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NORMALITY CONCERNING EXCEPTIONAL FUNCTIONS

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ABSTRACT. Let $\varphi(z)(\not\equiv 0)$ be a function holomorphic in a domain D, $k \in \mathbb{N}$, and let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+2$ such that, for every $f \in \mathcal{F}$, $f^{(k)}(z) \neq \varphi(z)$. The non-normal sequences in $\mathcal{F}$ are characterized.

1. Introduction and main results. Let D be a domain in $\mathbb{C}$, and let $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $\{f_n\} \in \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [**4, 10, 13**]).

The following well-known normality criterion was conjectured by Hayman [**5**] and proved by Gu [**3**].

Theorem A. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and let k be a positive integer. If, for every function $f \in \mathcal{F}$, $f \neq 0$, and $f^{(k)} \neq 1$ in D, then $\mathcal{F}$ is normal in D.

This result has undergone various extensions and improvements. In [**12**] (cf., [**7, 9**]) Xu obtained:

Theorem B. Let $\varphi(z)(\not\equiv 0)$ be a function holomorphic in a domain $D \subset \mathbb{C}$, $k \in \mathbb{N}$. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose poles are multiple and whose zeros all have multiplicity

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at least $k+2$. If, for every function $f \in \mathcal{F}$, $f^{(k)}(z) \neq \varphi(z)$, then $\mathcal{F}$ is normal in D.

Theorem C. Let $\varphi(z)$ ($\not\equiv 0$) be a function holomorphic in a domain $D \subset \mathbb{C}$, $k \in \mathbb{N}$. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros all have multiplicity at least $k+3$. If, for every function $f \in \mathcal{F}$, $f^{(k)}(z) \neq \varphi(z)$, then $\mathcal{F}$ is normal in D.

Theorem D. Let $\varphi(z)(\not\equiv 0)$ be a function holomorphic in a domain $D \subset \mathbb{C}$, $k \in \mathbb{N}$. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+2$. If, for every function $f \in \mathcal{F}$, $f^{(k)}(z) \neq \varphi(z)$, and $\varphi(z)$ has no simple zeros in D, then $\mathcal{F}$ is normal in D.

We remark that:

(1) the condition 'all of whose poles are multiple' in Theorem B is necessary;

(2) the number $k+3$ in Theorem C is best possible;

(3) the hypothesis '$\varphi(z)$ has no simple zeros' in Theorem D cannot be omitted.

These can be shown by the following example.

Example 1.1. Let $k \in \mathbb{N}$, $D = \{z : |z| < 1\}$, $\varphi(z) = z$ and

$$\mathcal{F} = \left\{ f_n(z) = \frac{1}{(k+1)!} \frac{(z-1/n)^{k+2}}{z-(k+2)/n} \right\}.$$

Since

$$f_n(z) = \frac{1}{(k+1)!} \left(z^{k+1} + P_{k-1}(z) + \frac{a}{z-(k+2)/n} \right),$$

where $P_{k-1}(z)$ is a polynomial of degree $k-1$ and $a \in \mathbb{C}\backslash\{0\}$, we have $f_n^{(k)}(z) \neq \varphi(z)$. Clearly, all zeros of f_n have multiplicity $k+2$, and all poles of f_n are simple. But $\mathcal{F}$ is not normal at $z = 0$.

In this paper, inspired by the idea in [**1, 6**], we prove the following result, which shows that the counterexample above is unique in some sense.

