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TWO-GRID METHODS FOR THE SOLUTION OF NONLINEAR WEAKLY SINGULAR INTEGRAL EQUATIONS BY PIECEWISE POLYNOMIAL COLLOCATION

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ABSTRACT. To solve nonlinear weakly singular integral equations by the piecewise polynomial collocation method, it is necessary to solve large nonlinear systems. This can be done straightforwardly only for comparatively rough discretizations. In this paper a two-grid iteration method is presented which enables us to find the solution of such systems for fine discretizations. We prove the convergence and establish the convergence rate of this method. So we generalize for nonlinear equations the results proved in [**10**] for linear equations.

1. Introduction. We shall deal with the nonlinear weakly singular integral equation

$$u(x) = \int_G K(x,y,u(y))\,dy + f(x), \quad x \in G, \tag{1}$$

where

$$G = \{x = (x_1, \dots, x_n) : \ 0 < x_k < b_k, \ k = 1, \dots, n\}$$

is an n-dimensional parallelepiped. The piecewise polynomial collocation method for the solution of such equations is considered in [**1, 5, 8, 12**]. In order to calculate the approximate solution by collocation method, large nonlinear systems must be solved. In the present paper a two-grid iteration scheme is presented for the solution of such systems. Fast convergence of this method is shown. Analogous results have been established for linear equations in [**10**] and for nonlinear equations in the case of piecewise constant collocation method in [**9**].

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2. Integral equation. We shall make the following assumptions (A1)–(A3).

(A1) The kernel $K(x,y,u)$ is m times, $m \geq 1$, continuously differentiable with respect to x,y and u for $x \in G$, $y \in G$, $x \neq y$, $u \in (-\infty,\infty)$, whereby there exists a real number $\nu \in (-\infty, n)$ such that, for any nonnegative integer $l \in \mathbf{Z}_+$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{Z}_+^n$ with $l + |\alpha| + |\beta| \leq m$, the following inequalities hold:

$$\begin{aligned}&\left|D_x^\alpha D_{x+y}^\beta \left(\frac{\partial}{\partial u}\right)^l K(x,y,u)\right| \\ &\qquad\qquad \leq \psi_1(|u|) \begin{cases} 1 & \nu + |\alpha| < 0 \\ 1 + |\log|x-y|| & \nu + |\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|} & \nu + |\alpha| > 0 \end{cases},\end{aligned}$$

$$\begin{aligned}&\left|D_x^\alpha D_{x+y}^\beta \left(\frac{\partial}{\partial u}\right)^l K(x,y,u_1) - D_x^\alpha D_{x+y}^\beta \left(\frac{\partial}{\partial u}\right)^l K(x,y,u_2)\right| \\ &\quad \leq \psi_2(\max\{|u_1|,|u_2|\})|u_1-u_2| \begin{cases} 1 & \nu + |\alpha| < 0 \\ 1 + |\log|x-y|| & \nu + |\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|} & \nu + |\alpha| > 0. \end{cases}\end{aligned}$$

Here $|\alpha| = \alpha_1 + \dots + \alpha_n$ for $\alpha \in \mathbf{Z}_+^n$, $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ for $x \in \mathbf{R}^n$,

$$D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

$$D_{x+y}^\beta = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n},$$

and the functions $\psi_1 : [0,\infty) \to [0,\infty)$ and $\psi_2 : [0,\infty) \to [0,\infty)$ are assumed to be monotonically increasing.

(A2) The righthand term $f \in C^{m,\nu}(G)$ with the same m and ν as in (A1), i.e., $f(x)$ is m times continuously differentiable on G and the estimates

$$|D^\alpha f(x)| \leq \text{const} \begin{cases} 1 & |\alpha| < n - \nu \\ 1 + |\log \rho(x)| & |\alpha| = n - \nu \\ \rho(x)^{n-\nu-|\alpha|} & |\alpha| > n - \nu \end{cases},$$

$$\left|\frac{\partial^l f(x)}{\partial x_k^l}\right| \le \text{const}\begin{cases} 1 & l < n-\nu \\ 1+|\log\rho_k(x)| & l = n-\nu \\ \rho_k(x)^{n-\nu-l} & l > n-\nu \end{cases}$$

hold for $x \in G$, $|\alpha| \le m$, $l = 1,\dots,m$, $k = 1,\dots,n$, where $\rho_k(x) = \min\{x_k, b_k - x_k\}$ and $\rho(x) = \min_{1\le k\le n}\rho_k(x)$ is the distance from x to ∂G, the boundary of G.

(A3) Integral equation (1) has a solution $u_0 \in L^\infty(G)$ and the linearized integral equation

$$v(x) = \int_G K_0(x,y)v(y)\,dy, \quad K_0(x,y) = \left[\frac{\partial K(x,y,u)}{\partial u}\right]_{u=u_0(y)},$$

has only the trivial solution $v = 0$ in $L^\infty(G)$.

Note that the assumption (A1) holds, for example, for the kernels $K(x,y,u) = K_1(x,y,u)|x-y|^{-\nu}$, $0 < \nu < n$, and $K(x,y,u) = K_1(x,y,u)\log|x-y|$, $\nu = 0$, where $K_1(x,y,u)$ is an $m+1$ times continuously differentiable function with respect to x, y, u for $x, y \in \overline{G}$, $u \in (-\infty,\infty)$.

From (A1)–(A3) it follows that the solution u_0 of (1) belongs to the space $C^{m,\nu}(G)$ [**7, 12**].

