambda)\eta\|\mathbf{u}\|}{8}\bigg[\int_{1/4}^{\sigma_i}\psi_2^{-1}\bigg(\int_s^{\sigma_i} h_i(\tau)\,d\tau\bigg)\,ds + \int_{\sigma_i}^{3/4}\psi_2^{-1}\bigg(\int_{\sigma_i}^{s} h_i(\tau)\,d\tau\bigg)\,ds\bigg] \\ &\geq \psi_2^{-1}(\lambda)\,\Gamma\,\eta\,\|\mathbf{u}\|.\end{aligned}$$

For $\sigma_i > 3/4$, it is easy to see

$$\|T_\lambda^i \mathbf{u}\| \geq \int_{1/4}^{3/4} \varphi^{-1}\biggl(\int_s^{3/4} \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\, d\tau \biggr)\, ds.$$

On the other hand, we have

$$\|T_\lambda^i \mathbf{u}\| \geq \int_{1/4}^{3/4} \varphi^{-1}\biggl(\int_{1/4}^{s} \lambda h_i(\tau)\, f^i(\mathbf{u}(\tau))\, d\tau \biggr)\, ds \quad \text{if} \quad \sigma_i < \frac{1}{4}.$$

Similar arguments show that $\|\mathbf{T}_\lambda \mathbf{u}\| \geq \psi_2^{-1}(\lambda)\, \Gamma\, \eta\, \|\mathbf{u}\|$ if $\sigma_i > 3/4$ or $\sigma_i < c\, 1/4$. $\square$

For each $i = 1, \ldots, n$, define a new function $\hat{f}^i(t) : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$\hat{f}^i(t) = \max\{ f^i(\mathbf{u}) : \mathbf{u} \in \mathbf{R}_+^n \text{ and } \|\mathbf{u} \leq \mathrm{t}\}.$$

Note that $\hat{f}_0^i = \lim_{t\to 0} \hat{f}^i(t)/\varphi(t)$ and $\hat{f}_\infty^i = \lim_{t\to\infty} \hat{f}^i(t)/\varphi(t)$.

Lemma 2.7 [**21, 22**]. *Assume* (A1)–(A2) *hold. Then* $\hat{f}_0^i = f_0^i$ *and* $\hat{f}_\infty^i = f_\infty^i$, $i = 1, \ldots, n$.

Proof. It is easy to see that $\hat{f}_0^i = f_0^i$. For the second part, we consider the two cases, (a) $f^i(\mathbf{u})$ is bounded, and (b) $f^i(\mathbf{u})$ is unbounded. For case (a), it follows from $\lim_{t\to\infty} \varphi_i(t) = \infty$, that $\hat{f}_\infty^i = 0 = f_\infty^i$. For case (b), for any $\delta > 0$, let $M^i = \hat{f}^i(\delta)$ and

$$N_\delta^i = \inf\{ \|\mathbf{u}\| : \mathbf{u} \in \mathbf{R}_+^n,\ \|\mathbf{u}\| \geq \delta,\ f^i(\mathbf{u}) \geq M^i \} \geq \delta.$$

Then

$$\begin{aligned} &\max\{ f^i(\mathbf{u}) : \|\mathbf{u}\| \leq N_\delta^i,\ \mathbf{u} \in \mathbf{R}_+^n \} \\ &\qquad\qquad = M^i = \max\{ f^i(\mathbf{u}) : \|\mathbf{u}\| = N_\delta^i,\ \mathbf{u} \in \mathbf{R}_+^n \}. \end{aligned}$$

Thus, for any $\delta > 0$, there exists an $N_\delta^i \geq \delta$ such that

$$\hat{f}^i(t) = \max\{ f^i(\mathbf{u}) : N_\delta^i \leq \|\mathbf{u}\| \leq t,\ \mathbf{u} \in \mathbf{R}_+^n \} \quad \text{for} \quad t > N_\delta^i.$$

Hence, the definitions of $\hat{f}^i_\infty$ and f^i_∞ imply that $\hat{f}^i_\infty = f^i_\infty$. $\square$

Lemma 2.8. *Assume* (A1)–(A3) *hold, and let* $r > 0$. *If there exists an* $\varepsilon > 0$ *such that*

$$\hat{f}^i(r) \le \psi_1(\varepsilon)\,\varphi(r), \quad i = 1, \dots, n,$$

then

$$\|\mathbf{T}_\lambda \mathbf{u}\| \le \psi_1^{-1}(\lambda)\,\varepsilon\,\chi\|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\Omega_r.$$