Theorem 1.2. *Let $\varphi(z)$ $(\not\equiv 0)$ be a function holomorphic in a domain D, $k \in \mathbb{N}$, and let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+2$ such that, for every function $f \in \mathcal{F}$, $f^{(k)}(z) \neq \varphi(z)$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then z_0 must be the simple zero of $\varphi(z)$, and there exist $\delta > 0$ and $\{f_n\} \subset \mathcal{F}$ such that*

$$f_n(z) = \frac{(z-\xi_n)^{k+2}}{(z-\eta_n)}\widehat{f}_n(z)$$

on $\Delta_\delta(z_0) = \{z : |z-z_0| < \delta\}$, where $(\xi_n - z_0)/\rho_n \to -c$, $(\eta_n - z_0)/\rho_n \to -(k+2)c$ for some sequence of positive numbers $\rho_n \to 0$ and some constant $c \neq 0$. Moreover, $\widehat{f}_n(z)$ is holomorphic and non-vanishing on $\Delta_\delta(z_0)$ such that $\widehat{f_n}(z) \to \widehat{f}(z)$ locally uniformly on $\Delta_\delta(z_0)$, where $\widehat{f}(z)$ satisfies $[(z-z_0)^{k+1}\hat{f}(z)]^{(k)} \equiv \varphi(z)$.

In this paper, we denote $\Delta_R = \{z : |z| < R\}$ and $\Delta'_R = \{z : 0 < |z| < R\}$ and drop the subscript when $R = 1$.

2. Lemmas. To prove our results, we need the following lemmas.

Lemma 2.1. ([**8**]). *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each function $f \in \mathcal{F}$ has only zeros with multiplicities at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \leq \alpha \leq k$, there exist a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^\alpha} \longrightarrow g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\#(\xi) \leq g^\#(0) = kA + 1$. Moreover, $g(\xi)$ has order at most 2.

Here, as usual, $g^\#(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

Lemma 2.2. ([**11**]). *Let f be a meromorphic function of finite order in the plane $\mathbb{C}$, k a positive integer. If all zeros of f are of multiplicity at least $k+2$ and $f^{(k)}(z) \neq 1$, then $f(z)$ is a constant.*

Lemma 2.3. ([**2**]). *Let f be a transcendental meromorphic function of finite order, and let $b(z)$ be a polynomial which does not vanish identically. If f has only multiple zeros, then $f'(z) - b(z)$ has infinitely many zeros.*

Lemma 2.4. ([**12**]). *Let f be a transcendental meromorphic function, $k \geq 2, l$ positive integers. If all zeros of f are of multiplicity at least 3, then $f^{(k)}(z) - z^l$ has infinitely many zeros.*

Lemma 2.5. ([**11**]). *Let f be a non-polynomial rational function and k a positive integer. If $f^{(k)}(z) \neq 1$, then*

$$f(z) = \frac{1}{k!} z^k + a_{k-1} z^{k-1} + \cdots + a_0 + \frac{a}{(z+b)^n},$$

where $a_{k-1}, \ldots, a_0, a(\neq 0), b$ are constants and n is a positive integer.

Lemma 2.6. *Let Q be a non-constant rational function and k, l positive integers. If all zeros of Q are of multiplicity at least $k+2$ and $Q^{(k)}(z) \neq z^l$, then $l = 1$ and*

$$Q(z) = \frac{1}{(k+1)!} \frac{(z+c)^{k+2}}{(z+(k+2)c)},$$

where c is a nonzero constant.

Proof. If Q is a polynomial, then $Q^{(k)}(z) - z^l$ is also a polynomial. Noting that $Q^{(k)}(z) \neq z^l$, then $Q^{(k)}(z) - z^l$ is a zero-free polynomial, and hence $\deg(Q^{(k)}(z) - z^l) = 0$ and $Q^{(k)}(z) - z^l$ is a nonzero constant. So, we may assume that $Q^{(k)}(z) = z^l + \alpha$, where α is a nonzero constant. Since all zeros of Q have multiplicity at least $k+2$, then $Q^{(k+1)}(z) = 0$ whenever $Q(z) = 0$. But $Q^{(k+1)}(z) = lz^{l-1}$ vanishes only for $z = 0$. Then $Q(0) = 0$, so that $\alpha = Q^{(k)}(0) = 0$, a contradiction. Thus Q is a non-polynomial rational function.

