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ON TOPOLOGICAL SPACES THAT HAVE A BOUNDED COMPLETE DCPO MODEL

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ABSTRACT. A dcpo model of a topological space X is a dcpo (directed complete poset) P such that X is homeomorphic to the maximal point space of P with the subspace topology of the Scott space of P. It has been previously proved by Xi and Zhao that every T_1 space has a dcpo model. It is, however, still unknown whether every T_1 space has a bounded complete dcpo model (a poset is bounded complete if each of its upper bounded subsets has a supremum). In this paper, we first show that the set of natural numbers equipped with the co-finite topology does not have a bounded complete dcpo model and then prove that a large class of topological spaces (including all Hausdorff k-spaces) have a bounded complete dcpo model. We shall mainly focus on the model formed by all of the nonempty closed compact subsets of the given space.

Introduction. In domain theory, one of the most useful intrinsic order topologies on a poset is the Scott topology. Although the definition of this topology was originally motivated mainly by problems in computer science, it soon found deep links with other mathematical structures. One of the classic results on the Scott topology was discovered by Dana Scott: the injective objects of the category of all T_0 spaces are exactly the continuous lattices equipped with their Scott topologies **[18]**. A modern link between the Scott topology and general topological spaces was established via the maximal point spaces of Scott spaces. A poset model of a topological space X is a poset P such that $\mathrm{Max}(P)$ is homeomorphic to X **[11]**. Every space that has a poset model must be T_1. Edalat and Heckmann **[2]** proved that every

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complete metric space has a poset model that is a domain (continuous dcpo). Lawson [**11**] proved that a space has a domain model with a countable base and satisfies the Lawson condition if and only if it is a Polish space. Kopperman, Künzi and Waszkiewicz [**10**] proved that every complete metric space has a bounded complete domain model.

Spaces with other special types of poset models have also been considered by many other authors [**1, 12, 15, 16, 17, 19, 20**].

A natural question which then arises is: which spaces have a poset model? Erné [**4**] and Zhao [**22**] proved that every T_1 space has a bounded complete algebraic poset model. Thus the T_1 spaces are exactly those spaces which have a poset model.

Recently, Zhao and Xi [**21, 23**] further proved that every T_1 space has a dcpo model.

A subset A of a poset P is upper bounded if there is an element $b \in P$ such that $x \leq b$ for all $x \in A$. A poset is bounded complete if all of its upper bounded nonempty subsets have a supremum. The dcpo model constructed in [**21, 23**] for a T_1 space is not bounded complete in general.

Hence, we have the following problem: does every T_1 topological space have a bounded complete dcpo model?

In this paper, we first prove that, if P is a bounded complete dcpo, then, for any $x \in P$,

$$\downarrow((\uparrow x) \cap \mathrm{Max}(P))$$

is a Scott closed set and then deduce that the T_1 space of the set all positive integers equipped with the co-finite topology does not have a bounded complete dcpo model. Next, we prove that a large class of topological spaces, including all Hausdorff k-spaces, have a bounded complete dcpo model.

Given a T_1 space X, the set $\mathrm{CK}(X)$ of all nonempty closed compact subsets of X is a bounded complete dcpo with respect to the reverse inclusion order, and the set $\mathrm{Max}(\mathrm{CK}(X))$ of maximal points of $\mathrm{CK}(X)$ consists of all singletons. Furthermore, there is a natural mapping

$$\eta_X : X \longrightarrow \mathrm{Max}(\mathrm{CK}(X)),$$

where $\eta_X(x) = \{x\}$ for each $x \in X$. In this paper, we shall investigate the topological spaces X where η_X is a homeomorphism; such spaces X have $\mathrm{CK}(X)$ as a bounded complete dcpo model.

1. Not every T_1 space has a bounded complete dcpo model. For any subset A of a poset P, let

$$\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$$

and

$$\uparrow A = \{x \in P : x \geq y \text{ for some } y \in A\}.$$

A nonempty subset D of a poset P is a directed set if every two elements in D have an upper bound in D. A poset P is called a *directed complete poset*, or dcpo for short, if, for any directed subset of $D \subseteq P$,

$$\sup D = \bigvee D$$

exists in P.