3. Collocation method. We use the same non-uniform grid as in [**10, 12**]. To define the partition of $\overline{G}$ into cells we choose a vector $N = (N_1,\dots,N_n)$ of natural numbers and introduce in the interval $[0,b_k]$, $k = 1,\dots,n$, the following $2N_k + 1$ grid points:

$$\begin{aligned} x_{k,N}^{j_k} &= \frac{b_k}{2}\left(\frac{j_k}{N_k}\right)^r, \quad j_k = 0,1,\dots,N_k, \\ x_{k,N}^{N_k+j_k} &= b_k - x_{k,N}^{N_k-j_k}, \quad j_k = 1,\dots,N_k. \end{aligned} \tag{2}$$

Here $r \in \mathbf{R}$, $r \ge 1$, characterizes the non-uniformity of the grid. If $r = 1$, then the grid points (2) are uniformly located. Using the points (2) we introduce the partition of $\overline{G}$ into closed cells G_N^j:

$$G_N^j = \{x = (x_1,\dots,x_n) : x_{k,N}^{j_k-1} \le x_k \le x_{k,N}^{j_k},\ k = 1,\dots,n\} \subset \overline{G},$$

$$j \in J_N = \{j = (j_1,\dots,j_n) :\ j_k = 1,\dots,2N_k,\ k = 1,\dots,n\}.$$

We determine the collocation points in the following way. We choose m points $\eta_1, \dots, \eta_m$ in the interval $[-1, 1]$:

$$-1 \le \eta_1 < \eta_2 < \cdots < \eta_m \le 1.$$

By affine transformations we transfer them into the interval $[x_{k,N}^{j_k-1}, x_{k,N}^{j_k}]$, $j_k = 1, \dots, 2N_k$, $k = 1, \dots, n$,

$$\xi_{k,N}^{j_k,q_k} = x_{k,N}^{j_k-1} + \frac{\eta_{q_k}+1}{2}(x_{k,N}^{j_k} - x_{k,N}^{j_k-1}), \quad q_k = 1, \dots, m.$$

Note that $\xi_{k,N}^{j_k,m} = \xi_{k,N}^{j_k+1,1} = x_{k,N}^{j_k}$ if $\eta_1 = -1$ and $\eta_m = 1$, $j_k = 1, \dots, 2N_k - 1$. We assign the collocation points

$$\xi_N^{j,q} = (\xi_{1,N}^{j_1,q_1}, \dots, \xi_{n,N}^{j_n,q_n}), \quad q \in Q,$$
$$Q = \{q = (q_1, \dots, q_n) : \ q_k = 1, \dots, m, \ k = 1, \dots, n\},$$

to the cells G_N^j, $j \in J_N$.

We define the interpolation projector $\mathcal{P}_N$ by the formula

$$(\mathcal{P}_N u)(x) = \sum_{q \in Q} u(\xi_N^{j,q}) \varphi_N^{j,q}(x), \quad x \in G_N^j, \ j \in J_N, \tag{3}$$

where

$$\varphi_N^{j,q}(x) = \varphi_{1,N}^{j_1,q_1}(x_1) \cdots \varphi_{n,N}^{j_n,q_n}(x_n)$$

and $\varphi_{k,N}^{j_k,q_k}(x_k)$, $k = 1, \dots, n$, are the polynomials of one variable of degree $m-1$ such that

$$\varphi_{k,N}^{j_k,q_k}(\xi_{k,N}^{j_k,p_k}) = \begin{cases} 1 & \text{if } \ p_k = q_k \\ 0 & \text{if } \ p_k \ne q_k \end{cases}, \quad p_k = 1, \dots, m. \tag{4}$$

Let us denote by E_N the range of the projection $\mathcal{P}_N$. This is the finite dimensional space of piecewise polynomial functions u_N on $\overline{G}$ which on any cell G_N^j, $j \in J_N$, are polynomials of the degree not exceeding $m-1$ with respect to any of arguments $x_1, \dots, x_n$.

We determine the approximate solution $u_N \in E_N$ of the integral equation (1) by the collocation method from the following conditions:

$$\left[u_N(x) - \int_G K(x, y, u_N(y))\, dy - f(x) \right]_{x = \xi_N^{i,p}} = 0, \quad p \in Q, \ i \in J_N. \tag{5}$$

By the representation (3), we can find $u_N \in E_N$ in the form

$$u_N(x) = \sum_{q\in Q} c^{j,q} \varphi_N^{j,q}(x) \quad \text{if } x \in G_N^j,\ j \in J_N,$$

where, as it follows from (4),

$$c^{j,q} = u_N(\xi_N^{j,q}).$$

Now the collocation conditions (5) will take the following form of a nonlinear system which determines the coefficients $c^{j,q} = u_N(\xi_N^{j,q})$:

$$u_N(\xi_N^{i,p}) = \sum_{j\in J_N} \int_{G_N^j} K\bigg(\xi_N^{i,p}, y, \sum_{q\in Q} u_N(\xi_N^{j,q})\varphi_N^{j,q}(y)\bigg)\, dy + f(\xi_N^{i,p}), \tag{6}$$
$$p \in Q,\ i \in J_N.$$

If $\eta_1 > -1$ or $\eta_m < 1$, then this system has $(2m)^n N_1 \dots N_n = \dim E_N$ equations and the same number of unknowns. If $\eta_1 = -1$ and $\eta_m = 1$, then some collocation points coincide. The number of different collocation points is $[2N_1(m-1)+1] \dots [2N_n(m-1)+1] = \dim E_N$ and the system (6) has the same number of equations and unknowns.

In [**12**] the following result about the convergence of such collocation methods is proved.