Set

$$f(z) = Q(z) - \frac{l!}{(k+l)!} z^{k+l} + \frac{1}{k!} z^k.$$

Then $f(z)$ is a non-polynomial rational function and $f^{(k)}(z) \neq 1$. By Lemma 2.5,

$$f(z) = \frac{1}{k!}z^k + a_{k-1}z^{k-1} + \cdots + a_0 + \frac{a}{(z+b)^n},$$

where $a_{k-1}, \ldots, a_0, a(\neq 0), b$ are constants and n is a positive integer. Thus,

$$Q(z) = \frac{l!}{(k+l)!}z^{k+l} + a_{k-1}z^{k-1} + \cdots + a_0 + \frac{a}{(z+b)^n}. \tag{1}$$

There exists a point z_0 such that $Q(z_0) = 0$. Since all zeros of Q have multiplicity at least $k+2$, we get

$$Q^{(k)}(z_0) = z_0^l + (-1)^k \frac{n(n+1)\cdots(n+k-1)}{(z_0+b)^{n+k}} = 0, \tag{2}$$

and

$$Q^{(k+1)}(z_0) = l z_0^{l-1} + (-1)^{k+1} \frac{n(n+1)\cdots(n+k)}{(z_0+b)^{n+k+1}} = 0. \tag{3}$$

We see that $z_0 \neq 0$ since $a \neq 0$. Solving for z_0 from (2) and (3), we obtain

$$z_0 = -\frac{bl}{n+k+l},$$

and $b \neq 0$. By (1), this is the only zero of $Q(z)$ of multiplicity $k+l+n$. From (1), we have $Q^{(k+l+1)}(z) \neq 0$. It follows that $n = 1$ and

$$Q(z) = \frac{l!}{(k+l)!}\frac{(z + bl/(k+l+1))^{k+l+1}}{(z+b)}.$$

Again, by (1), we get

$$\begin{aligned}(z + \frac{bl}{k+l+1})^{k+l+1} \equiv z^{k+l}(z+b) + \frac{(k+l)!a_{k-1}}{l!}z^{k-1}(z+b) + \cdots \\ + \frac{(k+l)!a_0}{l!}(z+b) + \frac{(k+l)!a}{l!}.\end{aligned}$$

Comparing the coefficients of z^{k+l} gives $bl = b$, so that $l = 1$ since $b \neq 0$. Then

$$Q(z) = \frac{1}{(k+1)!}\frac{(z + b/(k+2))^{k+2}}{(z+b)}.$$

Letting $c = b/(k+2)$, we get

$$Q(z) = \frac{1}{(k+1)!}\frac{(z+c)^{k+2}}{(z+(k+2)c)}.$$

Lemma 2.6 is thus proved. □

Lemma 2.7. *Let k be a positive integer, $\mathcal{F} = \{f_n\}$ a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+2$, and let $\{\varphi_n(z)\}$ be a sequence of holomorphic functions such that $\varphi_n(z) \to \varphi(z)(\neq 0)$ locally uniformly on D. If $f_n^{(k)}(z) \neq \varphi_n(z)$ for $z \in D$, then $\mathcal{F}$ is normal in D.*

Proof. Suppose $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 2.1, there exist a subsequence which we still denote by $\{f_n\}$ for convenience, complex points $z_n \to z_0$, and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \rho_n^{-k} f_n(z_n + \rho_n\zeta) \to g(\zeta),$$

locally uniformly on $\mathbb{C}$ with respect to the spherical metric, where $g(\zeta)$ is a nonconstant meromorphic function, all of whose zeros have multiplicity at least $k+2$, and $g(\zeta)$ has order at most 2.

Moreover, on every compact subset of $\mathbb{C}$ which contains no poles of $g(\zeta)$, we have

$$\begin{aligned} &f_n^{(k)}(z_n + \rho_n\zeta) - \varphi_n(z_n + \rho_n\zeta) \\ &\qquad = g_n^{(k)}(\zeta) - \varphi_n(z_n + \rho_n\zeta) \longrightarrow g^{(k)}(\zeta) - \varphi(z_0). \end{aligned}$$

Since $f_n^{(k)}(z_n+\rho_n\zeta) \neq \varphi_n(z_n+\rho_n\zeta)$, Hurwitz's theorem implies that either $g^{(k)}(\zeta) \equiv \varphi(z_0)$ or $g^{(k)}(\zeta) \neq \varphi(z_0)$ for any $\zeta \in \mathbb{C}\backslash\{g^{-1}(\infty)\}$. Clearly, these also hold for all $\zeta \in \mathbb{C}$.