A subset U of a poset P is Scott open if:

(i) $U = \uparrow U$ (called an upper set) and

(ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$, whenever $\bigvee D$ exists.

All Scott open sets of poset P form a topology on P, denoted by $\sigma(P)$ and called the Scott topology on P. The space $(P, \sigma(P))$ is denoted by ΣP, and called the Scott space of P. It follows that a subset F of P is Scott closed if:

(i) $F = \downarrow F$ (called a lower set), and

(ii) for any directed subset D of P, $D \subseteq F$ implies $\bigvee D \in F$ if $\bigvee D$ exists.

For more about the Scott topology and related structures, see **[7, 8]**.

In the following, we shall always assume that the topology on the set $\mathrm{Max}(P)$ of maximal points of a poset P is the inherited subspace topology from ΣP, and we call the space $\mathrm{Max}(P)$ the *maximal point space* of P.

A poset model of a topological space X is a poset P such that $\mathrm{Max}(P)$ is homeomorphic to X **[11]**. Every space having a poset model is T_1.

The poset of all nonempty closed intervals of real numbers with the reverse inclusion order is a dcpo model of the real line with the Euclidean topology (see [**7,** Example V-6.8] for a more general result).

In [**23**], it was proven that every T_1 space X has a dcpo model. However, the dcpo model of X constructed in [**23**] is generally not bounded complete. It is still unknown whether every T_1 space has a bounded complete dcpo model. We give a negative answer to this question. Firstly, we prove a general result on bounded complete dcpos.

Lemma 1.1. *If P is a bounded complete* dcpo*, then for any $x \in P$, the set $\downarrow((\uparrow x) \cap \mathrm{Max}(P))$ is Scott closed.*

Proof. Let
$$D \subseteq \downarrow((\uparrow x) \cap \mathrm{Max}(P))$$
be a directed set. For each $d \in D$, the subset $\{d, x\}$ has an upper bound in $(\uparrow x) \cap \mathrm{Max}(P)$; thus, $d \vee x$ exists. Then,
$$\{d \vee x : d \in D\}$$
is a directed set, and clearly,
$$\bigvee D \leq \bigvee \{d \vee x : d \in D\}.$$
The element $\bigvee\{d \vee x : d \in D\}$ is below some maximal element v, which is clearly in $(\uparrow x) \cap \mathrm{Max}(P)$. Thus,
$$\bigvee \{d \vee x : d \in D\} \in \downarrow((\uparrow x) \cap \mathrm{Max}(P)),$$
implying
$$\bigvee D \in \downarrow((\uparrow x) \cap \mathrm{Max}(P)).$$
Since $\downarrow(\uparrow x) \cap \mathrm{Max}(P)$ is clearly a lower set, it is Scott closed. $\square$

For any element x in a dcpo P,
$$\uparrow x \cap \mathrm{Max}(P) = (\downarrow((\uparrow x) \cap \mathrm{Max}(P))) \cap \mathrm{Max}(P).$$
Thus, by Lemma 1.1, we deduce the following.

Corollary 1.2. *For any element x in a bounded complete* dcpo *P, $\uparrow x \cap \mathrm{Max}(P)$ is a closed subset of* $\mathrm{Max}(P)$*.*

Remark 1.3. By Zorn's lemma, every element in a dcpo P is below some maximal point. It thus follows that, in any dcpo P,

$$\downarrow(\uparrow x) = \downarrow((\uparrow x) \cap \mathrm{Max}(P))$$

holds for every $x \in P$. Therefore, for any x in a bounded complete dcpo P, $\downarrow(\uparrow x)$ is a Scott closed subset of P.

