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SOLUTIONS AND MULTIPLE SOLUTIONS FOR SECOND ORDER PERIODIC SYSTEMS WITH A NONSMOOTH POTENTIAL

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ABSTRACT. A nonautonomous second order system with a nonsmooth potential is studied. Using the nonsmooth critical point theory, first an existence theorem is proved. Then, by strengthening the hypotheses on the nonsmooth potential, a multiplicity theorem is proved using the nonsmooth second deformation. The hypotheses on the nonsmooth potential make the Euler functional of the problem bounded below but do not make it coercive. Moreover, the analytical framework of the paper incorporates strongly resonant periodic systems.

1. Introduction. In this paper, we consider the following second order periodic system with a nonsmooth potential:

$$(1)\qquad \begin{cases} -x''(t) - A(t)x(t) \in \partial j(t, x(t)) & \text{a.e. on T=[0,b]}, \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases}$$

Here $t \to A(t)$ is a continuous map from $T = [0, b]$ with $b > 0$ into the space of $N \times N$-symmetric matrices and $j(t, x)$ is a measurable function defined on $T \times \mathbf{R}^N$, which is locally Lipschitz and in general nonsmooth in the $x \in \mathbf{R}^N$ variable. By $\partial j(t, x)$ we denote the generalized (Clarke) subdifferential of $x \to j(t, x)$ (see Section 2).

Rabinowitz [**17**] examined problem (1) under the assumptions that, for every $t \in T$, $A(t)$ is strictly negative definite, $j \in C^1(T \times \mathbf{R}^N)$ and $j(t, \cdot)$ exhibits a strictly superquadratic growth. He proved an existence result using the saddle point theorem. Note that the hypothesis that $A(t)$ is strictly negative definite implies that the spectral decomposition of the nonlinear differential operator has trivial negative and zero parts. Mawhin [**12**] (see also Mawhin and Willem [**13**]) assumed that, for all

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$t \in T$, $A(t) = A$ (i.e., the matrix is time-invariant), but did not impose any sign condition on A. He also assumed that the potential $j(t,x)$ is measurable, continuously differentiable in the $x \in \mathbf{R}^N$ variable, and

$$|j(t,x)| \leq h(t), \ \|\nabla j(t,x)\| \leq h(t) \text{for almost all } t \in T \text{ and all } x \in \mathbf{R}^N,$$

where $h \in L^1(T)_+$. His approach is variational, and he used the saddle point theorem. In Mawhin-Willem [**13**, page 63], the same problem was studied with $A(t) = m^2\omega^2 I$ for all $t \in T$, with $m \geq 1$ an integer, $\omega = 2\pi/b$ and I is the $N \times N$-unit matrix. In this case, the potential function $j(t,x)$ is measurable, and it is C^1 and convex in the $x \in \mathbf{R}^N$ variable. More recently, Tang and Wu [**20**] examined problem (1) with a potential function $j(t,x)$ which is measurable and $j(t,\cdot)$ is C^1 and strictly subquadratic. All the aforementioned works deal with the existence problem and do not address the multiplicity of solutions. Multiplicity results for problem (1) can be found in Barletta and Livrea [**1**], Bonanno and Livrea [**3**], Cordaro [**6**], Faraci [**8**] (problems with a smooth potential) and in Barletta and Papageorgiou [**2**] and Motreanu, Motreanu and Papageorgiou [**14, 15**] (problems with a nonsmooth potential). In [**1, 3, 6, 8**], the authors assume $(A(t)x,x)_{\mathbf{R}^N} \leq -\beta\|x\|^2$ for all $t \in T$, all $x \in \mathbf{R}^N$, with $\beta > 0$ (hence, in the spectral decomposition, the negative and zero parts are trivial) and the method of the proof is similar, based on an abstract multiplicity result of Ricceri [**18**] or variants of it. In [**14**], in the spectral decomposition, the negative part is trivial, the nonsmooth potential $j(t,\cdot)$ is quadratic, and the approach is variational based on the nonsmooth critical point theory. In [**15**], the authors allow all components in the spectral decomposition to be nontrivial, the potential $j(t,\cdot)$ is subquadratic and the method of proof relies on a nonsmooth version of the reduction technique. Finally, the very recent work of the authors [**2**] assumes an unbounded from below potential, and the approach is based on the nonsmooth local linking theorem of Kandilakis, Kourogenis and Papageorgiou [**11**]. Their hypotheses exclude from consideration strongly resonant systems. In contrast, our setting here incorporates them. Moreover, our approach here is different and exploits the nonsmooth second deformation theorem due to Corvellec [**7**]. We prove an existence theorem and a multiplicity theorem (existence of two nontrivial solutions) under hypotheses that make the Euler functional bounded from below, but not coercive. This is in sharp contrast with [**2**], where the Euler functional is coercive.

The structure of the paper is the following. In Section 2, we present the mathematical tools that we will use in the sequel. In Section 3, for a more general version of problem (1), we prove an existence theorem, guaranteeing a nontrivial solution. Finally, in Section 4, we state and prove the multiplicity theorem for problem (1).

2. Mathematical background. We make the following assumption on the matrix-valued map $t \to A(t)$.

$\underline{H(A)}$: $A : T \to \mathbf{R}^{N\times N}$ is a continuous map and, for every $t \in T$, $A(t)$ is symmetric.