If $g^{(k)}(\zeta) \equiv \varphi(z_0)$, then $g(\zeta)$ must be a polynomial of degree k, which contradicts the fact that all zeros of $g(\zeta)$ have multiplicity at least $k+2$. So $g^{(k)}(\zeta) \neq \varphi(z_0)$. Lemma 2.2 implies that $g(\zeta)$ is a constant, a contradiction. Lemma 2.7 is proved. □

3. Proof of Theorem 1. Since $\mathcal{F}$ is not normal at z_0, by Lemma 2.7, z_0 must be a zero of $\varphi(z)$. Without loss of generality, we assume

$D = \Delta = \{z : |z| < 1\}$, and

$$\varphi(z) = z^m \phi(z),$$

where $m \geq 1, \phi(0) = 1, \phi(z) \neq 0$ for all $z \in \Delta$. $\mathcal{F}$ is normal on Δ' but not normal at the origin.

Consider the family

$$\mathcal{G} = \left\{ g(z) = \frac{f(z)}{\varphi(z)} : f \in \mathcal{F} \right\}.$$

Since $f^{(k)}(0) \neq \varphi(0) = 0$, and all zeros of f have multiplicity at least $k+2$, we get that $f(0) \neq 0$. Thus, for each $g \in \mathcal{G}$, $g(0) = \infty$ with multiplicity at least m. Furthermore, for each $g \in \mathcal{G}$, $g(z)$ has zeros of multiplicity at least $k+2$.

Clearly, $\mathcal{G}$ is normal on Δ'. We claim that $\mathcal{G}$ is not normal at $z = 0$. Indeed, if $\mathcal{G}$ is normal at $z = 0$, then $\mathcal{G}$ is normal on the whole disk Δ and hence equicontinuous on Δ with respect to the spherical distance. On the other hand, $g(0) = \infty$ for each $g \in \mathcal{G}$, so there exists $\epsilon > 0$ such that, for every $g \in \mathcal{G}$ and every $z \in \Delta_\epsilon$, $|g(z)| \geq 1$. Then $f(z)$ is non-vanishing, and thus $1/f$ is holomorphic on Δ_ϵ for all $f \in \mathcal{F}$. Since $\mathcal{F}$ is normal on Δ' but not normal on Δ, the family $\mathcal{F}_1 = \{1/f, f \in \mathcal{F}\}$ is holomorphic on Δ_ϵ and normal on Δ'_ϵ, but it is not normal at $z = 0$. Therefore, there exists a sequence $\{1/f_n\} \subset \mathcal{F}_1$ which converges locally uniformly on Δ'_ϵ, but not in Δ_ϵ. Hence, by the maximum modulus principle, $1/f_n \to \infty$ on Δ'_ϵ. Thus, $f_n \to 0$ converges locally uniformly on Δ'_ϵ, and so does $\{g_n\} \subset \mathcal{G}$, where $g_n = f_n/\varphi$. But $|g_n(z)| \geq 1$ for $z \in \Delta_\epsilon$, a contradiction.

Then, by Lemma 2.1, there exist functions $\{g_n\} \subset \mathcal{G}$, complex points $z_n \to 0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n^k} \longrightarrow G(\zeta)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $G(\zeta)$ is a nonconstant meromorphic function with finite order, and all of whose zeros have multiplicity at least $k+2$.