Example 1.4. The set $\mathbb{N}$ of all positive integers equipped with the co-finite topology τ_{cof} does not have a bounded complete dcpo model. Here, $U \in \tau_{\mathrm{cof}}$ if and only if either $U = \emptyset$ or $\mathbb{N} - U$ is a finite set. In fact, suppose, on the contrary, that P is a bounded complete dcpo model of $(\mathbb{N}, \tau_{\mathrm{cof}})$. To simplify the argument, we assume $\mathbb{N} = \mathrm{Max}(P)$. As $\mathbb{N} - \{1\}$ is not closed in $(\mathbb{N}, \tau_{\mathrm{cof}})$, the set $\downarrow(\mathbb{N} - \{1\})$ is not a Scott closed set of P (otherwise $\mathbb{N} - \{1\} = \downarrow(\mathbb{N} - \{1\}) \cap \mathrm{Max}(P)$ would be closed). Hence, there is a directed set

$$D \subseteq \downarrow(\mathbb{N} - \{1\})$$

such that $\bigvee D \notin \downarrow(\mathbb{N} - \{1\})$. It follows that $\bigvee D \leq 1$ and $\bigvee D \not\leq 2$. Hence, there is a $d \in D$ such that $d \not\leq 2$. Then, by Corollary 1.2, $\uparrow d \cap \mathrm{Max}(P)$ is a closed (and proper) subset of $\mathrm{Max}(P) = \mathbb{N}$. By the definition of τ_{cof}, $\uparrow d \cap \mathrm{Max}(P)$ must be a finite set, say

$$\uparrow d \cap \mathrm{Max}(P) = \{u_1, u_2, \ldots, u_n\}.$$

Clearly, $1 \in \{u_1, u_2, \ldots, u_n\}$ since $\bigvee D \leq 1$. Assume that $u_1 = 1$ and

$$\{u_2, u_3, \ldots, u_n\} \subset \mathbb{N} - \{1\}.$$

Then,

$$D \subseteq \downarrow\{u_2, u_3, \ldots, u_n\} \neq \emptyset,$$

which implies that $D \subseteq \downarrow u_m$ holds for some fixed $m \geq 2$. However, this implies

$$\bigvee D \in \downarrow u_m \subseteq \downarrow(\mathbb{N} - \{1\}),$$

contradicting the assumption on D. This contradiction shows that $(\mathbb{N}, \tau_{\mathrm{cof}})$ does not have a bounded complete dcpo model.

Remark 1.5. It is well known that the co-finite topology τ_{cof} on a set X is the coarsest T_1 topology on X. If X is finite, then (X, τ_{cof}) is a discrete space; hence, it has a bounded complete dcpo model. If X is an infinite set, then using a similar proof as for $(\mathbb{N}, \tau_{\mathrm{cof}})$, it may also be shown that (X, τ_{cof}) does not have a bounded complete dcpo model.

At this time, it remains unknown whether the set $\mathbb{R}$ of all real numbers equipped with the co-countable topology has a bounded complete dcpo model.

2. Spaces whose closed compact sets form a model. In this section, we prove some positive results on spaces which have a bounded complete dcpo model.

For any T_1 space (X, τ), let $\mathrm{CK}(X)$ be the set of all nonempty closed compact subsets of X.

The poset $(\mathrm{CK}(X), \supseteq)$ is directed complete: for any directed subset $\mathcal{D} \subseteq \mathrm{CK}(X)$,

$$\bigcap \mathcal{D} = \bigvee_{\mathrm{CK}(X)} \mathcal{D}.$$

The set of maximal points of $\mathrm{CK}(X)$ are the singleton sets:

$$\mathrm{Max}(\mathrm{CK}(X)) = \{\{x\} : x \in X\}.$$

It is then natural to ask when the dcpo $\mathrm{CK}(X)$ is a model of X, or more specifically, when the following mapping is a homeomorphism:

$$\eta_X : X \longrightarrow \mathrm{Max}(\mathrm{CK}(X)), \quad \eta_X(x) = \{x\},\ x \in X.$$

Definition 2.1. A subset U of a topological space (X, τ) is called CK-open if, for any filter base $\mathcal{F} \subseteq \mathrm{CK}(X)$ with $|\bigcap \mathcal{F}| = 1$, that is, $\bigcap \mathcal{F}$ is a singleton, and $\bigcap \mathcal{F} \subseteq U$, then $F \subseteq U$ for some $F \in \mathcal{F}$.