We will use the Sobolev space:

$$W^{1,2}_{per}((0,b),\mathbf{R}^N) = \{x \in W^{1,2}((0,b),\mathbf{R}^N) : x(0) = x(b)\}.$$

Since $W^{1,2}((0,b),\mathbf{R}^N)$ is (compactly) embedded into $C(T,\mathbf{R}^N)$, we see that the evaluations at $t = 0$ and $t = b$ make sense. In what follows, by $\|\cdot\|$, we denote both the Sobolev norm and the Euclidean norm on $\mathbf{R}^N$. It will always be clear from the context which one we use. Also $\|\cdot\|_p$ is the usual norm on $L^p(T,\mathbf{R})$ or $L^p(T,\mathbf{R}^N)$, $1 \le p < \infty$ and $\|\cdot\|_\infty$ stands for the norm on $C(T,\mathbf{R}^N)$. Let $\widehat{A} \in \mathcal{L}(C(T,\mathbf{R}^N), C(T,\mathbf{R}^N))$ be the Nemytskii operator corresponding to $A(\cdot)$, i.e.,

$$(\widehat{A}x)(t) = A(t)x(t) \quad \text{for all } x \in C(T,\mathbf{R}^N) \text{ and all } t \in T.$$

From Mawhin-Willem [**13**, page 89] and Showalter [**19**, page 78], using the spectral theorem for compact, self-adjoint operators on a Hilbert space, for the differential operator $x \to -x'' - \widehat{A}x$, we know that there is a sequence of eigenfunctions which form an orthonormal basis for $\mathcal{L}^2(T,\mathbf{R}^N)$ and an orthogonal basis for $W^{1,2}_{per}((0,b),\mathbf{R}^N)$. Hence, we have the following orthogonal direct sum decomposition:

$$W^{1,2}_{per}((0,b),\mathbf{R}^N) = H_- \oplus H_0 \oplus H_+,$$

where

$$\begin{aligned} H_- &= \operatorname{span}\{x \in W^{1,2}_{per}((0,b),\mathbf{R}^N) : -x'' - \widehat{A}(x) \\ &= \lambda x \text{ for some } \lambda < 0\}, \\ H_0 &= \ker(-x'' - \widehat{A}x) \end{aligned}$$

and

$$H_+ = \overline{\text{span}}\,\{x \in W^{1,2}_{per}((0,b),\mathbf{R}^N) : -x'' - \widehat{A}(x) \\ = \lambda x \text{ for some } \lambda > 0\}.$$

Note that H_- and H_0 are both finite dimensional. In this paper, we assume that $\dim H_- = 0$. We know that there exists a $\xi_0 > 0$ such that

$$(2) \qquad \|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbf{R}^N}\, dt \geq \xi_0 \|x\|^2 \quad \text{for all } x \in H_+$$

(see [**14**]). Let $\{\lambda_n\}_{n\geq 1}$ be the eigenvalues of $x \to -x'' - \widehat{A}x$ repeated according to the multiplicity. Using (2), we can see that there is a smallest positive eigenvalue $\lambda_m > 0$.

Now, let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi : X \to \mathbf{R}$ be a locally Lipschitz function. The generalized directional derivative $\varphi^0(x;h)$ of φ at $x \in X$, in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $h \to \varphi^0(x;h)$ is sublinear continuous. Therefore, it is the support function of a nonempty, w^*-compact, convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq \varphi^0(x;h) \quad \text{for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is the generalized (Clarke) subdifferential of φ. We say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. The locally Lipschitz function $\varphi : X \to \mathbf{R}$ satisfies the Palais-Smale condition at level $c \in \mathbf{R}$ (the PS_c-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \quad \text{and } m(x_n) = \inf[\|x^*\|_* : x^* \in \partial\varphi(x_n)] \longrightarrow 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence. If the PS_c-condition holds at every level $c \in \mathbf{R}$, then we say that φ satisfies the PS-condition.

For a given locally Lipschitz function $\varphi : X \to \mathbf{R}$, we introduce the following sets:

$$\dot{\varphi}^c = \{x \in X : \varphi(x) < c\} \text{ (the strict sublevel set of } \varphi \text{ at } c \in \mathbf{R}),$$
$$K = \{x \in X : 0 \in \partial\varphi(x)\} \text{ (the critical set of } \varphi),$$
$$K_c = \{x \in K : \varphi(x) = c\} \text{ (the critical set of } \varphi \text{ at the level } c \in \mathbf{R}).$$

The next theorem is a nonsmooth version of the so-called "second deformation theorem" (see [**4**, page 23] and [**10**, page 628]), and it is due to Corvellec [**7**]. In fact, the result of Corvellec is formulated in the more general context of metric spaces for continuous functions and using the so-called weak slope. However, for our purposes the following particular version suffices.

Theorem 2.1. *If X is a Banach space, $\varphi : X \to \mathbf{R}$ is locally Lipschitz $-\infty < a < b < +\infty$ and φ satisfies the PS_c-condition for every $c \in (a, b)$, φ has no critical points in $\varphi^{-1}(a,b)$ and K_a is a finite set consisting of local minima, then there exists a continuous deformation $h : [0,1] \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that*

(a) $h(t,\cdot)_{|K_a} = id_{|K_a}$ *for all* $t \in [0,1]$;

(b) $h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a$;

(c) $\varphi(h(t,x)) \leq \varphi(x)$ *for all* $t \in [0,1]$, *and all* $x \in \dot{\varphi}^b$.

