ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 2, 2017

ON THE EXISTENCE OF CONTINUOUS SOLUTIONS FOR NONLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS WITH STRONGLY GROWING LOWER-ORDER TERMS

MYKHAILO V. VOITOVYCH

ABSTRACT. In this article, we consider nonlinear elliptic fourth-order equations with the monotone principal part satisfying the common growth and coerciveness conditions for Sobolev space $W^{2,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$. It is supposed that the lower-order term of the equations admits arbitrary growth with respect to an unknown function and is arbitrarily close to the growth limit with respect to the derivatives of this function. We assume that the lower-order term satisfies the sign condition with respect to the unknown function. We prove the existence of continuous generalized solutions for the Dirichlet problem in the case $n = 2p$.

1. Introduction. Let $n, m \in \mathbb{N}$, $p \in \mathbb{R}$ be numbers such that $n \geq 3$, $m \geq 2$ and $p > 1$. Let Ω be a bounded open set of $\mathbb{R}^n$, and let $W^{m,p}(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{m,p} = \bigg(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p dx \bigg)^{1/p},$$

where the $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-dimensional multi-index with nonnegative integer components α_i, $i = 1, \ldots, n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and where the $D^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ is a generalized derivative of order $|\alpha|$. We denote by $W_0^{m,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in

2010 AMS *Mathematics subject classification.* Primary 35B45, 35B65, 35J40, 35J62.

Keywords and phrases. Nonlinear elliptic fourth-order equations, lower-order term, strong growth, Dirichlet problem, continuous solutions, L^∞-estimate.

This research is supported by a grant from the Ministry of Education and Science of Ukraine, project No. 0115U000136 and by a grant from the State Fund for Fundamental Research of Ukraine, project No. 0116U007160.

Received by the editors on August 6, 2015, and in revised form on June 3, 2016.

DOI:10.1216/RMJ-2017-47-2-667

$W^{m,p}(\Omega)$. The norm

$$\|u\|_{W_0^{m,p}(\Omega)} = \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}$$

is equivalent to $\|\cdot\|_{m,p}$ in the Banach space $W_0^{m,p}(\Omega)$.

We consider the general $2m$th order equation in the divergence form

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \mathcal{A}_\alpha(x, u, \dots, D^m u) = 0, \tag{1.1}$$

where $x \in \Omega$, $u \in W^{m,p}(\Omega)$ is an unknown function,

$$D^k u = \{D^\alpha u : |\alpha| = k\}, \quad k = 1, \dots, m.$$

We assume that $n = mp$, for every multi-index α with $|\alpha| \le m$,

$$\mathcal{A}_\alpha : \Omega \times \mathbb{R}^{N_m} \longrightarrow \mathbb{R}$$

is a Carathéodory function (N_m is the number of all multi-indices α with $|\alpha| \le m$), and for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^{N_m}$ the following inequalities hold:

$$\sum_{|\alpha|=m} \mathcal{A}_\alpha(x,\xi)\xi_\alpha \ge c_1 \sum_{|\alpha|=m} |\xi_\alpha|^p - c_2 \sum_{|\beta|<m} |\xi_\beta|^{p_\beta} - g(x), \tag{1.2}$$

$$|\mathcal{A}_\alpha(x,\xi)| \le c_2 \sum_{|\beta|\le m} |\xi_\beta|^{p_{\alpha\beta}} + g_\alpha(x), \quad |\alpha| \le m. \tag{1.3}$$

Here c_1 and c_2 are positive constants, $p_{\alpha\beta} = p - 1$ if $|\alpha| = |\beta| = m$, $p_\beta \ge 1$, $p_{\alpha\beta} \ge 0$ and

$$\begin{aligned} p_\beta &< n/|\beta| \quad \text{if } |\beta| < m, \\ p_{\alpha\beta} &< (n - |\alpha|)/|\beta| \quad \text{if } |\alpha| + |\beta| < 2m, \end{aligned} \tag{1.4}$$

g, g_α are nonnegative functions such that $g \in L^\tau(\Omega)$, $\tau > 1$, $g_\alpha \in L^{\tau_\alpha}(\Omega)$, $\tau_\alpha > n/(n - |\alpha|)$.

Under these assumptions, a generalized solution of equation (1.1) is a function $u \in W^{m,p}(\Omega)$ such that, for every function $v \in W_0^{m,p}(\Omega)$,

$$\int_\Omega \left\{ \sum_{|\alpha|\le m} \mathcal{A}_\alpha(x, u, \dots, D^m u) D^\alpha v \right\} dx = 0. \tag{1.5}$$

As is known, see for instance [**5,** Chapter 7], $W_0^{m,p}(\Omega) \subset C^k(\overline{\Omega})$ if $n < mp$ and $0 \leq k < m - n/p$. In the case that $n = mp$, the embedding

$$W^{m,p}(\Omega) \subset L^{\varphi}(\Omega) \tag{1.6}$$

($L^{\varphi}(\Omega)$ denotes the Orlicz space generated by the function $\varphi(t) = \exp[|t|^{p/(p-1)}] - 1$, $t \in \mathbb{R}$ [**5,** Chapter 7]) does not provide the boundedness of generalized solutions of equation (1.1). In this situation, Frehse [**4**] has established the boundedness of the arbitrary generalized solution $u \in W_0^{m,p}(\Omega)$ of equation (1.1), and the continuity of the solution has been proved by Skrypnyk [**11,** Chapter 2]. Hölder continuity of solutions was studied by Widman [**17**] and Solonnikov [**12**] at similar assumptions. Finally, in the case where $n > mp$, there exist examples of equations in the form (1.1)–(1.3) with unbounded solutions, see [**2, 10**]. We also note that the existence of a generalized solution of equation (1.1) with growth condition (1.3), (1.4) can be set using the theory of monotone operators and additional assumptions on the coefficients.