Let τ_{CK} be the set of all CK-open sets of X. Obviously, $\emptyset$ and X are CK-open. It is easy to verify that τ_{CK} is indeed a topology on X and $\tau \subseteq \tau_{\mathrm{CK}}$. For the reader's convenience, we give here a brief explanation.

If U and V are CK-open and $\mathcal{F} \subseteq \mathrm{CK}(X)$ is a filter base such that $\bigcap \mathcal{F} = \{x\}$ and $x \in U \cap V$, then $x \in U$ and $x \in V$; thus, there are $F_1, F_2 \in \mathcal{F}$ satisfying $F_1 \subseteq U, F_2 \subseteq V$. Choose $F \in \mathcal{F}$ such that $F \subseteq F_1 \cap F_2$. Then $F \subseteq U \cap V$. Hence, $U \cap V$ is CK-open. It is more straightforward to verify that the union of any collection of CK-open sets of X is CK-open. Hence, τ_{CK} is a topology on X.

Now, let U be a nonempty open set of (X,τ), i.e., $U \in \tau$, and $\mathcal{F} \subseteq \mathrm{CK}(X)$ a filter base such that $\bigcap \mathcal{F} = \{x\}$ and $x \in U$. Choose an $F_0 \in \mathcal{F}$ and consider

$$\mathcal{F}_0 = \{F \in \mathcal{F} : F \subseteq F_0\}.$$

Clearly, $\bigcap \mathcal{F}_0 = \bigcap \mathcal{F} = \{x\}$. The set

$$F_0 - U = F_0 \cap (X - U),$$

as a closed subset of the compact set F_0, is compact. In addition, $\{X - F : F \in \mathcal{F}_0\}$ is an open cover of $F_0 - U$, so there is an $F \in \mathcal{F}_0$ such that $F_0 - U \subseteq X - F$ (note that $\{X - F : F \in \mathcal{F}_0\}$ is a directed family of open sets). Thus,

$$F \cap (F_0 - U) \subseteq F \cap (X - F) = \emptyset.$$

Since $F \subseteq F_0$, we have

$$F \cap (F_0 - U) = F - U = \emptyset,$$

implying $F \subseteq U$. Hence, every open set of (X,τ) is CK-open, that is, $\tau \subseteq \tau_{\mathrm{CK}}$.

Definition 2.2. A topological space (X,τ) is called CK-*filter defined* if $\tau_{\mathrm{CK}} = \tau$.

Example 2.3. Let $X = \mathbb{R}$ be the set of all real numbers and τ the topology on X, where $U \in \tau$ if and only if $U = V - A$ for some Euclidean open set V and a countable set A. Then, $CK(X)$ is the family of all nonempty finite subsets of $\mathbb{R}$. Thus, every subset is CK-open, so (X,τ) is not CK-filter defined.

A space X is a k-space (or compactly generated space) if a subset U of X is open if and only if for any compact set K, $U \cap K$ is open in the subspace K. Equivalently, a subset B is closed if and only if for any compact set K, $B \cap K$ is closed in the subspace K.

Theorem 2.4. *Every Hausdorff k-space is* CK-*filter defined.*

Proof. Let (X,τ) be a Hausdorff k-space, and let U be CK-open. Let K be a compact subset of (X,τ). Assume that $K \cap (X - U)$ is not closed in K. Then, it is not a closed set. Thus, there is a net

$$\{x_n : n \in D\} \subseteq K \cap (X - U)$$

that converges to an element x_0 and $x_0 \notin K \cap (X - U)$. However, as K is closed since it is a compact subset of a Hausdorff space, so $x_0 \in K$. It thus follows that $x_0 \in U$. For each $n \in D$, let $F_n = \mathrm{cl}(\{x_k : k \geq n\})$. Then, each F_n is a closed compact subset of (X, τ) since it is a closed subset of the Hausdorff compact subspace K. Furthermore, $\mathcal{F} = \{F_n : n \in D\}$ is a filter base, and clearly, $\bigcap \mathcal{F} = \{x_0\}$. However, there is no F_n contained in U. This contradiction shows that, for any compact subset K of (X, τ), $K \cap (X - U)$ is closed in K; thus, $K \cap U$ is open in K. Since (X, τ) is a k-space, it follows that U is open in (X, τ). Therefore, $\tau_{\mathrm{CK}} \subseteq \tau$, implying $\tau = \tau_{\mathrm{CK}}$. $\square$

Since every locally compact Hausdorff space is a k-space, we have the following.

