ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 3, 2013

ON THE SOLUTIONS OF DIFFERENCE EQUATIONS OF ORDER FOUR

H. EL-METWALLY, E.M. ELSAYED AND E.M. ELABBASY

ABSTRACT. In this paper we deal with the behavior of the solution of the following difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}^2}{cx_{n-1} + dx_{n-3}}, \quad n = 0, 1, \dots,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we obtain the solution of some special cases of this equation.

1. Introduction. In this paper we deal with the behavior of the solutions of the difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}^2}{cx_{n-1} + dx_{n-3}}, \quad n = 0, 1, \dots, \tag{1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we obtain the solution of some special cases of this equation.

Nonlinear rational difference equations are of great importance in their own right because diverse nonlinear phenomena occurring in science and engineering can be modeled by such equations. Furthermore, the results about such equations offer prototypes towards the development of the basic theory of nonlinear difference equations.

The long term behavior of the solutions of nonlinear difference equations of order greater than one has been extensively studied during the last decade. For example, various results about boundedness, stability and periodic character of the solutions of the second-order nonlinear difference equation see [**1–9, 11, 12**].

2010 AMS *Mathematics subject classification.* Primary 39A10.
Keywords and phrases. Periodicity, boundedness, solution of difference equations.
Received by the editors on November 9, 2010.

DOI:10.1216/RMJ-2013-43-3-877

Many researchers have investigated the behavior of the solution of difference equations; for example, Agarwal et al. [**2**] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Cinar [**4**] has got the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Elabbasy et al. [**5**] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Elabbasy et al. [**6**] also studied the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Furthermore, Elabbasy et al. [**7**] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$

In [**8**] Elabbasy et al. studied the global stability character and the boundedness of solutions of the difference equation

$$x_{n+1} = \frac{ax_n^p + b\prod\limits_{r=1}^{p} x_{n-r}}{cx_n^p + d\prod\limits_{r=1}^{p} x_{n-r}}.$$

Yang et al. [**31**] investigated the invariant intervals, the global attractivity of equilibrium points and the asymptotic behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}},$$

see also [**9–15**]. Other related results on rational difference equations can be found in references [**16–33**].

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers, and let

$$f : I^{k+1} \longrightarrow I,$$

be a continuously differentiable function. Then, for every set of initial conditions $x_{-k}, x_{-k+1}, \dots , x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots , x_{n-k}), \quad n = 0, 1, \dots , \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [**23**].

Definition 1 (Equilibrium point). A point $\overline{x} \in I$ is called an equilibrium point of equation (2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \dots , \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of equation (2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2 (Periodicity). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3 (Fibonacci sequence). The sequence $\{F_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, \dots\}$, i.e., $F_m = F_{m-1}+F_{m-2}$, $m \geq 0$, $F_{-2} = 0$, $F_{-1} = 1$ is called the *Fibonacci sequence*.

Definition 4 (Stability). (i) The equilibrium point $\overline{x}$ of equation (2) is locally stable if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x_{-k}, x_{-k+1}, \dots , x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \dots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of equation (2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of equation (2) and there exists a $\gamma > 0$ such that, for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of equation (2) is a global attractor if, for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of equation (2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of equation (2).

(v) The equilibrium point $\overline{x}$ of equation (2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of equation (2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [**22**]. *Assume that $p, q \in R$ and $k \in \{0, 1, 2, \dots\}$. Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \dots .$$

Remark. Theorem A can be easily extended to a general linear equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad n = 0, 1, \ldots, \tag{4}$$

where $p_1, p_2, \ldots, p_k \in R$ and $k \in \{1, 2, \ldots\}$. Then equation (4) is asymptotically stable, provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [**23**]. *Let $g : [p,q]^2 \to [p,q]$ be a continuous function, where p and q are real numbers with $p < q$, and consider the following equation*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{5}$$

Suppose that g satisfies the following conditions:

(a) *$g(x,y)$ is non-decreasing in $x \in [p,q]$ for each fixed $y \in [p,q]$ and $g(x,y)$ is non-increasing in $y \in [p,q]$ for each fixed $x \in [p,q]$*;