If, in condition (1.4) on p_β and $p_{\alpha\beta}$ we replace the inequalities on the equalities, the above-mentioned results of Frehse and Skrypnyk cease to be valid. At the same time, using the method of [**11,** Chapter 2], Todorov [**13**] proved the continuity of every bounded generalized solution of equation (1.1) in the case where $p_\beta = n/|\beta|$ if $|\beta| < m$, $p_{\alpha\beta} = (n - |\alpha|)/|\beta|$ if $|\beta| \neq 0$ and $|\alpha| + |\beta| \leq 2m$, and for every multi-index α with $|\alpha| \leq m$, the coefficient $\mathcal{A}_\alpha$ admits an arbitrary growth with respect to an unknown function.

Next we recall the precise formulation of the Todorov's result [**13**].

Theorem 1.1. *Suppose that the coefficients of equation* (1.1) *satisfy the following conditions.*

(1) *For every multi-index α with $|\alpha| \leq m$, $\mathcal{A}_\alpha : \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ is a Carathéodory function.*
(2) *For almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{N_m}$ the following inequalities hold*:
   (a) $\sum_{|\alpha|=m} \mathcal{A}_\alpha(x,\xi)\xi_\alpha \geq \lambda(|\xi_0|) \sum_{|\alpha|=m} |\xi_\alpha|^p - C(|\xi_0|) \sum_{1\leq|\beta|<m} |\xi_\beta|^{n/|\beta|} - g(x)$,
   (b) $|A_\alpha(x,\xi)| \leq C_\alpha(|\xi_0|) \sum_{1\leq|\beta|\leq m} |\xi_\beta|^{(n-|\alpha|)/|\beta|} + g_\alpha(x)$, $|\alpha| \leq m$.

Here, $p > 1$, $n = mp$; $\mathbb{R}_+ = [0,+\infty)$ and $C, C_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, $\lambda : \mathbb{R}_+ \to (0,+\infty)$ is a continuous nonincreasing function; g, g_α are nonnegative functions such that

$$g \in L^\tau(\Omega), \quad \tau > 1, \qquad g_\alpha \in L^{\tau_\alpha}(\Omega), \quad \tau_\alpha > n/(n - |\alpha|).$$

Let $u \in W^{m,p}(\Omega) \cap L^\infty(\Omega)$ be a generalized solution of equation (1.1), *that is, for every function $v \in W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$ equality* (1.5) *is true. Then the solution u is continuous at every interior point of the set Ω.*

The question on existence of a bounded generalized solution of equation (1.1) under conditions (A), (B) is still open.

In this article, we consider fourth order equations ($m = 2$) in the form (1.1) satisfying the condition $\mathcal{A}_\alpha \equiv 0$ if $|\alpha| = 1$, and all assumptions in **[4]** except for inequality (1.3) for the lower-order term $\mathcal{A}_0$. Instead of this, we suppose a more general condition admitting, unlike **[4, 11, 12, 17]**, an arbitrary growth of the term $\mathcal{A}_0$ with respect to the function u (even stronger than the growth of the function $\varphi(u) = \exp[|u|^{p/(p-1)}] - 1$) and a growth of $\mathcal{A}_0$ with respect to the derivatives $D^\alpha u$, $|\alpha| = 1, 2$, which is arbitrarily close to the limiting growth. This means that

$$|\mathcal{A}_0(x,u,Du,D^2u)| \le a(|u|)\Big\{1 + \sum_{|\alpha|=1,2} |D^\alpha u|^{n/|\alpha|}\, \psi(|D^\alpha u|)\Big\} + g_0(x),$$

where a, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, $\lim_{t\to+\infty} \psi(t) = 0$. At the same time, it is supposed that the lower-order term $\mathcal{A}_0$ satisfies the sign condition $\mathcal{A}_0(x,u,Du,D^2u)u \ge 0$. The main result of this article is a theorem on the existence and L^∞-estimate of continuous generalized solutions of the Dirichlet problem for the equations under investigation.

We remark that, in the situation $n > mp$ results on the existence of bounded generalized solutions for nonlinear elliptic equations with natural growth lower-order terms were established, for instance, in **[1, 3]** (the case of second-order equations, $m = 1$) and in **[14, 15, 16]** (the case of high-order equations with strengthened coercivity, $m \ge 2$).

2. Statement of the main result. Let $n \in \mathbb{N}$, $n \ge 3$, and let Ω be a bounded open set of $\mathbb{R}^n$.

We shall use the following notation: $C(\Omega)$ is the set of continuous functions on Ω, Λ is the set of all n dimensional multi-indices α such that $|\alpha| \le 2$, N_1 (correspondingly N_2) is the number of all multi-indices α with $|\alpha| \le 1$ ($|\alpha| \le 2$), $N = N_2 - N_1$. If $\tau \in [1, +\infty]$, then $\|\cdot\|_\tau$ is the norm in $L^\tau(\Omega)$. For every measurable set $E \subset \Omega$ we denote by $|E|$ (or by meas E) n-dimensional Lebesgue measure of the set E.