Corollary 2.5. *Every Hausdorff locally compact space is* CK-*filter defined.*

A space X is called a *sequential space* if a subset A is closed if and only if for any sequence $\{x_i\}$ in A, A contains all limits of $\{x_i\}$. Sequential spaces are precisely the quotient spaces of metric spaces [**5, 6**]. Every sequential Hausdorff space is a k-space [**3,** Theorem 3.3.20].

Corollary 2.6. *Every Hausdorff sequential space is* CK-*filter defined. In particular, every first countable Hausdorff space is* CK-*filter defined.*

At this time, we still do not have an example of a CK-filter defined Hausdorff space which is not a k-space.

We now show that, for any CK-filter defined space X, $(\mathrm{CK}(X), \supseteq)$ is a bounded complete dcpo model of X.

Lemma 2.7. *For any Hausdorff space* $(X, \tau), \mathrm{CK}(X, \tau) = \mathrm{CK}(X, \tau_{\mathrm{CK}})$.

Proof. First, note that every compact Hausdorff space is locally compact, and thus, CK-filter defined. Also, in a Hausdorff space, every compact set is closed.

Since τ_{CK} is finer than τ, if $A \in \mathrm{CK}(X, \tau_{\mathrm{CK}})$, then A is a compact subset of (X, τ). However, (X, τ) is Hausdorff; thus, A is also closed in (X, τ). Hence, $A \in \mathrm{CK}(X, \tau)$.

Now, let

$$F \in \mathrm{CK}(X, \tau)$$

and

$$\{U_i : i \in I\} \subseteq \tau_{\mathrm{CK}}$$

be an open cover of F. Then,

$$F = \bigcup\{F \cap U_i : i \in I\}.$$

The subspace F of (X, τ) is compact Hausdorff; thus, it is CK-filter defined. It is easily seen that each $F \cap U_i$ is a CK-open set of the space F (if $\mathcal{D} \subseteq \mathrm{CK}(F)$ is a filter base such that $\bigcap \mathcal{D} = \{x\} \subseteq F \cap U_i$, then $\mathcal{D} \subseteq \mathrm{CK}(X, \tau)$ and $\bigcap \mathcal{D} = \{x\} \subseteq U_i$). Therefore, each $F \cap U_i$ is an open set of F since F is CK-filter defined, that is,

$$F \cap U_i = F \cap V_i$$

for some $V_i \in \tau$. Then,

$$F \subseteq \bigcup\{V_i : i \in I\}.$$

As $F \in \mathrm{CK}(X, \tau)$ is compact, there exist $i_1, i_2, \ldots, i_n$ such that

$$F \subseteq \bigcup\{V_{i_k} : k = 1, 2, \ldots, n\},$$

which then implies

$$F \subseteq \bigcup\{U_{i_k} : k = 1, 2, \ldots, n\}.$$

Hence, F is a compact set of (X, τ_{CK}). In addition, as F is closed in (X, τ) it is also closed in (X, τ_{CK}); thus, $F \in \mathrm{CK}(X, \tau_{\mathrm{CK}})$.

In all, this shows that $\mathrm{CK}(X, \tau) = \mathrm{CK}(X, \tau_{\mathrm{CK}})$. □

Corollary 2.8. *For any Hausdorff space (X, τ), the space (X, τ_{CK}) is Hausdorff and* CK*-filter defined.*

Proof. That (X, τ_{CK}) is Hausdorff follows from the fact that τ_{CK} is finer than τ.