Remark 2.1. In particular, the set $\dot{\varphi}^a \cup K_a$ is a weak deformation retract of $\dot{\varphi}^b$. In the smooth second deformation theorem (see [**4**, page 23] and [**10**, page 628]), the conclusion is that φ^a is a strong deformation retract of $\varphi^b \setminus K_b$, where $\varphi^b = \{x \in X : \varphi(x) \leq b\}$ (the sublevel set of φ at b).

For more details on the subdifferential theory of locally Lipschitz functions and on the nonsmooth critical point theory, we refer to the books of Clarke [**5**], Gasinski and Papageorgiou [**9**] and Motreanu and Radulescu [**16**].

3. Existence theorem. Recall that, throughout this work, we assume that $\dim H_- = 0$. So, we have

$$W^{1,2}_{per}((0,b), \mathbf{R}^N) = H_0 \oplus H_+.$$

Then every $x \in W^{1,2}_{per}((0,b),\mathbf{R}^N)$ has a unique decomposition $x = \overline{x}+\widehat{x}$, with $\overline{x} \in H_0$ and $\widehat{x} \in H_+$.

Let $h \in L^1(T,\mathbf{R}^N)$ be such that

$$\int_0^b (h(t),u(t))_{\mathbf{R}^N}\,dt = 0 \quad \text{for all } u \in H_0. \tag{3}$$

We consider the following second order system

(4)
$$\begin{cases} -x''(t) - A(t)x(t) \in \partial j(t,x(t)+h(t)) & \text{almost everywhere on } T, \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases}$$

Of course, (1) is a particular case of (4), when $h \equiv 0$. In this section we prove an existence theorem for problem (4). To do this, we impose the following hypotheses on the nonsmooth potential $j(t,x)$.

$\underline{H(j)}$: $j : T \times \mathbf{R}^N \to \mathbf{R}$ is a function such that $j(t,0) = 0$ almost everywhere on T and

(i) for all $x \in \mathbf{R}^N$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) there exist functions $a \in L^1(T)_+$ and $c \in C(\mathbf{R}_+)_+$ such that, for almost all $t \in T$, all $x \in \mathbf{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \le a(t)c(\|x\|);$$

(iv) there exists a function $\xi \in L^1(T)_+$ such that $j(t,x) \le \xi(t)$ almost everywhere on T, for all $x \in \mathbf{R}^N$;

(v) $\beta = \int_0^b \limsup_{\|x\|\to\infty} j(t,x)dt < +\infty$ and there exists a $\delta_0 > 0$ such that

$$j(t,x) > 0 \quad \text{for almost all } t \in T \text{ and all } 0 < \|x\| \le \delta.$$

Example 3.1. The following function satisfies hypothesis $H(j)$. For the sake of simplicity, we drop the $t-$dependence.

$$j(x) = \begin{cases} c\|x\|^3/3 & \text{if } \|x\| \le 1 \\ c/3\|x\| & \text{if } \|x\| > 1, \end{cases} \qquad c > 0.$$

The Euler functional $\varphi_1 : W^{1,2}_{per}((0,b),\mathbf{R}^N) \to \mathbf{R}$ for problem (4) is defined by

$$\varphi_1(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbf{R}^N}\,dt \\ - \int_0^b j(t,x(t))\,dt - \int_0^b (h(t),x(t))_{\mathbf{R}^N}\,dt.$$

From Clarke [**5**, page 83], we know that φ_1 is Lipschitz continuous on bounded sets; hence, it is locally Lipschitz.

Consider also the following auxiliary system:

$$(5)\quad \begin{cases} -x''(t) - A(t)x(t) = h(t) & \text{almost everywhere on } T=[0,b], \\ x(0)=x(b),\ x'(0)=x'(b). \end{cases}$$

Using (2) and (3), we deduce the following existence result for problem (5):

Proposition 3.1. *Problem* (5) *admits a unique solution* $\widehat{x}_0 \in H_+$, *which is a global minimizer of the functional* $\Psi : H_+ \to \mathbf{R}$ *defined by*

$$\Psi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbf{R}^N}\,dt \\ - \int_0^b (h(t),x(t))_{\mathbf{R}^N}\,dt.$$

Observe that

$$(6)\qquad \varphi_1(x) = \Psi(\widehat{x}) - \int_0^b j(t,x(t))\,dt \quad \text{for all } x \in W^{1,2}_{per}((0,b),\mathbf{R}^N)$$

where $\widehat{x} = \mathrm{proj}_{H_+}(x)$ is the projection of x on H_+.

As we already mentioned in the introduction, our analytical framework incorporates strongly resonant systems. Therefore, we do not expect that φ_1 satisfies the global PS-condition. This is reflected in the next proposition.