Theorem 1. *Let the assumptions* (A1)–(A3) *hold. Then there exist* $N_0 > 0$ *and* $\delta_0 > 0$ *such that, for* $N_k \geq N_0$, $k = 1, \dots, n$, *the collocation conditions* (5) *define a unique approximation* $u_N \in E_N$ *to* u_0 *satisfying*

$$\sup_{x\in G} |u_N(x) - u_0(x)| < \delta_0.$$

The following error estimate holds:

$$\max_{j\in J_N} \max_{q\in Q} |u_N(\xi_N^{j,q}) - u_0(\xi_N^{j,g})|$$
$$\leq \text{const}\, h_N^m \text{ for } \begin{cases} r > m/(2(n-\nu)) & \text{if } n-\nu \leq 1 \\ r > m/(n-\nu+1) & \text{if } 1 < n-\nu \leq m-1 \\ r \geq 1 & \text{if } n-\nu > m-1 \end{cases}$$

where

$$h_N = \max_{1 \le k \le n} \frac{b_k}{N_k}.$$

It is shown in [**8**] that for special collocation points a more rapid convergence, the superconvergence, takes place.

To apply the collocation method it is necessary to solve the nonlinear system (6). We write this system in the form

$$\bar{u}_N = \mathcal{T}_N \bar{u}_N + \bar{f}_N \tag{7}$$

where $\bar{u}_N = (u_N(\xi_N^{j,q}))_{j \in J_N, q \in Q}$, $\bar{f}_N = (f(\xi_N^{j,q}))_{j \in J_N, q \in Q}$ are vectors and

$$(\mathcal{T}_N \bar{u}_N)(\xi_N^{i,p}) = \sum_{j \in J_N} \int_{G_N^j} K\Big(\xi_N^{i,p}, y, \sum_{q \in Q} u_N(\xi_N^{j,q}) \varphi_N^{j,q}(y)\Big)\, dy,$$
$$i \in J_N,\ p \in Q.$$

Usually the number of equations in (7) is large which makes solving the system directly rather costly if not impossible. An effective method for solving this system is a two-grid iteration method.

4. Two-grid method. In addition to the original grid corresponding to $N = (N_1, \dots, N_n)$, we define another grid, the coarse grid, corresponding to $M = (M_1, \dots, M_n)$ where M_k, $k = 1, \dots, n$, are integers such that N_k/M_k are integers greater than 1. Then every cell G_N^j, $j \in J_N$, of the original grid is fully contained in some cell G_M^i of the coarse grid.

For solving the system (7) the following two-grid iteration method is used:

$$\begin{aligned} \bar{v}_N^l &= \mathcal{T}_N \bar{u}_N^l + \bar{f}_N, \\ \bar{w}_M^l - \mathcal{T}_M \bar{w}_M^l &= \mathcal{R}_{NM} \bar{f}_N + \mathcal{R}_{NM} \mathcal{T}_N \bar{v}_N^l - \mathcal{T}_M \mathcal{R}_{NM} \bar{v}_N^l, \\ \bar{u}_N^{l+1} &= \bar{v}_N^l + \mathcal{P}_{MN}(\bar{w}_M^l - \mathcal{R}_{NM} \bar{v}_N^l), \quad l = 0, 1, \dots, \end{aligned} \tag{8}$$

where $\overline{u}_N^0$ is the initial guess of $\overline{u}_N$, $\mathcal{P}_{MN}\colon \mathbf{R}^{d_M} \to \mathbf{R}^{d_N}$ ($d_N = \dim E_N$) and $\mathcal{R}_{NM}\colon \mathbf{R}^{d_N} \to \mathbf{R}^{d_M}$ are the operators defined by the formulas:

$$\begin{aligned}
(\mathcal{P}_{MN}\bar{w}_M^l)(\xi_N^{i,p}) &= \sum_{q\in Q} w_M^l(\xi_M^{j,q})\varphi_M^{j,q}(\xi_N^{i,p}) \text{ if } \xi_N^{i,p} \in G_M^j,\\
(\mathcal{R}_{NM}\bar{f}_N)(\xi_M^{i,p}) &= f(\xi_M^{i,p}),\\
(\mathcal{R}_{NM}\mathcal{T}_N\bar{u}_N^l)(\xi_M^{i,p}) &= \sum_{j\in J_N}\int_{G_N^j} K(\xi_M^{i,p}, y, \sum_{q\in Q} u_N^l(\xi_N^{j,q})\varphi_N^{j,q}(y))\,dy,\\
\mathcal{R}_{NM}\bar{v}_N^l &= \mathcal{R}_{NM}\mathcal{T}_N\bar{u}_N^l + \mathcal{R}_{NM}\bar{f}_N.
\end{aligned}$$

For linear integral equations, method (8) coincides with the two-grid method in [**10**]. For nonlinear integral equations similar methods are considered in [**4, 9**]. Iteration method (8) resembles the two-grid method in [**9**]. An essentially new idea in the present two-grid method is the restriction operator $\mathcal{R}_{NM}$ as the collocation points of the coarse grid may not coincide with the collocation points of the fine grid.

To apply method (8), it is necessary for every l, to solve the nonlinear system in the form

$$\bar{w}_M - \mathcal{T}_M\bar{w}_M = \bar{g}_M \tag{9}$$

where

$$\bar{g}_M = \mathcal{R}_{NM}\bar{f}_N + \mathcal{R}_{NM}\mathcal{T}_N(\mathcal{T}_N\bar{u}_N + \bar{f}_N) - \mathcal{T}_M\mathcal{R}_{NM}(\mathcal{T}_N\bar{u}_N + \bar{f}_N).$$

Note that, compared to the system (7) which corresponds to a fine discretization, system (9) corresponds to coarse discretization, and thus the dimension $d_M = \dim E_M$ of this system is essentially less than the dimension d_N of (7). To solve the system (9), one can use some iterative methods, for example, Newton's method with initial guess $\mathcal{R}_{NM}\bar{v}_N^l$ for $\bar{w}_M^l$. If $\bar{u}_N^l$ is a sufficiently good approximation of the solution $\bar{u}_{N,0}$ of (7), we can use only one step of Newton's method [**9**]. Note that efficient two-grid methods for solving nonlinear integral equations are obtained using first Newton's method to solve system (7) and second exploiting at each step a coarse grid also [**1, 3, 6**].