Let U be a CK-open set of (X, τ_{CK}). We must show that $U \in \tau_{\mathrm{CK}}$. Let $\mathcal{D} \subseteq \mathrm{CK}(X, \tau)$ be a filter base such that

$$\bigcap \mathcal{D} = \{x\} \subseteq U.$$

Then, $\mathcal{D} \subseteq \mathrm{CK}(X, \tau_{\mathrm{CK}})$ by Lemma 2.7. As U is a CK-open set of (X, τ_{CK}), there is a $V \in \mathcal{D}$ such that $V \subseteq U$. Hence, $U \in \tau_{\mathrm{CK}}$. □

A subset U of a topological space (X, τ) is called CK*-*open* if, for any filter base $\mathcal{F} \subseteq \mathrm{CK}(X)$,

$$\bigcap \mathcal{F} \subseteq U \implies F \subseteq U \text{ for some } F \in \mathcal{F}.$$

Every open set of X is CK*-open, and every CK*-open set is CK-open.

The intersection of two CK*-open sets is clearly a CK*-open set. However, it seems impossible to show that, in general, the union of two CK*-open sets is CK*-open. Thus, it is unlikely that all CK*-open sets form a topology if no further condition is imposed.

Lemma 2.9. *A subset of a Hausdorff space is* CK-*open if and only if it is* CK*-*open.*

Proof. Let (X, τ) be a Hausdorff space. We only need prove that every CK-open set of X is CK*-open.

Let U be a CK-open set of X and $\mathcal{F} \subseteq \mathrm{CK}(X)$ a filter base such that $\bigcap \mathcal{F} \subseteq U$. Without loss of generality, we can assume that there is an $F_0 \in \mathcal{F}$ such that $F \subseteq F_0$ holds for all $F \in \mathcal{F}$. The subspace F_0 of X is Hausdorff and compact; therefore, it is CK-filter defined.

It is easy to verify that, if A is a closed set and U is a CK-open set, then $U \cap A$ is a CK-open subset of the subspace A. Now, $U \cap F_0$ is a CK-open set of the (Hausdorff compact) subspace F_0; thus, it is open. Therefore, it is a CK*-open set of F_0. Now,

$$\bigcap \mathcal{F} \subseteq U \cap F_0,$$

and each member of $\mathcal{F}$ is a closed compact subset of F_0. Thus, there is an $F \in \mathcal{F}$ with $F \subseteq U \cap F_0 \subseteq U$. In all, this shows that U is CK*-open. □

Theorem 2.10. *Let (X, τ) be a T_1 topological space. Consider the following statements*:

(1) *X is* CK-*filter defined.*

(2) *The mapping*

$$\eta_X : X \longrightarrow \mathrm{Max}(\mathrm{CK}(X))$$

is a homeomorphism.

(3) *Every* CK^*-*open set is an open set of X.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If X is Hausdorff, then the above three statements are equivalent.*

Proof.

(1) $\Rightarrow$ (2). Assume that (X, τ) is CK-filter defined. For any $U \in \tau$, we claim that the following set is a Scott open set of the dcpo $(\mathrm{CK}(X), \supseteq)$:

$$\{A \in \mathrm{CK}(X) : A \subseteq U\}.$$

In order to see this, let $\mathcal{E}$ be a directed subset of $\mathrm{CK}(X)$ and

$$\bigvee_{\mathrm{CK}(X)} \mathcal{E} = \bigcap \mathcal{E} \subseteq U.$$

Choose one $A_0 \in \mathcal{E}$, and let $\widehat{\mathcal{E}} = \{B \in \mathcal{E} : B \subseteq A_0\}$. Then, $\bigcap \widehat{\mathcal{E}} = \bigcap \mathcal{E} \subseteq U$. Further,

$$\bigcup \{B^c : B \in \widehat{\mathcal{E}}\} \supseteq U^c \supseteq A_0 \cap U^c;$$

thus, the directed family $\{B^c : B \in \widehat{\mathcal{E}}\}$ is an open cover of the compact set $A_0 \cap U^c$. There is a $B \in \widehat{\mathcal{E}}$ such that $A_0 \cap U^c \subseteq B^c$, which implies $B \subseteq U \cup A_0^c$. However, $B \subseteq A_0$; hence $B \subseteq U$. Since the set $\{A \in \mathrm{CK}(X) : A \subseteq U\}$ is clearly an upper set of $(\mathrm{CK}(X), \supseteq)$, it is thus a Scott open set of $(\mathrm{CK}(X), \supseteq)$. Furthermore,

$$\eta_X(U) = \{\{x\} : x \in U\} = \{A \in \mathrm{CK}(X) : A \subseteq U\} \bigcap \mathrm{Max}(\mathrm{CK}(X)),$$

which is an open set of $\mathrm{Max}(\mathrm{CK}(X))$. Thus, η_X is an open mapping.