Proposition 3.2. *If hypotheses $H(A)$ and $H(j)$ hold and $c < \Psi(\widehat{x}_0) - \beta$, then φ_1 satisfies the PS_c-condition.*

Proof. Consider a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbf{R}^N)$ such that

$$\varphi_1(x_n) \longrightarrow c \quad \text{and} \quad m_1(x_n) \longrightarrow 0 \text{ as } n \to \infty, \tag{7}$$

where $m_1(x_n) = \inf[\|x^*\|_* : x^* \in \partial\varphi_1(x_n)]$. Since $\partial\varphi_1(x_n) \subseteq W^{1,2}_{per}((0,b),\mathbf{R}^N)$ is weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi_1(x_n)$ such that $m_1(x_n) = \|x_n^*\|$, for all $n \geq 1$. We have

$$x_n^* = V(x_n) - \widehat{A}(x_n) - u_n - h \quad \text{with} \quad u_n \in N(x_n), \tag{8}$$

where $V \in \mathcal{L}(W^{1,2}_{per}((0,b),\mathbf{R}^N), W^{1,2}_{per}((0,b),\mathbf{R}^N)^*)$ is defined by

$$\langle V(x), y\rangle = \int_0^b (x'(t), y'(t))_{\mathbf{R}^N}\, dt \quad \text{for all} \quad x,\, y \in W^{1,2}_{per}((0,b),\mathbf{R}^N)$$

and $N : W^{1,2}_{per}((0,b),\mathbf{R}^N) \to 2^{L^1(T,\mathbf{R}^N)}$ is the multifunction defined by

$$N(x) = \{u \in L^1(T,\mathbf{R}^N) : u(t) \in \partial j(t,x(t)) \quad \text{almost everywhere on } T\}.$$

Note that, since $(t,x) \to \partial j(t,x)$ is a graph measurable multifunction (see [**9**]), for every $x \in W^{1,2}_{per}((0,b),\mathbf{R}^N$, $t \to \partial j(t,x)$ is a graph measurable multifunction and so, by the Yankov-von Neumann-Aumann selection theorem (see [**9**, page 23]), it admits a measurable selector $u : T \to \mathbf{R}^N$. By virtue of hypothesis $H(j)(iii)$, $u \in L^1(T,\mathbf{R}^N)$. So, the multifunction N has nonempty values.

Exploiting the orthogonality of component spaces H_0 and H_+ and using (3) and (7), we see that, for some $M_1 > 0$ and all $n \geq 1$, we have

$$M_1 \geq \varphi_1(x_n) = \frac{1}{2}\|\widehat{x}_n'\|_2^2 - \frac{1}{2}\int_0^b (A(t)\widehat{x}_n(t), \widehat{x}_n(t))_{\mathbf{R}^N}\, dt$$
$$- \int_0^b j((t), x_n(t))\, dt - \int_0^b (h(t), \widehat{x}_n(t))_{\mathbf{R}^N}\, dt,$$

(recall that $x_n = \overline{x}_n + \widehat{x}_n$, with $\overline{x}_n \in H_0$, $\widehat{x}_n \in H_+$). Hence, using hypothesis $H(j)(iv)$ and (2), we obtain

$$\begin{aligned} M_1 &\geq \Psi(\widehat{x}_n) - \int_0^b j((t), x_n(t))\,dt \\ &\geq \Psi(\widehat{x}_n) - \|\xi\|_1 \\ &\geq \xi_0\|\widehat{x}_n\|^2 - c_1\|\widehat{x}_n\| - \|\xi\|_1 \\ &\qquad\qquad \text{for some } c_1 > 0, \text{ all } n \geq 1; \end{aligned}$$

hence,

$$\{\widehat{x}_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbf{R}^N) \quad \text{is bounded.} \tag{9}$$

Suppose that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbf{R}^N)$ is not bounded. Then we may assume that $\|x_n\| \to +\infty$; hence, $\|\overline{x}_n\| \to +\infty$ (see (9)). Set $y_n = \overline{x}_n/\|\overline{x}_n\|$, $n \geq 1$. Then $y_n \in H_0$ and $\|y_n\| = 1$. Because H_0 is finite dimensional, by passing to a subsequence, if necessary, we may assume that $y_n \to y$ in $W^{1,2}_{per}((0,b),\mathbf{R}^N)$; from this, we deduce that $\|y\| = 1$ and $y \in H_0$. So $y(t) \neq 0$ for almost all $t \in T$ (see [**20**]); hence,

$$\lim_{n\to+\infty} \|\overline{x}_n(t)\| \to +\infty \quad \text{for almost all } t \in T. \tag{10}$$

On the other hand, since $W^{1,2}_{per}((0,b),\mathbf{R}^N)$ is compactly embedded in $C(T,\mathbf{R}^N)$, by (9) we can find an $M_2 > 0$ such that

$$\|\widehat{x}_n(t)\| \leq M_2 \quad \text{for all } t \in T \text{ and all } n \geq 1. \tag{11}$$

Then

$$\begin{aligned} \|x_n(t)\| \geq \|\overline{x}_n(t)\| - \|\widehat{x}_n(t)\| &\geq \|\overline{x}_n(t)\| - M_2 \\ &\text{for all } t \in T \text{ and all } n \geq 1, \end{aligned}$$

so, from (10) we deduce that

$$\lim_{n\to+\infty} \|x_n(t)\| \longrightarrow +\infty \quad \text{for almost all } t \in T. \tag{12}$$

Because of (6) and Proposition 3.2, we have

$$\varphi_1(x_n) = \Psi(\widehat{x}_n) - \int_0^b j(t, x_n(t))\,dt \geq \Psi(\widehat{x}_0) - \int_0^b j(t, x_n(t))\,dt.$$