For the convergence analysis of the two-grid method (8) we use the approach of [**2**] and consider the iteration method corresponding to (8) in function spaces.

We write the integral equation (1) in the form

$$u = \mathcal{T}u + f \tag{10}$$

where

$$(\mathcal{T}u)(x) = \int_G K(x, y, u(y))\, dy.$$

It is easy to see that the approximate solution $u_N \in E_N$ of (1), determined from the collocation conditions (5), is the solution of the equation

$$u_N = \mathcal{P}_N \mathcal{T} u_N + \mathcal{P}_N f. \tag{11}$$

Define the operators $\mathcal{R}_{\infty N} \colon E_N \to \mathbf{R}^{d_N}$ and $\mathcal{P}_{N\infty} \colon \mathbf{R}^{d_N} \to E_N$ by the equalities:

$$\begin{aligned} (\mathcal{R}_{\infty N} u)(\xi_N^{j,q}) &= u(\xi_N^{j,q}), \quad j \in J_N,\ q \in Q, \\ (\mathcal{P}_{N\infty} \bar{u}_N)(x) &= \sum_{q \in Q} u_N(\xi_N^{j,q}) \varphi_N^{j,q}(x), \quad x \in G_N^j,\ j \in J_N. \end{aligned}$$

These operators define one-to-one correspondence between elements of E_N and $\mathbf{R}^{d_N}$. From now on we use the operator $\mathcal{R}_{\infty N}$ for all functions defined in the collocation points $\xi_N^{j,q}$.

Denote

$$u_N^l = \mathcal{P}_{N\infty} \bar{u}_N^l, \qquad v_N^l = \mathcal{P}_{N\infty} \bar{v}_N^l, \qquad w_M^l = \mathcal{P}_{M\infty} \bar{w}_M^l.$$

Then

$$\bar{u}_N^l = \mathcal{R}_{\infty N} u_N^l, \qquad \bar{v}_N^l = \mathcal{R}_{\infty N} v_N^l, \qquad \bar{w}_M^l = \mathcal{R}_{\infty M} w_M^l.$$

Making use of the identities

$$\begin{gathered} \mathcal{R}_{\infty N} \mathcal{P}_{N\infty} = I, \qquad \mathcal{P}_{N\infty} \mathcal{R}_{\infty N} = \mathcal{P}_N, \\ \mathcal{T}_N = \mathcal{R}_{\infty N} \mathcal{T} \mathcal{P}_{N\infty}, \qquad \mathcal{P}_{N\infty} \mathcal{P}_{MN} = \mathcal{P}_{M\infty}, \end{gathered}$$

we rewrite the formulas (8) as follows:

$$\begin{aligned} v_N^l &= \mathcal{P}_N \mathcal{T} u_N^l + \mathcal{P}_N f, \\ w_M^l - \mathcal{P}_M \mathcal{T} w_M^l &= \mathcal{P}_M f + \mathcal{P}_M \mathcal{T} v_N^l - \mathcal{P}_M \mathcal{T} \mathcal{P}_M (\mathcal{T} u_N^l + f), \\ u_N^{l+1} &= v_N^l + w_M^l - \mathcal{P}_M (\mathcal{T} u_N^l + f), \quad l = 0, 1, \ldots \end{aligned} \tag{12}$$

Whereas $u_N^0 = \mathcal{P}_{N\infty}\bar{u}_N^0 \in E_N$ we also have $v_N^l \in E_N$, $w_M^l \in E_M \subset E_N$ and $u_N^{l+1} \in E_N$, $l = 0, 1\dots$ Therefore the methods (8) and (12) are equivalent. At the same time the method (12) is an iteration method to solve (11).

The equation (9) is equivalent to the equation

$$w_M - \mathcal{P}_M \mathcal{T} w_M = g_M \tag{13}$$

where $w_M = \mathcal{P}_{M\infty}\bar{w}_M$ and

$$g_M = \mathcal{P}_M f + \mathcal{P}_M \mathcal{T} \mathcal{P}_N(\mathcal{T} u_N + f) - \mathcal{P}_M \mathcal{T} \mathcal{P}_M(\mathcal{T} u_N + f).$$

In the discussion of solvability of equations (11) and (13) we use the following result which is proved in a more general setting, for example, in [**11**]. We consider the equations (10) and (13) in Banach space E and assume that $\mathcal{T}: E \to E$ is a nonlinear operator and $\mathcal{P}_M \in \mathcal{L}(E, E)$ are linear uniformly bounded operators, i.e., $||\mathcal{P}_M|| \le$ const.