(b) *If (m, M) is a solution of the system*

$$M = g(M, m) \quad \text{and} \quad m = g(m, M),$$

then

$$m = M.$$

Then there exists exactly one equilibrium $\overline{x}$ of equation (5), *and every solution of equation* (5) *converges to $\overline{x}$.*

2. Local stability of the equilibrium point. In this section we investigate the local stability character of the solutions of equation (1). Equation (1) has a unique equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2}{c\overline{x} + d\overline{x}},$$

or

$$\overline{x}^2(1-a)(c+d) = b\overline{x}^2;$$

if $(c+d)(1-a) \neq b$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = au + \frac{bu^2}{cu+dv}. \tag{6}$$

Therefore, it follows that

$$f_u(u,v) = a + \frac{bcu^2 + 2bduv}{(cu+dv)^2},$$
$$f_v(u,v) = \frac{-bdu^2}{(cu+dv)^2},$$

we see that

$$f_u(\overline{x},\overline{x}) = a + \frac{bc+2bd}{(c+d)^2},$$
$$f_v(\overline{x},\overline{x}) = \frac{-bd}{(c+d)^2}.$$

The linearized equation of equation (1) about $\overline{x}$ is

$$y_{n+1} - \left(a + \frac{bc+2bd}{(c+d)^2}\right)y_{n-1} + \frac{bd}{(c+d)^2}y_{n-3} = 0. \tag{7}$$

Theorem 1. *Assume that*

$$b(c+3d) < (c+d)^2(1-a).$$

Then the equilibrium point of equation (1) *is locally asymptotically stable.*

Proof. It follows by Theorem A that equation (7) is asymptotically stable if

$$\left|a + \frac{bc+2bd}{(c+d)^2}\right| + \left|\frac{bd}{(c+d)^2}\right| < 1,$$

or

$$a + \frac{bc + 3bd}{(c+d)^2} < 1,$$

and so

$$b(c + 3d) < (c + d)^2(1 - a).$$

The proof is complete. $\square$

3. Global attractor of the equilibrium point of equation (1). In this section we investigate the global attractivity character of solutions of equation (1).

Theorem 2. *The equilibrium point $\overline{x}$ of equation* (1) *is a global attractor if $c(1 - a) \neq b$.*

Proof. Let p, q be real numbers, and assume that $g : [p, q]^2 \to [p, q]$ is a function defined by $g(u, v) = au + (bu^2)/(cu + dv)$; then we can easily see that the function $g(u, v)$ is increasing in u and decreasing in v.

Suppose that (m, M) is a solution of the system

$$M = g(M, m) \quad \text{and} \quad m = g(m, M).$$

Then, from equation (1), we see that

$$M = aM + \frac{bM^2}{cM + dm}, \qquad m = am + \frac{bm^2}{cm + dM}.$$

Therefore,

$$M(1 - a) = \frac{bM^2}{cM + dm}, \qquad m(1 - a) = \frac{bm^2}{cm + dM},$$

or

$$c(1-a)M^2 + d(1-a)Mm = bM^2, \qquad c(1-a)m^2 + d(1-a)Mm = bm^2.$$

Subtracting, we obtain

$$c(1 - a)(M^2 - m^2) = b(M^2 - m^2), \qquad c(1 - a) \neq b.$$

Thus,

$$M = m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of equation (1) and then the proof is complete. □

4. Boundedness of solutions of equation (1). In this section we study the boundedness of solutions of equation (1).

Theorem 3. *Every solution of equation* (1) *is bounded if* $(a + (b/c)) < 1$.

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of equation (1). It follows from equation (1) that

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}^2}{cx_{n-1} + dx_{n-3}} \leq ax_{n-1} + \frac{bx_{n-1}^2}{cx_{n-1}} = \left(a + \frac{b}{c}\right)x_{n-1}.$$

Then

$$x_{n+1} \leq x_{n-1} \quad \text{for all } n \geq 0.$$

Then the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-3}, x_{-2}, x_{-1}, x_0\}$.

In order to confirm the results of this section, we consider a numerical example for $x_{-3} = 11$, $x_{-2} = 5$, $x_{-1} = 9$, $x_0 = 7$, $a = 0.6$, $b = 6$, $c = 9$, $d = 10$ (see Figure 1) and, for $x_{-3} = 11$, $x_{-2} = 5$, $x_{-1} = 9$, $x_0 = 7$, $a = 0.7$, $b = 6$, $c = 7$, $d = 12$ (see Figure 2).