Passing to the limit as $n \to +\infty$, using (7), Fatou's lemma and (12), we have

$$\begin{aligned} c &\geq \Psi(\widehat{x}_0) - \limsup_{n\to+\infty} \int_0^b j(t, x_n(t))\,dt \\ &\geq \Psi(\widehat{x}_0) - \int_0^b \limsup_{n\to+\infty} j(t, x_n(t))\,dt \\ &= \Psi(\widehat{x}_0) - \beta, \end{aligned}$$

which contradicts the choice of c. This proves that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b), \mathbf{R}^N)$ is bounded. So, we may assume that $x_n \rightharpoonup x$ in $W^{1,2}_{per}((0,b), \mathbf{R}^N)$ and $x_n \to x$ in $C(T, \mathbf{R}^N)$ (by $\rightharpoonup$ we denote the weak convergence of the sequence in the relevant space). From (7) and (8), we have

$$\tag{13} \begin{aligned} &\left| \langle V(x_n), x_n - x\rangle - \int_0^b (Ax_n, x_n - x)_{\mathbf{R}^N}\,dt - \int_0^b (u_n + h, x_n - x)_{\mathbf{R}^N}\,dt \right| \\ &\qquad\qquad\qquad\qquad \leq \varepsilon_n \|x_n - x\| \quad \text{with } \varepsilon_n \downarrow 0. \end{aligned}$$

Note that hypotheses $H(A)$ and $H(j)(iii)$ guarantee that

$$\begin{aligned} \lim_{n\to+\infty} \int_0^b (Ax_n, x_n - x)_{\mathbf{R}^N}\,dt &= \lim_{n\to+\infty} \int_0^b (u_n, x_n - x)_{\mathbf{R}^N}\,dt \\ &= \lim_{n\to+\infty} \int_0^b (h, x_n - x)_{\mathbf{R}^N}\,dt = 0, \end{aligned}$$

so, from (13), it follows that

$$\tag{14} \lim_{n\to+\infty} \langle V(x_n), x_n - x\rangle = 0.$$

We have $V(x_n) \rightharpoonup V(x)$ in $W^{1,2}_{per}((0,b), \mathbf{R}^N)^*$, so, from (14), we obtain

$$\lim_{n\to+\infty} \|x_n'\|_2 = \|x'\|_2.$$

Since $x_n' \rightharpoonup x'$ in $L^2(T, \mathbf{R}^N)$, from the Kadec-Klee property of Hilbert spaces, we have $x_n' \to x'$ in $L^2(T, \mathbf{R}^N)$, and so we conclude that $x_n \to x$ in $W^{1,2}_{per}((0,b), \mathbf{R}^N)$. Therefore, φ_1 satisfies the nonsmooth PS_c-condition for all $c < \Psi(\widehat{x}_0) - \beta$. $\square$

With this proposition, we can now state and prove an existence theorem for problem (4).

Theorem 3.1. *If* $\dim H_0 \neq 0$, *hypotheses* $H(A)$ *and* $H(j)$ *hold and* $\beta < \int_0^b j(t, \widehat{x}_0(t))\,dt$, *then problem* (4) *admits a nontrivial solution* $y_0 \in C^1_{per}(T, \mathbf{R}^N) = \{y \in C^1(T, \mathbf{R}^N) : y(0) = y(b), y'(0) = y'(b)\}$.

Proof. Taking into account the definition of φ_1 and using Proposition 3.1 and hypothesis $H(j)(iv)$, we see that

$$\varphi_1(x) = \Psi(\widehat{x}) - \int_0^b j(t, x(t))\,dt \geq \Psi(\widehat{x}_0) - \|\xi\|_1;$$

hence, φ_1 is bounded below. Therefore, $-\infty < \inf \varphi_1 = \widehat{m}_1$. Also

$$-\infty < \widehat{m}_1 \leq \varphi_1(\widehat{x}_0) = \Psi(\widehat{x}_0) - \int_0^b j(t, \widehat{x}_0(t))\,dt < \Psi(\widehat{x}_0) - \beta, \tag{15}$$

so Proposition 3.2 ensures that φ_1 satisfies the nonsmooth $PS_{\widehat{m}_1}$-condition.

From Gasinski and Papageorgiou [**9**, page 144], we infer that there exists a $y_0 \in W^{1,2}_{per}((0, b), \mathbf{R}^N)$ such that

$$\varphi_1(y_0) = \widehat{m}_1. \tag{16}$$

Since y_0 is a critical point, one has $\varphi_1'(y_0) = 0$, so we can find $u_0 \in N(y_0)$ such that

$$V(y_0) - \widehat{A}(y_0) = u_0 + h;$$

exploiting the previous inequality, we infer

$$-y_0''(t) - A(t)y_0(t) = u_0(t) + h(t) \quad \text{almost everywhere on } T,$$
$$y_0(0) = y_0(b), \quad y_0'(0) = y_0'(b).$$

Namely, y_0 is a solution of (4) and $y_0 \in C^1_{per}(T, \mathbf{R}^N)$ (see for example [**2**]).