Lemma 1. *Let the following conditions hold.*

(B1) *Equation* (10) *has a solution* $u_0 \in E$ *and*

$$||\mathcal{P}_M \mathcal{T} u_0 - \mathcal{T} u_0|| \to 0 \quad \text{as } \min_{1\le k\le n} M_k \to \infty.$$

(B2) *There is a positive* δ *such that the operator* $\mathcal{T}$ *is Fréchet differentiable in the ball* $S_\delta = \{u: ||u - u_0|| \le \delta\}$ *and for any* $\varepsilon > 0$ *there is a* δ_ε, $0 < \delta_\varepsilon < \delta$, *such that*

$$||\mathcal{T}'(u) - \mathcal{T}'(u_0)|| \le \varepsilon \quad \text{whenever } u \in S_{\delta_\varepsilon}.$$

(B3) *The derivative* $\mathcal{T}'(u_0)$ *is a compact operator,*

$$||\mathcal{P}_M \mathcal{T}'(u_0) - \mathcal{T}'(u_0)|| \longrightarrow 0 \quad \text{as } \min_{1\le k\le n} M_k \to \infty$$

and the homogeneous equation $v = \mathcal{T}'(u_0)v$ *has only the trivial solution in* E.

(B4) $||g_M - f|| \to 0$ *as* $\min_{1\le k\le n} M_k \to 0$.

Then there exist $M_0 > 0$ *and* δ_0, $0 < \delta_0 \le \delta$, *such that for* $M_k \ge M_0$, $k = 1, \dots, n$, *the equation* (13) *has a unique solution* $w_{M,0} = (I - \mathcal{P}_M\mathcal{T})^{-1} g_M$ *in the ball* S_{δ_0}. *Besides* $||u_{M,0} - u_0|| \to 0$ *if* $\min_{1\le k\le n} M_k \to \infty$ *with the error estimate*

$$||w_{M,0} - u_0|| \le \mathrm{const}\, ||(\mathcal{P}_M\mathcal{T}u_0 - \mathcal{T}u_0) + (g_M - f)||.$$

We can consider the method (12) as an iterative method

$$u_N^{l+1} = \Phi u_N^l, \quad l = 0, 1, \dots , \tag{14}$$

for solving the equation

$$u = \Phi u. \tag{15}$$

To study the convergence of this iterative method, the following well-known result is used.

Lemma 2. *Let equation* (15) *have a solution* $u_{N,0} \in E$, *and let* $S_{N,\delta} = \{u\colon ||u - u_{N,0}|| \le \delta\}$. *If* $\Phi\colon S_{N,\delta} \to E$ *and*

$$||\Phi'(u)|| \le q < 1 \quad \textit{as } u \in S_{N,\delta}$$

then $u_{N,0}$ *is the unique solution of equation* (15) *in* $S_{N,\delta}$. *For every initial guess* $u_N^0 \in S_{N,\delta}$ *the iterative method* (14) *converges to* $u_{N,0}$ *with the rate*

$$||u_N^{l+1} - u_{N,0}|| \le q||u_N^l - u_{N,0}||, \quad l = 0, 1, \dots .$$

In the following we take $E = L^\infty(G)$ with norm

$$||u|| = \sup_{x\in G} |u(x)|.$$

From assumption (A1) it follows that $\mathcal{T}\colon L^\infty(G) \to L^\infty(G)$ and that the Fréchet derivative

$$(\mathcal{T}'(u)\Delta u)(x) = \int_G \frac{\partial K(x,y,u(y))}{\partial u} \Delta u(y)\, dy$$

exists for any $u \in L^\infty(G)$. We use the following estimates for studying of the convergence of the two-grid iteration method.

Lemma 3. *Let assumptions* (A1) *and* (A2) *hold and* $u \in \{u\colon ||u|| \leq \text{const}\}$. *Then*

$$||\mathcal{P}_N f - f|| \leq \text{const}\, \varepsilon_{\nu,h_N}, \tag{16}$$

$$||\mathcal{P}_N \mathcal{T}u - \mathcal{T}u|| \leq \text{const}\, \varepsilon_{\nu,h_N}, \tag{17}$$

and

$$||\mathcal{P}_N \mathcal{T}'(u) - \mathcal{T}'(u)|| \leq \text{const}\, \varepsilon_{\nu,h_N}, \tag{18}$$

where

$$h_N = \max_{1\leq k\leq n} \frac{b_k}{N_k}$$

and

$$\varepsilon_{\nu,h_N} = \begin{cases} h_N & \nu < n-1 \\ h_N(1+|\log h_N|) & \nu = n-1\,. \\ h_N^{n-\nu} & \nu > n-1 \end{cases}$$

Proof. Let $x \in G_N^j$. Then

$$\begin{aligned}(\mathcal{P}_N f)(x) - f(x) &= \sum_{q\in Q} f(\xi_N^{j,q})\varphi_N^{j,q}(x) - f(x) \\ &= \sum_{q\in Q} \varphi_N^{j,q}(x)[f(\xi_N^{j,q}) - f(x)].\end{aligned}$$

Note that, see [**10**] and Lemma 2.4 in [**12**],

$$\sup_{x\in G_N^j} |f(\xi_N^{j,q}) - f(x)| \leq \text{const}\, \varepsilon_{\nu,h_N}$$

and

$$\max_{x\in G_N^j} |\varphi_N^{j,q}(x)| \le c^n \tag{19}$$

where

$$c = \max_{1\le q_k\le m} \max_{-1\le\eta\le 1} \Bigg| \prod_{\substack{s=1\\ s\ne q_k}}^{m} \frac{\eta-\eta_s}{\eta_{q_k}-\eta_s} \Bigg|.$$

Thus

$$\sup_{x\in G_N^j} |(\mathcal{P}_N f)(x) - f(x)| \le \text{const}\, \varepsilon_{\nu,h_N}, \quad j\in J_N,$$

from which follows the estimate (16). The proof of (17) is analogous using a generalization of Lemma 2.3 in [**12**] for nonlinear operators. The estimate (18) follows from the lemma in [**10**]. □

Remark. From (19) it follows that $||\mathcal{P}_N|| \le c^n = \text{const}$.