By **[12,** pages 410–411], we can assume that $z_n/\rho_n \to \alpha$, a finite complex number. Then

$$\begin{aligned}\frac{g_n(\rho_n\zeta)}{\rho_n^k} &= \frac{g_n(z_n+\rho_n(\zeta - z_n/\rho_n))}{\rho_n^k}\\ &= G_n(\zeta - z_n/\rho_n) \longrightarrow G(\zeta-\alpha) = \widetilde{G}(\zeta).\end{aligned}$$

spherically uniformly on compact subsets of $\mathbb{C}$. Clearly, all zeros of $\widetilde{G}(\zeta)$ have multiplicity at least $k+2$, and $\widetilde{G}(0)=\infty$ with multiplicity at least m.

Set

$$H_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{k+m}}. \tag{4}$$

Then

$$H_n(\zeta) = \frac{\varphi(\rho_n\zeta)}{\rho_n^m}\frac{g_n(\rho_n\zeta)}{\rho_n^k} \longrightarrow \zeta^m\widetilde{G}(\zeta) = H(\zeta) \tag{5}$$

spherically uniformly on compact subsets of $\mathbb{C}$. Obviously, all zeros of $H(\zeta)$ have multiplicity at least $k+2$ and $H(0)\neq 0$ since $\widetilde{G}(0)=\infty$ with multiplicity at least m.

Now, we claim that $H^{(k)}(\zeta)\neq\zeta^m$. Indeed, by (4), we have

$$\begin{aligned}0 &\neq \frac{f_n^{(k)}(\rho_n\zeta)-\varphi(\rho_n\zeta)}{\rho_n^m}\\ &= H_n^{(k)}(\zeta) - \frac{\varphi(\rho_n\zeta)}{\rho_n^m} \longrightarrow H^{(k)}(\zeta)-\zeta^m\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$.

If there exists $\zeta_0\in\mathbb{C}$ such that $H^{(k)}(\zeta_0)=\zeta_0^m$, then H is holomorphic at ζ_0, and Hurwitz's theorem implies that $H^{(k)}(\zeta)\equiv\zeta^m$. Hence, $H(\zeta)$ is a polynomial with degree of $k+m$. $H^{(k)}(\zeta)=0$ whenever $H(\zeta)=0$, since all zeros of $H(\zeta)$ have multiplicity at least $k+2$. But $H^{(k)}(\zeta)=\zeta^m$ vanishes only for $\zeta=0$. Then we get $H(0)=0$, a contradiction.

Thus, $H^{(k)}(\zeta)\neq\zeta^m$. Lemma 2.3 (for $k=1$) and Lemma 2.4 (for $k\geq 2$) imply that $H(\zeta)$ must be a rational function. Then by

Lemma 2.6, we have $m = 1$, and

$$H(\zeta) = \frac{(\zeta + c)^{k+2}}{(k+1)!(\zeta + (k+2)c)}, \quad c \in \mathbb{C}\backslash\{0\}.$$

This together with (4) and (5) gives that

$$\frac{f_n(\rho_n\zeta)}{\rho_n^{k+1}} \longrightarrow \frac{(\zeta + c)^{k+2}}{(k+1)!(\zeta + (k+2)c)}. \tag{6}$$

Noting that all zeros of f_n have multiplicity at least $k+2$, there exists $\zeta_n \to -c$ and $\zeta_n' \to -(k+2)c$ such that $\xi_n = \rho_n\zeta_n$ is the zero of f_n with exact multiplicity $k+2$ and $\eta_n = \rho_n\zeta_n'$ is the simple pole of f_n.

Now write

$$f_n(z) = \frac{(z - \xi_n)^{k+2}}{z - \eta_n}\widehat{f}_n(z). \tag{7}$$

Then by (6) and (7), we get

$$\widehat{f}_n(\rho_n\zeta) \longrightarrow \frac{1}{(k+1)!} \tag{8}$$

on $\zeta \in \mathbb{C}$.

Claim 3.1. *There exists $\delta > 0$ such that $\widehat{f}_n(z) \neq 0$ on Δ_δ.*

Suppose not, taking a sequence and renumbering if necessary. $\widehat{f}_n$ has zeros tending to 0. Assume $\widehat{z}_n \to 0$ is the zero of $\widehat{f}_n$ with the smallest modulus. Then by (8), we see that $\widehat{z}_n/\rho_n \to \infty$.