Let $\delta > 0$ be as in hypothesis $H(j)(v)$, and let $v \in H_0$, $v \neq 0$, be such that $\|v\|_\infty \leq \delta$. Then we have $v(t) \neq 0$ almost everywhere on T and

$$\varphi(v) = -\int_0^b j(t, v(t))\, dt < 0.$$

Therefore, $\widehat{m}_1 < 0 = \varphi_1(0)$ and so, from (16), we conclude that y_0 is nontrivial. $\square$

4. Multiplicity theorem. In this section, by strengthening hypotheses $H(j)$ and assuming that $h \equiv 0$, we prove a multiplicity theorem for problem (1). Since $h \equiv 0$, we have $\widehat{x}_0 = 0$ and $\Psi(\widehat{x}_0) = \int_0^b j(t, \widehat{x}_0(t))\, dt = 0$. We will need two new sets of hypotheses for the nonsmooth potential $j(t,x)$, depending on whether $\dim H_0 \neq 0$ or $\dim H_0 = 0$.

If $\dim H_0 \neq 0$, then we will need the following stronger version of hypotheses $H(j)$.

$\underline{H(j)'}$: $j : T \times \mathbf{R}^N \to \mathbf{R}$ is a function such that $j(t,0) = 0$ almost everywhere on T, hypotheses $H(j)'(i) \to (v)$ are the same as the corresponding hypotheses $H(j)(i) \to (v)$, and

(vi) $\beta = \int_0^b \limsup_{\|x\| \to +\infty} j(t,x)\, dt \leq 0$ and $\liminf_{x \to 0} (j(t,x))/\|x\|^2 \geq \mu > 0$ uniformly for almost all $t \in T$.

(vii) $j(t,x) \leq (\lambda_m/2)|x\|^2$ for almost all $t \in T$ and all $x \in \mathbf{R}^N$.

If $\dim H_0 = 0$, then $\lambda_m = \lambda_1 > 0$, and we will need the following set of hypotheses on $j(t,x)$.

$\underline{H(j)''}$: $j : T \times \mathbf{R}^N \to \mathbf{R}$ is a function such that $j(t,0) = 0$ almost everywhere on T, hypotheses $H(j)''$(i), (ii), (iii), (v) are the same as the corresponding hypotheses $H(j)'(i)$, (ii), (iii), (v), and

(iv) $\limsup_{\|x\| \to +\infty} (2j(t,x))/\|x\|^2 \leq \theta(t)$ uniformly for almost all $t \in T$, with $\theta \in L^1(T)_+$, $\theta(t) \leq \lambda_1$ almost everywhere on T, $\theta \neq \lambda_1$.

(vi) there exist $\delta > 0$ and $\eta \in L^1(T)_+$ such that $\eta(t) \geq \lambda_1$ almost everywere on T, $\eta \neq \lambda_1$ and $(\lambda_2/2)\|x\|^2 \geq j(t,x) \geq (\eta(t)/2)\|x\|^2$ for almost all $t \in T$ and all $\|x\| \leq \delta$.

Example 4.1. The following potential function satisfies hypotheses $H(j)'$:

$$j_1(x) = \begin{cases} (\lambda_m/2)\|x\|^2 & \text{if } \|x\| \leq 1 \\ (\lambda_m)/2\|x\| & \text{if } \|x\| > 1. \end{cases}$$

If θ, $\eta \in L^1(T)_+$ are such that $\theta(t) \leq \lambda_1 \leq \eta(t) \leq \lambda_2$ almost everywhere on T, $\theta \neq \lambda_1$, $\eta \neq \lambda_1$ then the function

$$j_2(t,x) = \begin{cases} (\eta(t)/2)\|x\|^2 & \text{if } \|x\| \leq 1 \\ (\theta(t)/2)\|x\|^2 + (\eta(t) - \theta(t))/2 & \text{if } \|x\| > 1, \end{cases}$$

satisfies hypotheses $H(j)''$. Note that both $j_1(x)$ and $j_2(t,x)$ fail to satisfy hypotheses $H(j)_3$ in the multiplicity result of [**2**].

The Euler functional $\varphi : W^{1,2}_{per}((0,b), \mathbf{R}^N) \to \mathbf{R}$ for problem (1) is defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbf{R}^N}\, dt - \int_0^b j(t, x(t))dt.$$

From Clarke [**5**, page 83], we know that φ is locally Lipschitz.

Theorem 4.1. *If* $\dim H_0 \neq 0$ *and hypotheses* $H(A)$, $H(j)'$ *hold, or* $\dim H_0 = 0$ *and hypotheses* $H(A)$, $H(j)''$ *hold, then problem* (1) *has at least two nontrivial solutions* y_0, $v_0 \in C^1_{per}(T, \mathbf{R}^N)$.

Proof. First suppose that $\dim H_0 \neq 0$ and hypotheses $H(j)'$ are in effect. Then, from Theorem 3.1, we already have one nontrivial solution $y_0 \in C^1_{per}(T, \mathbf{R}^N)$. Hypothesis $H(j)'(v)$ implies that, given $\varepsilon \in (0, \mu)$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$j(t,x) \geq (\mu - \varepsilon)\|x\|^2 \quad \text{for almost all } t \in T \text{ and all } \|x\| \leq \delta. \tag{17}$$

Since $H_0 \subseteq C(T, \mathbf{R}^N)$, we can find $r > 0$ such that, if $x \in H_0$ and $\|x\| \leq r$, then $\|x\|_\infty \leq \delta$. Hence, for such an x, we have

$$\varphi(x) = -\int_0^b j(t, x(t))\, dt \leq (\varepsilon - \mu)\|x\|_2^2,$$

so

$$\eta_r = \max_{\partial B_r \cap H_0} \varphi < 0, \tag{18}$$

where $\partial B_r = \{x \in W^{1,2}_{per}((0,b), \mathbf{R}^N) : \|x\| = r\}$. On the other hand, if we take $x \in H_+$ and use hypothesis $H(j)'(vii)$, then