We are now ready to prove the following result about the convergence of the two-grid method (8).

Theorem 2. *Let the assumptions* (A1)–(A3) *hold. Then there exist* $M_0 > 0$ *and* $\delta_0 > 0$ *such that, for* $N_k \ge M_0$, $k = 1,\dots,n$, *the system* (7) *has a unique solution* $\bar u_{N,0}$ *satisfying* $||\mathcal{P}_{N\infty}\bar u_{N,0} - u_0|| \le \delta_0$. *The two-grid iteration method* (8) *converges for* $M_k \ge M_0$, $k = 1,\dots,n$, *and for sufficiently good initial guess* $\overline{u}_N^0$ *to this solution with the rate*

$$||\bar u_N^{l+1} - \bar u_{N,0}|| \le c_1\varepsilon_{\nu,h_M}||\bar u_N^l - \bar u_{N,0}||, \quad l = 0,1,\dots, \tag{20}$$

where

$$||\bar u_N|| = \max_{j\in J_N}\max_{q\in Q}|u_N(\xi_N^{j,q})|.$$

Proof. We get by Lemma 3 that the conditions (B1)–(B3) of Lemma 1 are fulfilled with $E = L^\infty(G)$. On the grounds of Lemma 1 there exist $M_0' > 0$ and $\delta_0 > 0$ such that for $N_k \ge M_0'$, $k = 1,\dots,n$, equation (11) has a unique solution $u_{N,0}$ in the ball S_{δ_0}. From this the first assertion of the theorem follows because $u_{N,0} \in E_N$ and $\bar u_{N,0} = \mathcal{R}_{\infty N}u_{N,0}$ is the solution of (7).

Method (12) is an iterative method of the form (14) where

$$\Phi u = (\mathcal{P}_N - \mathcal{P}_M)(\mathcal{T}u + f) + (I - \mathcal{P}_M\mathcal{T})^{-1}[\mathcal{P}_M f + \mathcal{P}_M\mathcal{T}\mathcal{P}_N(\mathcal{T}u + f) - \mathcal{P}_M\mathcal{T}\mathcal{P}_M(\mathcal{T}u + f)].$$

It is easy to check that the solution $u_{N,0}$ of equation (11) is the solution of equation (15) too. The derivative of Φ is expressed

$$\Phi'(u)\Delta u = (\mathcal{P}_N - \mathcal{P}_M)\mathcal{T}'(u)\Delta u + [I - \mathcal{P}_M\mathcal{T}'(w)]^{-1}[\mathcal{P}_M\mathcal{T}'(\mathcal{P}_N(\mathcal{T}u + f))\mathcal{P}_N - \mathcal{P}_M\mathcal{T}'(\mathcal{P}_M(\mathcal{T}u + f))\mathcal{P}_M]\mathcal{T}'(u)\Delta u$$

where w is the solution of the equation

$$w - \mathcal{P}_M\mathcal{T}w = g \tag{21}$$

with

$$g = \mathcal{P}_M f + \mathcal{P}_M\mathcal{T}\mathcal{P}_N(\mathcal{T}u + f) - \mathcal{P}_M\mathcal{T}\mathcal{P}_M(\mathcal{T}u + f).$$

Denote $v_N = \mathcal{P}_N(\mathcal{T}u + f)$. Then

$$g = \mathcal{P}_M f + \mathcal{P}_M\mathcal{T}v_N - \mathcal{P}_M\mathcal{T}v_M.$$

Let $||u|| \le \text{const}$. Then by Lemma 3 we get

$$||g - f|| \le ||\mathcal{P}_M f - f|| + ||\mathcal{P}_M\mathcal{T}v_N - \mathcal{P}_M\mathcal{T}v_M|| \le \text{const}\,\varepsilon_{\nu,h_M} + \text{const}\,||v_N - v_M|| \le \text{const}\,\varepsilon_{\nu,h_M}.$$

On the grounds of Lemma 1 there exists $M_0'' \ge M_0'$ such that for $M_k \ge M_0''$, $k = 1, \dots, n$, equation (21) has a unique solution w in the ball S_{δ_0} and

$$||w - u_0|| \le \text{const}\,\varepsilon_{\nu,h_M} \to 0 \text{ as } \min_{1\le k\le n} M_k \to \infty. \tag{22}$$

Analogously we get that the formulas (12) and (8) define sequences $u_N^l \in E_N$ and $\bar{u}_N^l = \mathcal{R}_{\infty N}u_N^l$, $l = 1, 2, \dots$, for $M_k \ge M_0''$, $k = 1, \dots, n$, and for sufficiently good initial guess $u_N^0 \in E_N$ to $u_{N,0}$.

Due to (A1) and (A3) there exists the inverse operator $[I-\mathcal{T}'(u_0)]^{-1}$. By Lemma 3 and (22) we get

$$\begin{aligned}||\mathcal{P}_M\mathcal{T}'(w)-\mathcal{T}'(u_0)|| &\le ||\mathcal{P}_M\mathcal{T}'(w)-\mathcal{P}_M\mathcal{T}'(u_0)|| \\ &\quad + ||\mathcal{P}_M\mathcal{T}'(u_0)-\mathcal{T}'(u_0)|| \le \text{const}\,\varepsilon_{\nu,h_M} \longrightarrow 0 \\ &\quad \text{as } \min_{1\le k\le n} M_k \to \infty.\end{aligned}$$