We set $p = n/2$ and note that, by (1.6) ($m = 2$) and Sobolev inequality, see for instance, [**5,** Theorem 7.10], for every $\lambda \ge 1$ and for every function $u \in W_0^{2,p}(\Omega)$,

$$\begin{aligned} c_{\lambda,n,\Omega}\bigg(\int_\Omega |u|^\lambda dx\bigg)^{1/\lambda} &\le \bigg(\sum_{|\alpha|=1}\int_\Omega |D^\alpha u|^n dx\bigg)^{1/n} \\ &\le c_n\bigg(\sum_{|\alpha|=2}\int_\Omega |D^\alpha u|^p dx\bigg)^{1/p}, \end{aligned} \tag{2.1}$$

where $c_{\lambda,n,\Omega}$ is a positive constant depending only on λ, n and $|\Omega|$, and c_n is a positive constant depending only on n.

Next, let $c_1, c_2 > 0$, let g_1 and g_2 be nonnegative summable functions on Ω, and let p_0, p_1, $\widetilde{p}$ be arbitrary numbers satisfying the inequalities $p_0 \ge 0$,

$$0 \le p_1 < n, \tag{2.2}$$

$$1 \le \widetilde{p} < p. \tag{2.3}$$

For every $\alpha \in \Lambda$ with $|\alpha| = 2$, let $A_\alpha : \Omega \times \mathbb{R}^{N_2} \to \mathbb{R}$ be a Carathéodory function. We assume that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{N_2}$, the following inequalities hold:

$$\sum_{|\alpha|=2} A_\alpha(x,\xi)\xi_\alpha \ge c_1 \sum_{|\alpha|=2} |\xi_\alpha|^p - c_2 \sum_{|\alpha|\le 1} |\xi_\alpha|^{\tilde{p}} - g_1(x), \tag{2.4}$$

$$\begin{aligned} &\sum_{|\alpha|=2} |A_\alpha(x,\xi)|^{p/(p-1)} \\ &\qquad \le c_2\bigg\{|\xi_0|^{p_0} + \sum_{|\alpha|=1} |\xi_\alpha|^{p_1} + \sum_{|\alpha|=2} |\xi_\alpha|^p\bigg\} + g_2(x). \end{aligned} \tag{2.5}$$

Next, let g_3 and g_4 be nonnegative summable functions on Ω, let $b : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions,

$$\lim_{t\to+\infty} \psi(t) = 0, \tag{2.6}$$

and let $B : \Omega \times \mathbb{R}^{N_2} \to \mathbb{R}$ be a Carathéodory function such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^{N_2}$, the following inequalities hold:

$$|B(x,\xi)| \leq b(|\xi_0|)\Big\{1 + \sum_{|\alpha|=1,2} |\xi_\alpha|^{n/|\alpha|}\psi(|\xi_\alpha|)\Big\} + g_3(x), \tag{2.7}$$

$$B(x,\xi)\xi_0 \geq -g_4(x). \tag{2.8}$$

Further, let $\tau > 1$ and

$$f \in L^\tau(\Omega). \tag{2.9}$$

We consider the Dirichlet problem

$$\sum_{|\alpha|=2} D^\alpha A_\alpha(x,u,Du,D^2u) + B(x,u,Du,D^2u) = f \quad \text{in } \Omega, \tag{2.10}$$

$$D^\alpha u = 0, \quad |\alpha| = 0,1, \text{ on } \partial\Omega. \tag{2.11}$$

The following remark provides correctness of the definition of a generalized solution to problem (2.10), (2.11).

Remark 2.1. By (2.1), (2.2), (2.5) and imbedding (1.6), for every $u, v \in \overset{\circ}{W}{}^{2,p}_0(\Omega)$ and every $\alpha \in \Lambda$ with $|\alpha| = 2$, the function $A_\alpha(x,u,Du,D^2u)D^\alpha v$ is summable on Ω, and by (2.1) and (2.7), for every $u, v \in W^{2,p}_0(\Omega) \cap L^\infty(\Omega)$, the function $B(x,u,Du,D^2u)v$ is summable on Ω. Moreover, it follows from (1.6) and (2.9) that, for every $v \in W^{2,p}_0(\Omega)$, the function fv is summable on Ω.

Definition 2.2. A generalized solution of problem (2.10), (2.11) is a function $u \in W^{2,p}_0(\Omega) \cap L^\infty(\Omega)$ such that, for every function $v \in$

$W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{|\alpha|=2} \int_\Omega A_\alpha(x,u,Du,D^2u) D^\alpha v\, dx + \int_\Omega B(x,u,Du,D^2u) v\, dx = \int_\Omega fv\, dx. \tag{2.12}$$

The next theorem is the main result of the present article.

Theorem 2.3. *Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded, with $n \geq 3$. Suppose also that the assumptions in (2.2)–(2.9) hold with $p = n/2$, and with the functions g_1, g_2, g_3, g_4 and f belonging to $L^\tau(\Omega)$, $\tau > 1$. Let M be a majorant for the norms $\|g_1\|_\tau$, $\|g_2\|_\tau$, $\|g_4\|_\tau$ and $\|f\|_\tau$, and let, for almost every $x \in \Omega$ and for every $\eta \in \mathbb{R}^{N_1}$ and $\zeta, \zeta' \in \mathbb{R}^N$, $\zeta \neq \zeta'$, the following inequality holds*:

$$\sum_{|\alpha|=2} [A_\alpha(x,\eta,\zeta) - A_\alpha(x,\eta,\zeta')](\zeta_\alpha - \zeta'_\alpha) > 0. \tag{2.13}$$

Then there exists a generalized solution $u_0 \in C(\Omega)$ of problem (2.10), (2.11) such that $\|u_0\|_\infty \leq C_1$ where C_1 is a positive constant depending only on n, $\widetilde{p}$, p_0, p_1, $|\Omega|$, c_1, c_2, τ and M.