Set

$$\widehat{f}_n^*(z) = \widehat{f}_n(\widehat{z}_n z). \tag{9}$$

Then $\widehat{f}_n^*(z)$ is well-defined on $\mathbb{C}$ and non-vanishing on Δ. Moreover, $\widehat{f}_n^*(1) = 0$.

Now, let

$$M_n(z) = \frac{(z - \xi_n/\widehat{z}_n)^{k+2}}{z - \eta_n/\widehat{z}_n}\widehat{f}_n^*(z). \tag{10}$$

By (7), (9) and (10), we have

$$M_n(z) = \frac{(z\widehat{z}_n - \xi_n)^{k+2}}{(z\widehat{z}_n - \eta_n)} \frac{\widehat{f}_n(\widehat{z}_n z)}{(\widehat{z}_n)^{k+1}} = \frac{f_n(\widehat{z}_n z)}{(\widehat{z}_n)^{k+1}}.$$

Obviously, all zeros of $M_n(z)$ have multiplicity at least $k+2$. Since $f_n^{(k)}(z) \neq \varphi(z)$, we obtain

$$M_n^{(k)}(z) - z\phi(\widehat{z}_n z) = (\widehat{z}_n)^{-1}(f_n^{(k)}(\widehat{z}_n z) - \varphi(\widehat{z}_n z)) \neq 0. \tag{11}$$

Hence, by applying Lemma 2.7, $\{M_n(z)\}$ is normal on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Noting that

$$\frac{\xi_n}{\widehat{z}_n} = \frac{\xi_n}{\rho_n} \frac{\rho_n}{\widehat{z}_n} \longrightarrow 0$$

and

$$\frac{\eta_n}{\widehat{z}_n} = \frac{\eta_n}{\rho_n} \frac{\rho_n}{\hat{z}_n} \longrightarrow 0,$$

we deduce from (10) that $\{\widehat{f}_n^*\}$ is also normal on $\mathbb{C}^*$. Thus, by taking a subsequence, we assume that $\widehat{f}_n^* \to \widehat{f}^*$ spherically locally uniformly on $\mathbb{C}^*$. Clearly, $\widehat{f}^*(z)$ has a zero at 1 with multiplicity at least $k+2$ since $\widehat{f}_n^*(1) = 0$.

Set

$$K_n(z) = M_n^{(k)}(z) - z\phi(\widehat{z}_n z). \tag{12}$$

Then, from (11), $K_n \neq 0$.

Now we prove that $\widehat{f}^*(z) \not\equiv 0$. Otherwise, $\widehat{f}_n^*(z) \to 0$; thus, $K_n(z) \to -z$ and $K_n'(z) \to -1$ locally uniformly on $\mathbb{C}^*$. By the argument principle, we have

$$\left| n(1, K_n) - n\left(1, \frac{1}{K_n}\right) \right| = \frac{1}{2\pi} \left| \int_{|z|=1} \frac{K_n'}{K_n} \, dz \right| \longrightarrow \frac{1}{2\pi} \left| \int_{|z|=1} \frac{1}{z} \, dz \right| = 1, \tag{13}$$

where $n(r, f)$ denotes the number of poles of f in Δ_r, counting multiplicity. It follows that $n(1, K_n) = 1$, which means that $K_n(z) = M_n^{(k)}(z) - z\phi(\widehat{z}_n z)$ has one simple pole, a contradiction.

Then $1/\widehat{f}_n^* \to 1/\widehat{f}^* \not\equiv \infty$ spherically locally uniformly on $\mathbb{C}^*$. Recalling that $\widehat{f}_n^*$ is non-vanishing on Δ, then $1/\widehat{f}_n^*$ is holomorphic on Δ. The maximum modulus principle yields $1/\widehat{f}_n^* \to 1/\widehat{f}^*$, and then $\widehat{f}_n^* \to \widehat{f}^*$ on Δ. Hence, $\widehat{f}_n^* \to \widehat{f}^*$ on $\mathbb{C}$.