Therefore we can find $M_0''' \ge M_0''$ so that for $M_k \ge M_0'''$, $k=1,\dots,n$, there exist the inverse operators $[I-\mathcal{P}_M\mathcal{T}'(w)]^{-1}$ which are uniformly bounded:

$$||[I-\mathcal{P}_M\mathcal{T}'(w)]^{-1}|| \le \text{const}.$$

For $M_k \ge M_0'''$, $k=1,\dots,n$, we can estimate

$$\begin{aligned}||\Phi'(u)|| &\le ||(\mathcal{P}_N-\mathcal{P}_M)\mathcal{T}'(u)|| \\ &\quad + \text{const}\,||\mathcal{P}_M\mathcal{T}'(v_N)\mathcal{P}_N\mathcal{T}'(u)-\mathcal{P}_M\mathcal{T}'(v_M)\mathcal{P}_M\mathcal{T}'(u)|| \\ &\le \text{const}\,[\varepsilon_{\nu,h_M} + ||\mathcal{P}_M[\mathcal{T}'(v_N)-\mathcal{T}'(v_M)]\mathcal{P}_N\mathcal{T}'(u)|| \\ &\qquad + ||\mathcal{P}_M\mathcal{T}'(v_M)(\mathcal{P}_N-\mathcal{P}_M)\mathcal{T}'(u)||] \\ &\le c_0\varepsilon_{\nu,h_M} \quad \text{if } u\in S_{N,\delta}\end{aligned}$$

where $\delta>0$ and $S_{N,\delta}\subset S_{\delta_0}$. We choose $M_0 \ge M_0'''$ so that $c_0\varepsilon_{\nu,h_M} < 1$ as $M_k \ge M_0$, $k=1,\dots,n$. Then from Lemma 2 it follows that the iteration method (12) converges, for $x_N^0 \in S_{N,\delta}$ and $M_k \ge M_0$, $k=1,\dots,n$, and

$$(23) \qquad ||u_N^{l+1}-u_{N,0}|| \le c_0\varepsilon_{\nu,h_M}||u_N^l-u_{N,0}||, \quad l=1,2,\dots .$$

The estimate (20) with $c_1=c^nc_0$ follows from (23) because

$$||\bar u_N^{l+1}-\bar u_{N,0}|| = ||\mathcal{R}_{\infty N}(u_N^{l+1}-u_{N,0})|| \le ||u_N^{l+1}-u_{N,0}||$$

and

$$||u_N^l-u_{N,0}|| = ||\mathcal{P}_{N\infty}(\bar u_N^l-\bar u_{N,0})|| \le c^n||\bar u_N^l-\bar u_{N,0}||.$$

The last inequality is a consequence of (19). Theorem 2 is proved. □

From the estimate (20) we see that the two-grid iteration method (8) converges quite quickly provided that M_k, $k = 1, \dots, n$, are chosen sufficiently large. Thus we get for the nonlinear equations the same rate of convergence as is proved for linear equations in [**10**]. In [**9**] a two-grid iteration method for the solution of nonlinear weakly singular integral equations on arbitrary bounded domain G by piecewise constant collocation method is considered. In the special case when the domain G and the cells G_N^j are parallelepipeds the main result of [**9**] follows from Theorem 2 proved above.

Example. Consider the integral equation

$$u(x) = \int_0^1 |x-y|^{-1/2} u^2(y)dy + f(x), \quad 0 < x < 1,$$

where $f(x)$ is selected so that $u_0(x) = \sqrt{x(1-x)}$ is the solution to be approximated, see [**8**]. It is easy to see that the assumptions (A1)–(A3) hold for this equation with $\nu = 1/2$. Tables 1–4 present the norm ε_N of the errors of the approximate solutions $\overline{u}_N$ calculated by the collocation method:

$$\varepsilon_N = ||\bar{u}_N - \bar{u}_0|| = \max_{j \in J_N, q \in Q} |u_N(\xi_N^{j,q}) - u_0(\xi_N^{j,q})|.$$

TABLE 1.

	$r=1$		$r=2$	
N	ε_N	ρ_N	ε_N	ρ_N
2	1.0 E-1		8.7 E-2	
4	3.0 E-2	3.4	2.2 E-2	4.0
8	8.7 E-3	3.4	5.3 E-3	4.1
16	2.4 E-3	3.6	1.3 E-3	4.0
32	6.8 E-4	3.6	3.3 E-4	4.0
64	2.5 E-4	2.8	8.2 E-5	4.0
128	8.8 E-5	2.8	2.0 E-5	4.0

TABLE 2.

		$r=1$		$r=2$	
M	N	l_0	ε_N	l_0	ε_N
4	8	4	8.7 E-3	4	5.5 E-3
4	16	5	2.5 E-3	5	1.4 E-3
8	16	3	2.4E-3	3	1.3E-3
4	64	6	2.5E-4	8	8.2 E-5
8	64	4	2.5E-4	5	8.0E-5
16	64	3	2.5E-4	3	8.3E-5
32	64	3	2.5E-4	2	8.3E-5
4	256	8	3.2E-5	12	4.9 E-6
8	256	5	3.1E-5	8	5.1 E-6
16	256	3	3.2E-5	5	5.1 E-6
32	256	3	3.1E-5	4	5.1 E-6
64	256	3	3.1E-5	3	5.1 E-6

TABLE 3.