We will prove Theorem 2.3 in Section 3. First, we give some remarks and an example of functions satisfying conditions (2.4)–(2.8) and (2.13).

Remark 2.4. The proof of the existence of the solution u_0 is based on the consideration of a sequence of approximate problems for equations with bounded lower-order terms, obtaining the uniform boundedness of their solutions $\{u_i\}$ and the subsequent limit passage. At the same time, solvability of the approximate problems is established using the results of **[9]** on solvability of equations with pseudomonotone operators. By virtue of condition (2.8) the proof of the uniform boundedness of the solutions $\{u_i\}$ follows the proof of the boundedness of arbitrary generalized solution $u \in W_0^{m,p}(\Omega)$ for equation (1.1) in **[4]**. So we omit this proof here. The limit passage in the approximate problems is justified using ideas of **[7, 8]**.

Remark 2.5. Suppose that conditions (2.2)–(2.7) and (2.9) hold with $p = n/2$ and with the functions g_1, g_2, g_3, $f \in L^\tau(\Omega)$, $\tau > 1$. Let $u \in W^{2,p}(\Omega) \cap L^\infty(\Omega)$ be a generalized solution of equation (2.10), that is, for every function $v \in \overset{\circ}{W}{}^{2,p}_0(\Omega) \cap L^\infty(\Omega)$ equality (2.12) is true. Then, by Theorem 1.1, we have the inclusion $u \in C(\Omega)$.

Example 2.6. Let, for every $\alpha \in \Lambda$ with $|\alpha| = 2$, $A_\alpha : \Omega \times \mathbb{R}^{N_2} \to \mathbb{R}$ be the function defined by

$$A_\alpha(x, \xi) = \left(\sum_{|\beta|=2} \xi_\beta^2 \right)^{(p-2)/2} \xi_\alpha + \sum_{|\beta|\le 1} |\xi_\beta|^{\tilde{p}-1}.$$

Then the functions $\{A_\alpha : |\alpha| = 2\}$ satisfy inequalities (2.4) and (2.5) (with the exponents $p_0 = p_1 = (\widetilde{p} - 1)p/(p-1)$) and (2.13). Next, for every $(x, \xi) \in \Omega \times \mathbb{R}^{N_2}$, we set

$$B(x, \xi) = \xi_0 b_1(|\xi_0|)$$
$$\times \left\{ 1 + \sum_{|\alpha|=1} |\xi_\alpha|^n \,[\,\ln(2 + |\xi_\alpha|)\,]^{-1} + \sum_{|\alpha|=2} |\xi_\alpha|^p \,[\,\ln\ln(3 + |\xi_\alpha|)\,]^{-1} \right\}$$

where b_1 is an arbitrary nonnegative continuous function on $\mathbb{R}_+$, for example $b_1(t) = \exp(t^\lambda)$, $\lambda > 0$. Then the function B satisfies inequalities (2.7), (2.8) and $\psi(t) = [\,\ln\ln(3 + t)\,]^{-1}$, $t \in \mathbb{R}_+$.

3. Proof of Theorem 2.3.

Step 1. Suppose that conditions (2.2)–(2.9) and (2.13) are satisfied with $p = n/2$ and with the functions g_1, g_2, g_3, g_4, $f \in L^\tau(\Omega)$, $\tau > 1$. Let M be a majorant for $\|g_1\|_\tau$, $\|g_2\|_\tau$, $\|g_4\|_\tau$ and $\|f\|_\tau$.

By c_i, $i = 3, 4, \ldots$, we shall denote positive constants, depending only on n, $\widetilde{p}$, p_0, p_1, $|\Omega|$, c_1, c_2, τ and M.

For every $i \in \mathbb{N}$, we define the function $B_i : \Omega \times \mathbb{R}^{N_2} \to \mathbb{R}$ by

$$B_i(x, \xi) = \frac{B(x, \xi)}{1 + |B(x, \xi)|/i}, \quad (x, \xi) \in \Omega \times \mathbb{R}^{N_2}.$$

Obviously, for every $i \in \mathbb{N}$ and for every $(x, \xi) \in \Omega \times \mathbb{R}^{N_2}$,

$$|B_i(x,\xi)| \leq i, \tag{3.1}$$

$$B_i(x,\xi)\xi_0 \geq -g_4(x), \tag{3.2}$$

$$|B_i(x,\xi)| \leq b(|\xi_0|)\left\{1 + \sum_{|\alpha|=1,2} |\xi_\alpha|^{n/|\alpha|}\,\psi(|\xi_\alpha|)\right\} + g_3(x). \tag{3.3}$$

From (2.1)–(2.5), (2.13), (3.1) and embedding (2.9) and the results of **[9]** on solvability of equations with pseudomonotone operators, it follows that, if $i \in \mathbb{N}$, then there exists a function $u_i \in \overset{\circ}{W}{}^{2,p}(\Omega)$ such that, for every function $v \in \overset{\circ}{W}{}^{2,p}(\Omega)$,

$$\sum_{|\alpha|=2} \int_\Omega A_\alpha(x, u_i, Du_i, D^2u_i) D^\alpha v\, dx + \int_\Omega B_i(x, u_i, Du_i, D^2u_i) v\, dx = \int_\Omega f v\, dx. \tag{3.4}$$