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbf{R}^N}\, dt \\ &\quad - \int_0^b j(t, x(t))\, dt \\ &\geq \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbf{R}^N}\, dt \\ &\quad - \frac{\lambda_m}{2}\|x\|_2^2 \geq 0,\end{aligned}$$

so

$$\inf_{H_+} \varphi = 0. \tag{19}$$

Let $B_r = \{x \in W^{1,2}_{per}((0,b), \mathbf{R}^N) : \|x\| < r\}$, and consider the set

$$\Gamma = \{\gamma \in C\left(\overline{B}_r \cap H_0, W^{1,2}_{per}((0,b), \mathbf{R}^N)\right) : \gamma_{|\partial B_r \cap H_0} = \mathrm{id}_{|\partial B_r \cap H_0}\}.$$

We define

$$\widehat{c}_r = \inf_{\gamma \in \Gamma} \sup_{v \in \overline{B}_r \cap H_0} \varphi(\gamma(x)).$$

We know that the pair $\{\partial B_r \cap H_0, \overline{B}_r \cap H_0\}$ is linking with H_+ in $W^{1,2}_{per}((0,b), \mathbf{R}^N)$ (see [**10**, page 642]). Therefore, for all $\gamma \in \Gamma$, we have $\gamma(\overline{B}_r \cap H_0) \cap H_+ \neq \varnothing$ and so, from (19), we infer

$$\widehat{c}_r \geq 0. \tag{20}$$

Suppose that $\{0,\, y_0\}$ were the only critical points of φ. Let

$$a = \widehat{m} = \inf \varphi = \varphi(y_0) \quad \text{and} \quad b = 0 = \varphi(0).$$

We know (see (18)) that $a < 0 = b$. By virtue of Proposition 3.2, φ satisfies the PS_c-condition for every $c \in (a,b)$ (recall that now $\varphi(\widehat{x}_0) = 0$). Also, $K_a = \{y_0\}$, so we can apply Theorem 2.1 and obtain a continuous deformation $h : [0,1] \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that

$$\begin{aligned} h(t,\cdot)_{|K_a} &= \mathrm{id}_{|K_a} \quad \text{for all } t \in [0,1] \\ h(1, \dot{\varphi}^b) &\subseteq \dot{\varphi}^a \cup K_a = \{y_0\} \end{aligned} \tag{21}$$

and

$$\varphi(h(t,x)) \le \varphi(x) \quad \text{for all } t \in [0,1] \quad \text{and all } x \in \dot{\varphi}^b. \tag{22}$$

We introduce the map $\gamma_0 : \overline{B}_r \cap H_0 \to W^{1,2}_{per}((0,b),\mathbf{R}^N)$, defined by

$$\gamma_0(x) = \begin{cases} y_0 & \text{if } \|x\| \le r/2 \\ h[(2(r-\|x\|)/r),(rx/\|x\|)] & \text{if } \|x\| > r/2, \end{cases} \quad x \in \overline{B}_r \cap H_0. \tag{23}$$

If $\|x\| = r/2$, then $\varphi(2x) < 0$ (see (18)); from (21) we deduce $h[(2(r-\|x\|)/r),(rx/\|x\|)] = h(1,2x) = y_0$, so γ_0 is continuous. If $\|x\| = r$, then $\gamma_0(x) = h(0,x) = x$ (since h is a deformation). Hence, $\gamma_0 \in \Gamma$.

Moreover, from (22), (23) and since $\varphi(y_0) \le \eta_r$, we have $\varphi(\gamma_0(x)) \le \eta_r < 0$ for all $x \in \overline{B}_r \cap H_0$, so

$$\widehat{c}_r < 0. \tag{24}$$

Comparing (20) and (24), we reach a contradiction. This means that there is one more critical point $v_0 \notin \{0, y_0\}$ of φ. Then v_0 solves (1) and $v_0 \in C^1_{per}(T,\mathbf{R}^N)$ (see [**2**]).

Now assume that $\dim H_0 = 0$ and hypotheses $H(j)''$ are in effect. Hypotheses $H(j)''(iii)$ and (iv) imply that, given $\varepsilon > 0$, we can find $\xi_\varepsilon \in L^1(T_+)$ such that

$$j(t,x) \le \frac{\theta(t)+\varepsilon}{2}\|x\|^2 + \xi_\varepsilon(t) \quad \text{for almost all } t \in T \text{ and all } x \in \mathbf{R}^N. \tag{25}$$

Then, using (25) and Lemma 2 of [**15**], for every $x \in W^{1,2}_{per}((0,b),\mathbf{R}^N)$, we have

$$\begin{aligned} \varphi(x) &\ge \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbf{R}^N}\,dt \\ &\quad - \frac{1}{2}\int_0^b \theta(t)\|x(t)\|^2 dt - \frac{\varepsilon}{2}\|x\|_2^2 - \|\xi\|_1 \\ &\ge \frac{\widehat{\xi}-\varepsilon}{2}\|x\|^2 - \|\xi\|_1 \quad \text{for some } \widehat{\xi} > 0. \end{aligned} \tag{26}$$

Choose $\varepsilon \in (0, \widehat{\xi})$. Then, from (26), we see that φ is coercive. Therefore, it satisfies the nonsmooth PS-condition and it is bounded below.