5. Special cases of equation (1).

5.1. First equation $x_{n+1} = x_{n-1} + (x_{n-1}^2)/(x_{n-1} + x_{n-3})$. In this section we study the following special case of equation (1):

$$x_{n+1} = x_{n-1} + \frac{x_{n-1}^2}{x_{n-1} + x_{n-3}}, \tag{8}$$

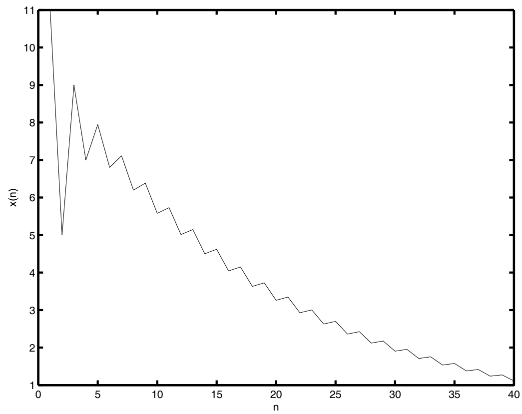

FIGURE 1. Plot of $x(n+1) = ax(n-1) + (bx(n-1)^{*}x(n-1))/(cx(n-1) + dx(n-3))$ vs. n.

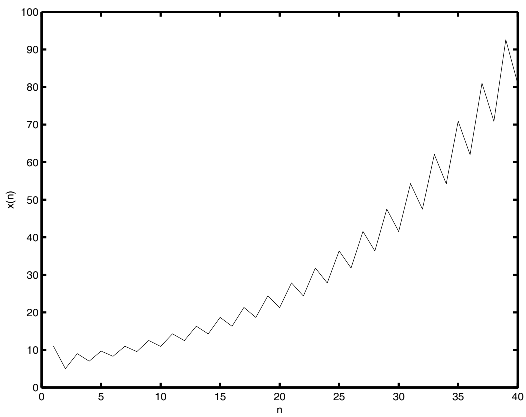

FIGURE 2. Plot of $x(n+1) = ax(n-1) + (bx(n-1)^{*}x(n-1))/(cx(n-1) + dx(n-3))$ vs. n.

where the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers.

Theorem 4. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of equation* (8). *Then, for* $n = 0, 1, 2, \ldots,$

$$x_{2n-1} = k \prod_{i=1}^{n} \left(\frac{f_{2i+1}k + f_{2i}p}{f_{2i}k + f_{2i-1}p} \right), \qquad x_{2n} = h \prod_{i=1}^{n} \left(\frac{f_{2i+1}h + f_{2i}r}{f_{2i}h + f_{2i-1}r} \right),$$

where $x_{-3} = p$, $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$, $\{f_m\}_{m=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$, $n - 2$. That is,

$$x_{2n-5} = k \prod_{i=1}^{n-2} \left(\frac{f_{2i+1}k + f_{2i}p}{f_{2i}k + f_{2i-1}p} \right), \qquad x_{2n-4} = h \prod_{i=1}^{n-2} \left(\frac{f_{2i+1}h + f_{2i}r}{f_{2i}h + f_{2i-1}r} \right),$$

$$x_{2n-3} = k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k + f_{2i}p}{f_{2i}k + f_{2i-1}p} \right), \qquad x_{2n-2} = h \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}h + f_{2i}r}{f_{2i}h + f_{2i-1}r} \right).$$

Now, it follows from equation (8) that

$$\begin{aligned}
x_{2n-1} &= x_{2n-3} + \frac{x_{2n-3}^2}{x_{2n-3} + x_{2n-5}} \\
&= k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k + f_{2i}p}{f_{2i}k + f_{2i-1}p} \right) \\
&\quad + \frac{k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p} \right) k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p} \right)}{k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p} \right) + k \prod_{i=1}^{n-2} \left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p} \right)} \\
&= k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k + f_{2i}p}{f_{2i}k + f_{2i-1}p} \right) + \frac{k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p} \right) \left(\frac{f_{2n-1}k+f_{2n-2}p}{f_{2n-2}k+f_{2n-3}p} \right)}{\left(\frac{f_{2n-1}k+f_{2n-2}p}{f_{2n-2}k+f_{2n-3}p} \right) + 1} \\
&= k \prod_{i=1}^{n-1} \left(\frac{f_{2i+1}k + f_{2i}p}{f_{2i}k + f_{2i-1}p} \right) \left(1 + \frac{f_{2n-1}k + f_{2n-2}p}{f_{2n-1}k + f_{2n-2}p + f_{2n-2}k + f_{2n-3}p} \right)
\end{aligned}$$