Proof. From the definition of T_λ, for $\mathbf{u} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &= \sum_{i=1}^n \sup_{t\in[0,1]} |T^i_\lambda\,\mathbf{u}(t)| \\
&\le \sum_{i=1}^n \varphi^{-1}\bigg(\int_0^1 \lambda h_i(\tau)\,f^i(\mathbf{u}(\tau))\,d\tau\bigg) \\
&\le \sum_{i=1}^n \varphi^{-1}\bigg(\int_0^1 h_i(\tau)\,d\tau\,\lambda\,\hat{f}^i(r)\bigg) \\
&\le \sum_{i=1}^n \varphi^{-1}\bigg(\int_0^1 h_i(\tau)\,d\tau\,\lambda\,\psi_1(\varepsilon)\,\varphi(r)\bigg).
\end{aligned}$$

Note that $\lambda = \psi_1(\psi_1^{-1}(\lambda))$. Then (A1) and Lemma 2.5 imply that

$$\begin{aligned}
\|\mathbf{T}_\lambda \mathbf{u}\| &\le \sum_{i=1}^n \varphi^{-1}\bigg(\int_0^1 h_i(\tau)\,d\tau\,\varphi(\psi_1^{-1}(\lambda)\,\varepsilon r)\bigg) \\
&\le \psi_1^{-1}(\lambda)\,\varepsilon r \sum_{i=1}^n \psi_1^{-1}\bigg(\int_0^1 h_i(\tau)\,d\tau\bigg) \\
&= \psi_1^{-1}(\lambda)\,\varepsilon\,\chi\,\|\mathbf{u}\|. \qquad \square
\end{aligned}$$

3. Proof of Theorem 1. We now provide the proof for this paper's main result.

Proof. Part (a). Let $f^i_0 = \mathbf{f}_0 > 0$ for some fixed i. It follows that

$$\psi_2\bigg(\frac{1}{\Gamma\,\psi_2^{-1}(f^i_0)}\bigg) < \lambda < \psi_1\bigg(\frac{1}{\chi\,\psi_1^{-1}(\mathbf{f}_\infty)}\bigg).$$

Condition (A1) implies that there exists an $0 < \varepsilon < f_0^i$ such that

$$\psi_2\bigg(\frac{1}{\Gamma\,\psi_2^{-1}(f_0^i - \varepsilon)}\bigg) < \lambda < \psi_1\bigg(\frac{1}{\chi\,\psi_1^{-1}(\mathbf{f}_\infty + \varepsilon)}\bigg).$$

Beginning with f_0^i, there is an $r_1 > 0$ such that

$$f^i(\mathbf{u}) \geq (f_0^i - \varepsilon)\,\varphi(\|\mathbf{u}\|)$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbf{R}_+^n$ and $\|\mathbf{u}\| \leq r_1$. Note that

$$(f_0^i - \varepsilon)\,\varphi(\|\mathbf{u}\|) = \psi_2(\psi_2^{-1}(f_0^i - \varepsilon))\,\varphi(\|\mathbf{u}\|).$$

If $\mathbf{u} \in \partial\,\Omega_{r_1}$, then

$$f^i(\mathbf{u}(t)) \geq \psi_2(\psi_2^{-1}(f_0^i - \varepsilon))\,\varphi\bigg(\sum_{j=1}^n u_j(t)\bigg) \geq \varphi\bigg(\psi_2^{-1}(f_0^i - \varepsilon)\sum_{j=1}^n u_j(t)\bigg)$$

for $t \in [0,1]$. Lemma 2.6 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \geq \psi_2^{-1}(\lambda)\,\Gamma\,\psi_2^{-1}(f_0^i - \varepsilon)\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\,\Omega_{r_1}.$$

It remains to consider $\mathbf{f}_\infty$. It follows from Lemma 2.7 that $\hat{f}_\infty^j = f_\infty^j$, $j = 1, \dots, n$. Therefore, there is an $r_2 > 2r_1$ such that, for $j = 1, \dots, n$,

$$\hat{f}^j(r_2) \leq (f_\infty^j + \varepsilon)\,\varphi(r_2) \leq (\mathbf{f}_\infty + \varepsilon)\,\varphi(r_2) = \psi_1(\psi_1^{-1}(\mathbf{f}_\infty + \varepsilon))\,\varphi(r_2).$$