Observe that, for every $i \in \mathbb{N}$,

$$\|u_i\|_{\overset{\circ}{W}{}^{2,p}(\Omega)} \leq c_3. \tag{3.5}$$

In fact, fixing an arbitrary $i \in \mathbb{N}$ and putting into (3.4) the function u_i instead of v, we obtain

$$\sum_{|\alpha|=2} \int_\Omega A_\alpha(x, u_i, Du_i, D^2u_i) D^\alpha u_i dx + \int_\Omega B_i(x, u_i, Du_i, D^2u_i) u_i dx = \int_\Omega f u_i dx.$$

This, along with (2.4) and (3.2), implies that

$$c_1 \int_\Omega \left\{ \sum_{|\alpha|=2} |D^\alpha u_i|^p \right\} dx \leq c_2 \int_\Omega \left\{ \sum_{|\alpha|\leq 1} |D^\alpha u_i|^{\tilde{p}} \right\} dx + \int_\Omega f u_i dx + \int_\Omega (g_1 + g_4)\, dx.$$

From this inequality, estimating the first addend on the right-hand side by means of Hölder's and Young's inequalities and (2.1), (2.3), and the second addend by means of Hölder's, Young's inequalities and (2.1), we deduce (3.5).

Taking into account inequalities (3.1), (3.2) and (3.5), inclusions g_1, g_2, g_4, $f \in L^\tau(\Omega)$, $\tau > 1$, and using the reasoning of **[4]**, we establish that, for every $i \in \mathbb{N}$,

$$\|u_i\|_\infty \leq c_4. \tag{3.6}$$

By virtue of (3.5) and the compactness of the embedding $W_0^{2,p}(\Omega) \subset W_0^{1,\lambda}(\Omega)$ with $\lambda < n$, there exist an increasing sequence $\{i_j\} \subset \mathbb{N}$ and a function $u_0 \in W_0^{2,p}(\Omega)$ such that

$$u_{i_j} \longrightarrow u_0 \qquad \text{weakly in } W_0^{2,p}(\Omega), \tag{3.7}$$

$$u_{i_j} \longrightarrow u_0 \qquad \text{almost everywhere in } \Omega, \tag{3.8}$$

$$D^\alpha u_{i_j} \longrightarrow D^\alpha u_0 \qquad \text{almost everywhere in } \Omega,\ \text{if } |\alpha| = 1. \tag{3.9}$$

Now, from (3.6) and (3.8) we deduce the estimate

$$\|u_0\|_\infty \leq c_4. \tag{3.10}$$

Step 2. For every $i \in \mathbb{N}$, we set

$$\Phi_i = \sum_{|\alpha|=2} \big[A_\alpha(x, u_i, Du_i, D^2u_i) - A_\alpha(x, u_i, Du_i, D^2u_0)\big](D^\alpha u_i - D^\alpha u_0).$$

Let us demonstrate that

$$\lim_{j\to\infty} \int_\Omega \Phi_{i_j}\, dx = 0. \tag{3.11}$$

Let $j \in \mathbb{N}$. Since $u_{i_j} - u_0 \in W_0^{2,p}(\Omega)$, by virtue of (3.4), we have

$$\begin{aligned} \int_\Omega \Phi_{i_j}\, dx &= \int_\Omega f(u_{i_j} - u_0)\, dx \\ &\quad - \int_\Omega B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2u_{i_j})(u_{i_j} - u_0)\, dx \\ &\quad - \int_\Omega \Big\{ \sum_{|\alpha|=2} A_\alpha(x, u_{i_j}, Du_{i_j}, D^2u_0)(D^\alpha u_{i_j} - D^\alpha u_0) \Big\}\, dx. \end{aligned} \tag{3.12}$$

The integrals on the right-hand side of (3.12) tend to zero as $j \to \infty$. In fact, by (3.6) and (3.8), we have

$$\lim_{j\to\infty} \int_\Omega f(u_{i_j} - u_0)\, dx = 0. \tag{3.13}$$

Next, we fix an arbitrary $\varepsilon > 0$. By virtue of (2.6) and the non-negativeness of the function ψ, there exists the number $K > 1$ depending only on ψ and ε such that

$$0 \le \psi(t) < \varepsilon \quad \text{if } t > K. \tag{3.14}$$

We set $\widetilde{b} = \max_{s\in[0,\,c_4]} b(s)$. By (3.3) and (3.6), we have

$$\begin{aligned}
&\left| \int_\Omega B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2u_{i_j})(u_{i_j} - u_0)\, dx \right| \\
&\quad\le \int_\Omega |B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2u_{i_j})||u_{i_j} - u_0|\, dx \\
&\quad\le \widetilde{b} \sum_{|\alpha|=1,2} \int_\Omega |D^\alpha u_{i_j}|^{n/|\alpha|}\psi(|D^\alpha u_{i_j}|)|u_{i_j} - u_0|\, dx \\
&\qquad + \int_\Omega (g_3 + \widetilde{b})|u_{i_j} - u_0|\, dx.
\end{aligned} \tag{3.15}$$