By (10) and (12), we see that

$$K_n(z) \longrightarrow K(z) = (z^{k+1}\widehat{f}^*(z))^{(k)} - z$$

on $\mathbb{C}$. Since $K_n(z) \neq 0$, Hurwitz's theorem implies that either $K(z) \equiv 0$ or $K(z) \neq 0$. Since $\widehat{f}^*(z)$ has a zero at 1 with multiplicity at least $k+2$, we know that $K(1) = -1$. On the other hand, $\widehat{f}_n^*(0) = \widehat{f}_n(0) \to 1/(k+1)! = \widehat{f}^*(0)$, it follows that $K(0) = 0$. We arrive at a contradiction, and thus prove our claim.

We now proceed with our proof. Since $\{f_n\}$, and hence $\{\widehat{f}_n\}$ is normal on Δ', taking a subsequence and renumbering, we have $\widehat{f}_n \to \widehat{f}$ spherically locally uniformly on Δ'.

The proof follows our previous argument rather closely. We prove that $\widehat{f}(z) \not\equiv 0$ on Δ'. Otherwise, we have $f_n^{(k)}(z) \to 0$ and $f_n^{(k+1)}(z) \to 0$ locally uniformly on Δ'. Then the argument principle yields that:

$$\begin{aligned}\left| n\left(\frac{1}{2}, f_n^{(k)} - \varphi\right) - n\left(\frac{1}{2}, \frac{1}{f_n^{(k)} - \varphi}\right) \right| \\ = \frac{1}{2\pi}\left| \int_{|z|=\frac{1}{2}} \frac{f_n^{(k+1)} - \varphi'}{f_n^{(k)} - \varphi} dz \right| \to \frac{1}{2\pi}\left| \int_{|z|=\frac{1}{2}} \frac{\varphi'}{\varphi} dz \right| = 1.\end{aligned}$$

Since $f_n^{(k)}(z) \neq \varphi(z)$, we have $n(\frac{1}{2}, f_n^{(k)}) = n(\frac{1}{2}, f_n^{(k)} - \varphi) = 1$, which is impossible.

Hence, $1/\widehat{f}_n \to 1/\widehat{f} \not\equiv \infty$ spherically locally uniformly on Δ'. Recall that $\widehat{f}_n(z) \neq 0$ on Δ_δ, $1/\widehat{f}_n$ is holomorphic on Δ_δ. By the maximum modulus principle, $1/\widehat{f}_n \to 1/\widehat{f}$, and hence $\widehat{f}_n \to \widehat{f}$ spherically locally uniformly on Δ. Since $\widehat{f}_n(0) \to 1/(k+1)!$, we have $\widehat{f}(0) = 1/(k+1)!$, so $\widehat{f}$ is holomorphic at 0. Moreover, there exists $\delta' > 0$ such that each $\widehat{f}_n$ is holomorphic on $\Delta_{\delta'}$.

By (7), we obtain $f_n(z) \to z^{k+1}\widehat{f}(z)$ on Δ. Thus,

$$f_n^{(k)}(z) - \varphi(z) \to [z^{k+1}\widehat{f}(z)]^{(k)} - \varphi(z), \tag{14}$$

on $\Delta\backslash(\widehat{f}^{-1}(\infty))$.

If $[z^{k+1}\widehat{f}(z)]^{(k)} - \varphi(z) \not\equiv 0$, by the maximum modulus principle (14) still holds on Δ since $f_n^{(k)}(z) \neq \varphi(z)$. Hurwitz's theorem implies that $[z^{k+1}\widehat{f}(z)]^{(k)} - \varphi(z) \neq 0$, violating the fact that $[(z^{k+1}\widehat{f}(z))^{(k)} - \varphi(z)]|_{z=0} = 0$. Hence, $[z^{k+1}\widehat{f}(z)]^{(k)} \equiv \varphi(z)$. The proof of Theorem 1 is completed. □

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