	$r=1$		$r=2$		$r=4$	
N	ε_N	ρ_N	ε_N	ρ_N	ε_N	ρ_N
2	7.8E-4		1.9E-3		6.7E-3	
4	1.9E-4	4.0	2.2E-4	8.9	1.5E-3	4.5
8	7.0E-5	2.8	3.4 E-5	6.5	2.1E-4	7.0
16	2.3E-5	3.0	5.2 E-6	6.5	2.7E-5	8.0
32	7.7E-6	3.0	1.0 E-6	5.2	6.7E-6	4.0
64	2.6E-6	2.9	1.7 E-7	5.9	1.4E-6	4.8

The results of Tables 1 and 3 have been obtained by the solution of the nonlinear system (6) with Newton's method and of Tables 2 and 4 by the solution of this system with the two-grid method (8) using l_0 iterative steps. The number of steps l_0 is chosen so that the errors in Tables 2 and 4 are within 3% at the error of the corresponding

TABLE 4.

		$r=1$		$r=2$		$r=4$	
M	N	l_0	ε_N	l_0	ε_N	l_0	ε_N
4	8	4	6.9E-5	11	3.4E-5	13	2.2 E-4
4	16	4	2.4 E-5	11	5.1 E-6	12	2.6 E-5
8	16	3	2.3 E-5	7	5.1 E-6	11	2.7 E-5
4	32	6	7.9 E-6	13	1.0 E-6	13	6.8 E-6
8	32	3	7.7 E-6	8	1.0 E-6	10	6.3 E-6
16	32	3	7.7 E-6	4	1.0 E-6	6	6.5 E-6
4	64	7	2.7 E-6	16	1.7 E-7	18	1.3 E-6
8	64	4	2.7 E-6	10	1.7 E-7	12	1.3 E-6
16	64	2	2.5 E-6	7	1.7 E-7	8	1.4 E-6
32	64	2	2.6 E-6	4	1.7 E-7	3	1.4 E-6

collocation method. All the integrals needed for the construction of the system (6) were found analytically.

We have chosen $m=2$ and $-\eta_1=\eta_2=1$ in the case of Tables 1 and 2 and $-\eta_1=\eta_2=1/\sqrt{3}$ in the case of Tables 3 and 4 (the last values of η_1 and η_2 determine the collocation points of the superconvergence). The number of the equations and unknowns in system (6) is in the first case $2N+1$ and in the second case $4N$. The ratio $\rho_N=\varepsilon_{N/2}/\varepsilon_N$ characterizes the rate of the convergence of the collocation method. It is proved [**8, 12**] that the error of the collocation method $\varepsilon_N=O(N^{-2})$ in the first case for $r>2$ and $\varepsilon_N=O(N^{-5/2})$ in the second case for $r>4$. Thus the value of the ratio ρ_N ought to be respectively $2^2=4$ and $2^{5/2}\approx 5.7$. From Tables 1 and 3 we see that such rate of convergence is achieved already for $r=2$. For $r=4$ the errors of the collocation method are bigger and the rate of the convergence of the two-grid iteration method is slower than for $r=2$.

The initial guess of the two-grid method has been taken $\bar{u}_N^0=\mathcal{P}_{MN}\bar{u}_M$ in the case of Table 2 and $\bar{u}_N^0=\mathcal{R}_{MN}\mathcal{T}_M\bar{u}_M+\bar{f}_N$ in the case of Table 4. In the case of Table 4 one can also use the initial guess $\bar{u}_N^0=\mathcal{P}_{MN}\bar{u}_M$ but then in most cases we need one extra approximation step of the two-grid method to get the same precision. Such initial guesses are so good that it is sufficient to make only one step of Newton's method with initial guess $\mathcal{R}_{NM}v_N^l$ for w_M^l for the solution of the system (9).

From numerical examples it follows that the estimate (20) of Theorem 2 expresses quite well the convergence rate of the two-grid method. This method converges approximately with the rate of the geometric progression, which factor essentially behaves like const $M^{-1/2}$. It appears that one can use rather coarse grids even for a quite fine initial grid. A good strategy will be $M \approx \sqrt{N}$.

REFERENCES

1. K.E. Atkinson, *A survey of numerical methods for solving nonlinear integral equations*, J. Integral Equations Appl. **4** (1992), 15–46.

2. ———, *Two-grid iteration methods for linear integral equations of the second kind on piecewise smooth surfaces in* $\mathbf{R}^3$, SIAM J. Sci. Comput. **15** (1994), 1083–1104.

3. ———, *The numerical solution of a non-linear boundary integral equation on smooth surfaces*, IMA J. Numer. Anal. **14** (1994), 461–483.

4. W. Hackbusch, *Multi-grid methods and applications*, Springer-Verlag, Berlin, 1985.

5. H. Kaneko, R. Noren and Y. Xu, *Numerical solutions for weakly singular Hammerstein equations and their superconvergence*, J. Integral Equations Appl. **4** (1992), 391–407.

6. C.T. Kelley, *A fast multilevel algorithm for integral equations*, SIAM J. Numer. Anal. **32** (1995), 501–513.

7. A. Pedas and G. Vainikko, *Tangential derivatives of solutions to nonlinear multidimensional weakly singular integral equations*, in *Beiträge zur angewandten Analysis und Informatik*, Shaker Verlag, Aachen, 1994, 271–287.

8. ———, *Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations*, J. Integral Equations Appl. **9** (1997), 379–406.

9. E. Tamme, *Two-grid method for nonlinear multi-dimensional weakly singular integral equations*, J. Integral Equations Appl. **7** (1995), 99–113.

10. ———, *Two-grid method for the solution of weakly singular integral equations by piecewise polynomial approximation*, Proc. Estonian Acad. Sci. Phys. Math. **46** (1997), 241–250.

11. G. Vainikko, *Approximative methods for nonlinear equations*, Nonlin. Analysis **2** (1978), 647–687.

12. ———, *Multidimensional weakly singular integral equations*, Lecture Notes in Math. **1549** (1993).

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