Let $\alpha \in \Lambda$, $|\alpha| = 1,2$ and $\psi_K = \max_{t\in[-K,K]} \psi(t)$. Using (3.6), (3.10) and (3.14), we obtain

$$\begin{aligned}
&\int_\Omega |D^\alpha u_{i_j}|^{n/|\alpha|}\, \psi(|D^\alpha u_{i_j}|)|u_{i_j} - u_0|\, dx \\
&\quad= \int_{\{|D^\alpha u_{i_j}|\le K\}} |D^\alpha u_{i_j}|^{n/|\alpha|}\, \psi(|D^\alpha u_{i_j}|)|u_{i_j} - u_0|\, dx \\
&\qquad + \int_{\{|D^\alpha u_{i_j}|> K\}} |D^\alpha u_{i_j}|^{n/|\alpha|}\, \psi(|D^\alpha u_{i_j}|)|u_{i_j} - u_0|\, dx \\
&\le K^n \psi_K \int_\Omega |u_{i_j} - u_0|\, dx + 2c_4\varepsilon \int_\Omega |D^\alpha u_{i_j}|^{n/|\alpha|}\, dx.
\end{aligned} \tag{3.16}$$

From (3.15), (3.16), (3.5) and (2.1) we deduce the inequality

$$\left| \int_\Omega B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2u_{i_j})(u_{i_j} - u_0)dx \right|$$

$$\leq \int_\Omega (g_3 + \widetilde{b} + \widetilde{b}\psi_K K^n N_2)|u_{i_j} - u_0|\,dx + 2c_4c_5\widetilde{b}\,\varepsilon.$$

From this and (3.6), (3.8) and an arbitrary choice of ε, it follows that

$$\lim_{j\to\infty} \int_\Omega B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2u_{i_j})(u_{i_j} - u_0)\,dx = 0. \tag{3.17}$$

By virtue of (3.8) and (3.9), for every $\alpha \in \Lambda$ with $|\alpha| = 2$, we have

$$A_\alpha(x, u_{i_j}, Du_{i_j}, D^2u_0) \longrightarrow A_\alpha(x, u_0, Du_0, D^2u_0) \quad \text{a.e. in } \Omega. \tag{3.18}$$

From (2.2), (2.5) and (3.7), the property of absolute continuity of the Lebesgue integral and the compact embeddings $W_0^{2,p}(\Omega) \subset W_0^{1,\lambda}(\Omega)$ with $\lambda < n$ and $W_0^{2,p}(\Omega) \subset L^\kappa(\Omega)$ with $1 \leq \kappa < +\infty$, it follows that

$$\lim_{|E|\to 0} \sup_{j\in\mathbb{N}} \int_E \Big\{ \sum_{|\alpha|=2} |A_\alpha(x, u_{i_j}, Du_{i_j}, D^2u_0)|^{p/(p-1)} \Big\}\,dx = 0, \tag{3.19}$$

$E \subset \Omega$. Using (3.18), (3.19) and the convergence theorem of Vitali, we establish the following assertion:

if, $\alpha \in \Lambda$ and $|\alpha| = 2$, then

$$A_\alpha(x, u_{i_j}, Du_{i_j}, D^2u_0) \longrightarrow A_\alpha(x, u_0, Du_0, D^2u_0) \quad \text{strongly in } L^{p/(p-1)}(\Omega).$$

From this and (3.7), it follows that

$$\lim_{j\to\infty} \int_\Omega \Big\{ \sum_{|\alpha|=2} A_\alpha(x, u_{i_j}, Du_{i_j}, D^2u_0)(D^\alpha u_{i_j} - D^\alpha u_0) \Big\}\,dx = 0. \tag{3.20}$$

Now, the validity of equality (3.11) follows from (3.12), (3.13), (3.17) and (3.20).

Step 3. We now demonstrate that, for every $\alpha \in \Lambda$ with $|\alpha| = 2$,

$$D^\alpha u_{i_j} \longrightarrow D^\alpha u_0 \quad \text{in measure.} \tag{3.21}$$

For this purpose, we introduce some auxiliary functions and sets.

Let $\Phi : \Omega \to \mathbb{R}$ be the function defined by $\Phi(x) = \inf_{j\in\mathbb{N}} \Phi_{i_j}(x)$. Then Φ is an infimum of countably many measurable functions, and hence measurable (see [**6,** Section 20]); moreover, by virtue of (2.5), we have

$$\Phi \in L^1(\Omega). \tag{3.22}$$

Further, let for every $x \in \Omega$, $A_x : \mathbb{R}^{N_1} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the function such that, for every triplet $(\eta, \zeta, \zeta') \in \mathbb{R}^{N_1} \times \mathbb{R}^N \times \mathbb{R}^N$,

$$A_x(\eta, \zeta, \zeta') = \sum_{|\alpha|=2} \big[A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \zeta')\big](\zeta_\alpha - \zeta'_\alpha).$$

Since, for every $\alpha \in \Lambda$ with $|\alpha| = 2$, A_α is a Carathéodory function and for almost every $x \in \Omega$ and for every $\eta \in \mathbb{R}^{N_1}$ and $\zeta, \zeta' \in \mathbb{R}^N$, $\zeta \neq \zeta'$, inequality (2.13) holds, there exists a set $E \subset \Omega$ of measure zero such that

(i) for every $x \in \Omega \setminus E$ the function A_x is continuous in $\mathbb{R}^{N_1} \times \mathbb{R}^N \times \mathbb{R}^N$;
(ii) for every $x \in \Omega \setminus E$, $\eta \in \mathbb{R}^{N_1}$ and $\zeta, \zeta' \in \mathbb{R}^N$, $\zeta \neq \zeta'$, we have $A_x(\eta, \zeta, \zeta') > 0$.