Let $E(\lambda_1)$ be the eigenspace corresponding to the eigenvalue $\lambda_1 > 0$. If $x \in E(\lambda_1) \subseteq C(T, \mathbf{R}^N)$, $x \neq 0$ and $\|x\|_\infty \leq \delta$, then hypothesis $H(j)''(vi)$ implies

$$\varphi(x) \leq \frac{1}{2} \int_0^b (\lambda_1 - \eta(t)) \|x\|^2 dt.$$

But $\|x(t)\| \neq 0$ almost everywhere on T. So, from the hypothesis on η, we see that

$$\varphi(x) < 0 \quad \text{for all } x \in E(\lambda_1),\ x \neq 0,\ \|x\|_\infty \leq \delta. \tag{27}$$

On the other hand, if $y \in E(\lambda_1)^\perp \subseteq C(T, \mathbf{R}^N)$ and $\|y\|_\infty \leq \delta$, then again by virtue of hypothesis $H(j)''(vi)$, we have

$$\varphi(y) \geq \frac{\lambda_2}{2}\|y\|_2^2 - \frac{\lambda_2}{2}\|y\|_2^2 = 0,\ y \in E(\lambda_1)^\perp, \|y\|_\infty \leq \delta. \tag{28}$$

Since $W^{1,2}_{per}((0,b), \mathbf{R}^N)$ is embedded compactly in $C(T, \mathbf{R}^N)$, we can always find $r > 0$ small such that $v \in W^{1,2}_{per}((0,b), \mathbf{R}^N)$ with $\|v\| \leq r$, implies $\|v\|_\infty \leq \delta$. Hence, (27) and (28) permit the use of the nonsmooth local linking theorem for the decomposition of $W^{1,2}_{per}((0,b), \mathbf{R}^N)$ (see [**2**, Theorem 2.2]). So, we obtain two nontrivial critical points $y_0,\ v_0 \in W^{1,2}_{per}((0,b), \mathbf{R}^N)$ of φ. Then $y_0,\ v_0 \in C^1_{per}(T, \mathbf{R}^N)$ and (1) is solved. $\square$

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REFERENCES

1. G. Barletta and R. Livrea, *Existence of three periodic solutions for a non autonomous second order system*, Le Matem. **57** (2002), 205–215.

2. G. Barletta and N.S. Papageorgiou, *Nonautonomous second order periodic systems*: *Existence and multiplicity of solutions*, J. Nonlin. Convex Anal. **8** (2007), 373–390.

3. G. Bonanno and R. Livrea, *Periodic solutions for a class of second order Hamiltonian systems*, Electr. J. Diff. Eqns. **2005** (2005), 1–13.

4. K.C. Chang, *Infinite dimensional morse theory and multiple solutions problems*, Birkhäuser, Boston, 1993.

5. F.H. Clarke, *Optimization and nonsmooth analysis*, Class. Appl. Math. **5**, SIAM, Philadelphia, 1990.

6. G. Cordaro, *Three periodic solutions to an eigenvalue problem for a class of second order Hamiltonian systems*, Abstr. Appl. Anal. **2003** (2003), 1037–1045.

7. J.N.-Corvellec, *On the second deformation theorem*, Top. Meth. Nonlin. Anal. **17** (2001), 55–66.

8. F. Faraci, *Three periodic solutions for a second order non autonomous system*, J. Nonl. Convex Anal. **3** (2002), 393–399.

9. L. Gasinski and N.S. Papageorgiou, *Nonsmooth critical point theory and nonlinear boundary value problems*, Chapman Hall/CRC Press, Boca Raton, FL, 2005.

10. ———, *Nonlinear analyisis*, Chapman Hall/CRC Press, Boca Raton, FL, 2006.

11. D. Kandilakis, N. Kourogenis and N.S. Papageorgiou, *Two nontrivial critical points for nonsmooth functionals via local linking and applications*, J. Global Optim. **34** (2006), 219–244.

12. J. Mawhin, *Forced second order conservative systems with periodic nonlinearity*, Ann. Inst. Poincaré, Anal. Non Lin. **6** (1989), 415–434.

13. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, New York, 1989.

14. D. Motreanu, V. Motreanu and N.S. Papageorgiou, *Periodic solutions for non autonomous systems with nonsmooth quadratic or superquadratic potential*, Top. Meth. Nonlin. Anal. **24** (2004), 269–296.

15. ———, *Two nontrivial solutions for periodic systems with indefinite linear part*, Discr. Cont. Dynam. Systems **19** (2007), 197–210.

16. D. Motreanu and V. Radulescu, *Variational and non variational methods in nonlinear analysis and boundary value problems*, Kluwer, Dordrecht, 2003.

17. P. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **31** (1978), 157–184.

18. B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113** (2000), 401–410.

19. R. Showalter, *Hilbert space methods for partial differential equations*, Pitman, London, 1977.

20. C.L. Tang and X.P. Wu, *Periodic solutions for a class of nonautonomous subquadratic second order Hamiltonian systems*, J. Math. Anal. Appl. **275** (2002), 870–882.

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