Now, let $\mathcal{U}$ be a Scott open set in $\mathrm{CK}(X)$. We prove that $\eta_X^{-1}(\mathcal{U} \cap \mathrm{Max}(\mathrm{CK}(X)))$ is open in (X, τ). By assumption (1), we only need verify that $\eta_X^{-1}(\mathcal{U} \cap \mathrm{Max}(\mathrm{CK}(X)))$ is a CK-open set of (X, τ). Assume that $\mathcal{F} \subseteq \mathrm{CK}(X)$ is a filter base such that

$$\bigcap \mathcal{F} = \{x\} \subseteq \eta_X^{-1}\Big(\mathcal{U} \bigcap \mathrm{Max}(\mathrm{CK}(X))\Big).$$

Then, $\mathcal{F}$ is a directed subset of $\mathrm{CK}(X)$ and

$$\bigvee_{\mathrm{CK}(X)} \mathcal{F} = \{x\} = \eta_X(x) \in \mathcal{U}$$

since $x \in \eta_X^{-1}(\mathcal{U} \bigcap \mathrm{Max}(\mathrm{CK}(X)))$. Since $\mathcal{U}$ is Scott open, there is a $F \in \mathcal{F}$ with $F \in \mathcal{U}$. Note that $\mathcal{U}$ is an upper set of $(\mathrm{CK}(X), \supseteq)$; thus, for any $y \in F$, it holds that $\{y\} \in \mathcal{U}$. Therefore,

$$\eta_X(F) = \{\{y\} : y \in F\} \subseteq \mathcal{U} \cap \mathrm{Max}(\mathrm{CK}(X)).$$

It follows that $F \subseteq \eta_X^{-1}(\mathcal{U} \cap \mathrm{Max}(\mathrm{CK}(X))$, showing that $\eta_X^{-1}(\mathcal{U} \cap \mathrm{Max}(\mathrm{CK}(X))$ is CK-open in (X, τ). Therefore, η_X is continuous. Since η_X is also clearly bijective, it is a homeomorphism.

(2) $\Rightarrow$ (3). Let $U \subseteq X$ be a CK^*-open set. By the definition of CK^*-open sets, it follows that the set $\widehat{U} = \{A \in \mathrm{CK}(X) : A \subseteq U\}$ is a Scott open set of $(\mathrm{CK}(X), \supseteq)$; thus,

$$\eta_X^{-1}(\mathrm{Max}(\mathrm{CK}(X)) \cap \widehat{U}) = U$$

must be an open set of (X, τ). Thus, (3) is proved.

Now assume that (X, τ) is Hausdorff and every CK^*-open set is open in (X, τ). From Lemma 2.9, every CK-open set of (X, τ) is CK^*-open, so every CK-open set of (X, τ) is open, showing that X is CK-filter defined. Hence, (3) implies (1); therefore, all three statements are equivalent. $\square$

Theorem 2.11. *For any Hausdorff space* (X, τ)*,* $\mathrm{Max}(\mathrm{CK}(X))$ *is homeomorphic to* (X, τ_{CK})*.*

Proof. We prove that the mapping

$$g : (X, \tau_{\mathrm{CK}}) \longrightarrow \mathrm{Max}(\mathrm{CK}(X))$$

is a homeomorphism, where $g(x) = \{x\}, x \in X$.