For every $\theta > 0$, $\sigma > 0$ and for every $\nu > \sigma$, we set

$$G_{\theta,\sigma,\nu} = \Big\{(\eta, \zeta, \zeta') \in \mathbb{R}^{N_1} \times \mathbb{R}^N \times \mathbb{R}^N : \\ \sum_{|\alpha|=2} |\zeta_\alpha - \zeta'_\alpha| \geq \sigma, \sum_{|\alpha|=2} |\zeta_\alpha| \leq \nu, \sum_{|\alpha|=2} |\zeta'_\alpha| \leq \nu, \sum_{|\alpha|\leq 1} |\eta_\alpha| \leq \theta\Big\}.$$

Evidently, for every $\theta > 0$, $\sigma > 0$ and for every $\nu > \sigma$, the set $G_{\theta,\sigma,\nu}$ is nonempty, closed and bounded in $\mathbb{R}^{N_1} \times \mathbb{R}^N \times \mathbb{R}^N$.

Let, for every $\theta > 0$, $\sigma > 0$ and for every $\nu > \sigma$, $\mu_{\theta,\sigma,\nu} : \Omega \to \mathbb{R}$ be the function such that

$$\mu_{\theta,\sigma,\nu}(x) = \begin{cases} \min\limits_{G_{\theta,\sigma,\nu}} A_x & \text{if } x \in \Omega \setminus E, \\ 0 & \text{if } x \in E. \end{cases}$$

Using properties (i) and (ii), we establish that, if $\theta > 0$, $\sigma > 0$ and $\nu > \sigma$, then

$$\mu_{\theta,\sigma,\nu}(x) > 0 \quad \text{for every } x \in \Omega \setminus E. \tag{3.23}$$

Now, we pass to the immediate proof of assertion (3.21). We fix $\sigma > 0$ and $\varepsilon > 0$. Using (2.1) and (3.5), we obtain that, for every $\theta > 0$, $\nu > 0$ and for every $i \in \mathbb{N}$,

$$\theta \operatorname{meas}\Big\{ \sum_{|\alpha|\le 1} |D^\alpha u_i| \ge \theta \Big\} \le \int_{\{\sum_{|\alpha|\le 1} |D^\alpha u_i| \ge \theta\}} \Big(\sum_{|\alpha|\le 1} |D^\alpha u_i| \Big)\, dx \le c_6,$$

$$\nu \operatorname{meas}\Big\{ \sum_{|\alpha|=2} |D^\alpha u_i| \ge \nu \Big\} \le \int_{\{\sum_{|\alpha|=2} |D^\alpha u_i| \ge \nu\}} \Big(\sum_{|\alpha|=2} |D^\alpha u_i| \Big)\, dx \le c_7.$$

Therefore, there exist $\theta > 0$ and $\nu > \max(1, \sigma)$ such that

$$\begin{aligned} &\sup_{j\in\mathbb{N}} \operatorname{meas}\Big\{ \sum_{|\alpha|\le 1} |D^\alpha u_{i_j}| \ge \theta \Big\} \le \varepsilon, \\ &\sup_{j\in\mathbb{N}} \operatorname{meas}\Big\{ \sum_{|\alpha|=2} |D^\alpha u_{i_j}| \ge \nu \Big\} \le \varepsilon, \\ &\operatorname{meas}\Big\{ \sum_{|\alpha|=2} |D^\alpha u_0| \ge \nu \Big\} \le \varepsilon. \end{aligned} \tag{3.24}$$

For every $j \in \mathbb{N}$, we set

$$E_j = \Big\{ \sum_{|\alpha|\le 1} |D^\alpha u_{i_j}| \le \theta, \quad \sum_{|\alpha|=2} |D^\alpha u_{i_j}| \le \nu, \ \sum_{|\alpha|=2} |D^\alpha u_0| \le \nu, \quad \sum_{|\alpha|=2} |D^\alpha u_{i_j} - D^\alpha u_0| \ge \sigma \Big\}.$$

Let $j \in \mathbb{N}$ and $x \in E_j \setminus E$. We have

$$\sum_{|\alpha|\le 1} |D^\alpha u_{i_j}| \le \theta, \qquad \sum_{|\alpha|=2} |D^\alpha u_{i_j}| \le \nu,$$

$$\sum_{|\alpha|=2} |D^\alpha u_0| \le \nu, \qquad \sum_{|\alpha|=2} |D^\alpha u_{i_j} - D^\alpha u_0| \ge \sigma.$$

Hence, $(u_{i_j}(x), Du_{i_j}(x), D^2u_{i_j}(x), D^2u_0(x)) \in G_{\theta,\sigma,\nu}$. Then, by virtue of the definition of $\mu_{\theta,\sigma,\nu}$ and A_x, we have $\mu_{\theta,\sigma,\nu}(x) \le \Phi_{i_j}(x)$, and hence, $\mu_{\theta,\sigma,\nu}(x) \le \Phi(x)$, which together with (3.23) yields

$$\Phi > 0 \quad \text{almost everywhere in } \bigcup_{j=1}^{\infty} E_j \setminus E. \tag{3.25}$$

Now, taking into account (2.13), we conclude that for every $j \in \mathbb{N}$,

$$\int_{E_j} \Phi\, dx \le \int_{E_j} \Phi_{i_j}\, dx \le \int_{\Omega} \Phi_{i_j}\, dx.$$

This and (3.11) imply that

$$\lim_{j\to\infty} \int_{E_j} \Phi\, dx = 0.$$

Hence, taking into account (3.22) and (3.25) and applying [**7,** Lemma 5], we deduce that

$$\lim_{j\to\infty} \operatorname{meas} E_j = 0. \tag{3.26}$$

Obviously, for every $j \in \mathbb{N}$,

$$\begin{aligned}\operatorname{meas}\Big\{\sum_{|\alpha|=2} |D^\alpha u_{i_j} - D^\alpha u_0| \ge \sigma\Big\} &\le \operatorname{meas}\Big\{\sum_{|\alpha|\le 1} |D^\alpha u_{i_j}| > \theta\Big\} \\ &\quad + \operatorname{meas}\Big\{\sum_{|\alpha|=2} |D^\alpha u_{i_j}| > \nu\Big\} \\ &\quad + \operatorname{meas}\Big\{\sum_{|\alpha|=2} |D^\alpha u_0| > \nu\Big\} + \operatorname{meas} E_j.\end{aligned}$$

From this and (3.24) and (3.26), we infer (3.21).