Lemma 2.8 implies that, for $\mathbf{u} \in \partial\,\Omega_{r_2}$, we have

$$\begin{aligned}\|\mathbf{T}_\lambda \mathbf{u}\| &\leq \psi_1^{-1}(\lambda)\,\chi\,\psi_1^{-1}(\mathbf{f}_\infty + \varepsilon)\|\mathbf{u}\| \\ &< \|\mathbf{u}\|.\end{aligned}$$

By Lemma 2.1,

$$i(\mathbf{T}_\lambda, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_2}, K) = 1.$$

It follows from the additivity of the fixed point index that $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K) = 1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_2} \setminus \overline{\Omega}_{r_1}$, which is the desired positive solution of (1.1)–(1.2).

Part (b). Let $f^i_\infty = \mathbf{f}_\infty > 0$ for some fixed i. It follows that

$$\psi_2\left(\frac{1}{\Gamma\,\psi_2^{-1}(f^i_\infty)}\right) < \lambda < \psi_1\left(\frac{1}{\chi\,\psi_1^{-1}(\mathbf{f}_0)}\right).$$

Condition (A1) implies that there exists an $0 < \varepsilon < f^i_\infty$ such that

$$\psi_2\left(\frac{1}{\Gamma\,\psi_2^{-1}(f^i_\infty - \varepsilon)}\right) < \lambda < \psi_1\left(\frac{1}{\chi\,\psi_1^{-1}(\mathbf{f}_0 + \varepsilon)}\right).$$

Since $\hat{f}^j_0 = f^j_0$, $j = 1, \dots, n$, there exists a $r_3 > 0$ such that

$$\hat{f}^j(r_3) \le (f^j_0 + \varepsilon)\,\varphi(r_3) \le (\mathbf{f}_0 + \varepsilon)\,\varphi(r_3), \quad j = 1, \dots, n.$$

Lemma 2.8 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \le \psi_1^{-1}(\lambda)\,\chi\,\psi_1^{-1}(\mathbf{f}_0 + \varepsilon)\|\mathbf{u}\| < \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\,\Omega_{r_3}.$$

Next, considering f^i_∞, there is an $\widehat{H} > 0$ such that

$$f^i(\mathbf{u}) \ge (f^i_\infty - \varepsilon)\,\varphi(\|\mathbf{u}\|)$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbf{R}^n_+$ and $\|\mathbf{u}\| \ge \widehat{H}$. Let $r_4 = \max\{2r_3, 4\widehat{H}\}$. If $\mathbf{u} = (u_1, \dots, u_n) \in \partial\,\Omega_{r_4}$, then

$$\min_{1/4 \le t \le 3/4} \sum_{j=1}^n u_j(t) \ge \frac{1}{4}\,\|\mathbf{u}\| \ge \widehat{H},$$

and hence,

$$f^i(\mathbf{u}(t)) \ge (f^i_\infty - \varepsilon)\,\varphi\left(\sum_{j=1}^n u_j(t)\right) \ge \varphi\left(\psi_2^{-1}(f^i_\infty - \varepsilon)\sum_{j=1}^n u_j(t)\right) \quad \text{for} \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Lemma 2.6 implies that

$$\|\mathbf{T}_\lambda \mathbf{u}\| \ge \psi_2^{-1}(\lambda)\,\Gamma\,\psi_2^{-1}(f^i_\infty - \varepsilon)\|\mathbf{u}\| > \|\mathbf{u}\| \quad \text{for} \quad \mathbf{u} \in \partial\,\Omega_{r_4}.$$

Again it follows from Lemma 2.1 that

$$i(\mathbf{T}_\lambda, \Omega_{r_3}, K) = 1 \quad \text{and} \quad i(\mathbf{T}_\lambda, \Omega_{r_4}, K) = 0.$$

Hence, $i(\mathbf{T}_\lambda, \Omega_{r_4} \backslash \overline{\Omega}_{r_3}, K) = -1$. Thus, $\mathbf{T}_\lambda$ has a fixed point in $\Omega_{r_4} \backslash \overline{\Omega}_{r_3}$, which is the desired positive solution of (1.1)–(1.2). $\square$

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Department of Mathematics, Baylor University, Waco, Texas 76798-7328
E-mail address: Johnny_Henderson@baylor.edu

Department of Mathematical Sciences & Applied Computing, Arizona State University, Phoenix, AZ 85069-7100
E-mail address: wangh@asu.edu