For any open set E of $\mathrm{Max}(\mathrm{CK}(X))$, there is a Scott open set $\widehat{E}$ of $\mathrm{CK}(X)$ such that $E = \{\{y\} : \{y\} \in \widehat{E}\}$. Then, $g^{-1}(E) = \{y : \{y\} \in \widehat{E}\}$

is a CK-open set of X. As a matter of fact, let $\mathcal{F} \subseteq \mathrm{CK}(X)$ be a filter base such that

$$\bigcap \mathcal{F} = \{y\} \subseteq g^{-1}(E).$$

Then,

$$\bigvee_{\mathrm{CK}(X)} \mathcal{F} \in \widehat{E}.$$

Thus, there is an $F \in \mathcal{F} \cap \widehat{E}$. Again, since $\widehat{E}$ is an upper set of $(\mathrm{CK}(X), \supseteq)$ we have that $\{y\} \in \widehat{E}$ holds for any $y \in F$, which implies $g(F) \subseteq E$. Thus, $F \subseteq g^{-1}(E)$. Hence, $g^{-1}(E)$ is an open set of (X, τ_{CK}); therefore, g is continuous.

Now, let $U \in \tau_{\mathrm{CK}}$. From Lemma 2.9, U is a CK*-open set of X. We can verify that

$$H = \{A \in \mathrm{CK}(X) : A \subseteq U\}$$

is a Scott open set of $\mathrm{CK}(X)$ and

$$H \cap \mathrm{Max}(\mathrm{CK}(X)) = g(U).$$

Thus, g is also an open mapping, and therefore, a homeomorphism. □

Corollary 2.12.

(1) *For any Hausdorff space X,* $\mathrm{Max}(\mathrm{CK}(X))$ *is* CK-*filter defined.*

(2) *A Hausdorff space X is* CK-*filter defined if and only if it is homeomorphic to* $\mathrm{Max}(\mathrm{CK}(Y))$ *for some Hausdorff space Y.*

By the implication of (1) ⇒ (2) in Theorem 2.10, we deduce the main positive result in this paper on spaces that have a bounded complete dcpo model.

Theorem 2.13. *Every* CK-*filter defined T_1 space has a bounded complete* dcpo *model.*

Corollary 2.14. *Every Hausdorff k-space has a bounded complete* dcpo *model.*

Another possible bounded complete dcpo model of a space X formed by some subsets is the set $\mathrm{CKC}(X)$ of all nonempty *closed compact* and *connected* subsets of X. With respect to the reverse inclusion order, $\mathrm{CKC}(X)$ is a bounded complete dcpo, and the maximal points are the singletons. For the real line $\mathbb{R}$ with the Euclidean topology, $\mathrm{CKC}(\mathbb{R})$ is the set of all closed intervals and it is indeed a dcpo model of $\mathbb{R}$ (see [**7,** Example V-6.3]).

The next theorem can be proved using a similar method as for locally compact Hausdorff spaces.

Proposition 2.15. *If X is a locally compact and locally connected T_1 space, then* $(\mathrm{CKC}(X), \supseteq)$ *is a bounded complete* dcpo *model of X.*

3. Conclusions and remarks for further work. In this paper, we introduced and studied CK-filter defined spaces and used this to characterize the Hausdorff spaces whose nonempty compact subsets form a dcpo model. One of the main results is that every Hausdorff k-space has a bounded complete dcpo model.

The following are some related problems and tasks for further study on this topic.

(1) Example 1.1 shows that not every T_1 space has a bounded complete dcpo model. We do not know, at this moment, whether assuming a stronger separation axiom will guarantee the existence of such a dcpo model. In particular, we are interested in knowing whether every Hausdorff space has a bounded complete dcpo model.

(2) From Theorem 2.4, if a space is a Hausdorff k-space, it is CK-filter defined. We do not know whether the converse conclusion for Hausdorff spaces are true. We conjecture it is not true.

(3) It is well known that the category of all Hausdorff k-spaces is Cartesian closed. Now, the category of Hausdorff k-spaces is a sub-category of Hausdorff CK-filtered spaces, and they seem very close to each other. We wonder whether the category of all Hausdorff CK-filter defined space also owns some closure properties. For example, one problem is: is the product of two Hausdorff CK-filter defined spaces CK-filter defined?

(4) In [**9**], Hofmann and Lawson introduced q-spaces and proved that every Hausdorff k-space is a q-space. One of the characteristics of sober q-spaces was given in terms of collections of quasicompact saturated subsets of the space [**9,** Proposition 2.9]. It would be desirable to find more links between CK-filter defined spaces and q-spaces.

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