We remark that, in the proof of assertion (3.21) we used some ideas of [**7, 8**].

Step 4. We now prove that the following assertions hold:

(iii) for every function $v \in W_0^{2,p}(\Omega)$,

$$\lim_{|E|\to 0} \sup_{j\in\mathbb{N}} \int_E \Big| \sum_{|\alpha|=2} A_\alpha(x, u_{i_j}, Du_{i_j}, D^2u_{i_j}) D^\alpha v \Big|\, dx = 0, \quad E \subset \Omega;$$

(iv) for every function $v \in W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{|E| \to 0} \sup_{j \in \mathbb{N}} \int_E |B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j}) v| \, dx = 0, \quad E \subset \Omega.$$

In fact, let $j \in \mathbb{N}$, $v \in W_0^{2,p}(\Omega)$, and let $E \subset \Omega$ be an arbitrary measurable set. Using Hölder's inequality for sums and integrals along with (2.1), (2.2), (2.5) and (3.5), we obtain

$$\begin{aligned}
&\int_E \Big| \sum_{|\alpha|=2} A_\alpha(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j}) D^\alpha v \Big| \, dx \\
&\quad \le \bigg[\int_E \bigg\{ \sum_{|\alpha|=2} |A_\alpha(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j})|^{p/(p-1)} \bigg\} \, dx \bigg]^{(p-1)/p} \\
&\quad \times \bigg[\int_E \bigg\{ \sum_{|\alpha|=2} |D^\alpha v|^p \bigg\} dx \bigg]^{1/p} \\
&\quad \le c_8 \bigg[\int_E \bigg\{ \sum_{|\alpha|=2} |D^\alpha v|^p \bigg\} dx \bigg]^{1/p}.
\end{aligned}$$

This and the property of absolute continuity of Lebesgue integral imply that assertion (iii) holds.

Now, let $v \in W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$ and $\varepsilon > 0$. By analogy with (3.15) and (3.16), we establish that

$$\begin{aligned}
&\int_E |B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j}) v| dx \\
&\quad \le \widetilde{b} \sum_{|\alpha|=1,2} \int_E |D^\alpha u_{i_j}|^{n/|\alpha|} \, \psi(|D^\alpha u_{i_j}|) |v| \, dx \\
&\qquad + \int_E (g_3 + \widetilde{b}) |v| \, dx,
\end{aligned}$$

$$\begin{aligned}
&\int_E |D^\alpha u_{i_j}|^{n/|\alpha|} \, \psi(|D^\alpha u_{i_j}|) |v| \, dx \\
&\quad \le K^n \psi_K \int_E |v| \, dx \\
&\qquad + \varepsilon \|v\|_\infty \int_\Omega |D^\alpha u_{i_j}|^{n/|\alpha|} dx, \quad |\alpha| = 1, 2.
\end{aligned}$$

From the last two inequalities and (2.1) and (3.5), it follows that

$$\int_E |B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j})v|\, dx \leq \int_E (g_3 + \widetilde{b} + \widetilde{b}\,\psi_K K^n N_2)|v|\, dx + \varepsilon c_5 \widetilde{b}\|v\|_\infty.$$

This and the property of absolute continuity of the Lebesgue integral and an arbitrary choice of ε imply that assertion (iv) holds.

Using (3.8), (3.9), (3.21), assertions (iii) and (iv) and the convergence theorem of Vitali, we establish that, for every function $v \in W_0^{2,p}(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega \Big\{ \sum_{|\alpha|=2} A_\alpha(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j}) D^\alpha v \Big\}\, dx = \int_\Omega \Big\{ \sum_{|\alpha|=2} A_\alpha(x, u_0, Du_0, D^2 u_0) D^\alpha v \Big\}\, dx,$$

and, for every function $v \in W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega B_{i_j}(x, u_{i_j}, Du_{i_j}, D^2 u_{i_j}) v\, dx = \int_\Omega B(x, u_0, Du_0, D^2 u_0) v\, dx.$$

From this and (3.4), it follows that, for every function $v \in W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{|\alpha|=2} \int_\Omega A_\alpha(x, u_0, Du_0, D^2 u_0) D^\alpha v\, dx + \int_\Omega B(x, u_0, Du_0, D^2 u_0) v\, dx = \int_\Omega f v\, dx.$$

The properties obtained of the function u_0 allow us to conclude that u_0 is a generalized solution of problem (2.10), (2.11). By Remark 2.5, this solution is continuous at every interior point of the set Ω.

Theorem 2.3 is proved. $\square$

Acknowledgments. I wish to thank the Referee for useful suggestions.

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Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine, 19 Gen. Batiouk str., Sloviansk, 84116, Ukraine and Mariupol State University, Faculty of Economics and Law, 129a Budivelnykiv Ave., Mariupol, 87500, Ukraine and Vasyl Stus Donetsk National University, 21, 600-richya str., Vinnytsia, 21021, Ukraine
Email address: **voitovichmv76@gmail.com**