$$
\begin{aligned}
&= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p}\right)\left(1+\frac{f_{2n-1}k+f_{2n-2}p}{f_{2n}k+f_{2n-1}p}\right)\\
&= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p}\right)\left(\frac{f_{2n+1}k+f_{2n}p}{f_{2n}k+f_{2n-1}p}\right).
\end{aligned}
$$

Therefore,

$$
x_{2n-1} = k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}p}{f_{2i}k+f_{2i-1}p}\right).
$$

Also, we see from equation (8) that

$$
\begin{aligned}
x_{2n} &= x_{2n-2} + \frac{x_{2n-2}^2}{x_{2n-2}+x_{2n-4}}\\
&= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}r}{f_{2i}h+f_{2i-1}r}\right)\\
&\quad + \frac{h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}r}{f_{2i}h+f_{2i-1}r}\right)\left(\frac{f_{2n-1}h+f_{2n-2}r}{f_{2n-2}h+f_{2n-3}r}\right)}{\left(\frac{f_{2n-1}h+f_{2n-2}r}{f_{2n-2}h+f_{2n-3}r}\right)+1}\\
&= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}r}{f_{2i}h+f_{2i-1}r}\right)\left(1+\frac{f_{2n-1}h+f_{2n-2}r}{f_{2n}h+f_{2n-1}r}\right)\\
&= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}r}{f_{2i}h+f_{2i-1}r}\right)\left(\frac{f_{2n+1}h+f_{2n}r}{f_{2n}h+f_{2n-1}r}\right).
\end{aligned}
$$

Therefore,

$$
x_{2n} = h\prod_{i=1}^{n}\left(\frac{f_{2i+1}h+f_{2i}r}{f_{2i}h+f_{2i-1}r}\right).
$$

Hence, the proof is completed. $\square$

In order to confirm the results of this section, we consider a numerical example for $x_{-3}=2$, $x_{-2}=9$, $x_{-1}=5$, $x_0=3$ (see Figure 3).

5.2. Second equation $x_{n+1} = x_{n-1} + (x_{n-1}^2)/(x_{n-1} - x_{n-3})$. In this section we give a specific form of the solutions of the difference

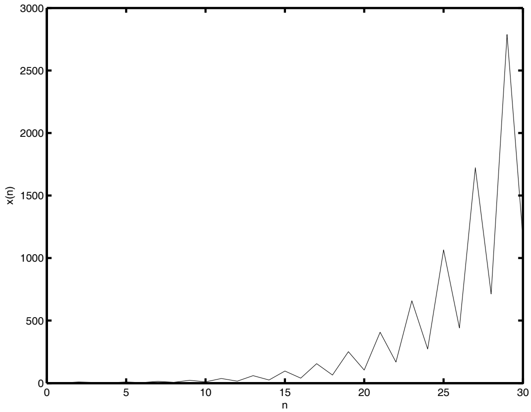

FIGURE 3. Plot of $x(n+1) = x(n-1) + (x(n-1)^* x(n-1))/(x(n-1) + x(n-3))$ vs. n.

equation

$$x_{n+1} = x_{n-1} + \frac{x_{n-1}^2}{x_{n-1} - x_{n-3}}, \tag{9}$$

where the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers with $x_{-1} \neq x_0$.

Theorem 5. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of equation* (9). *Then, for* $n = 0, 1, 2, \ldots,$

$$x_{2n-1} = k \prod_{i=1}^{n} \left(\frac{f_{i+2}k - f_i p}{f_i k - f_{i-2} p} \right), \qquad x_{2n} = h \prod_{i=1}^{n} \left(\frac{f_{i+2}h - f_i r}{f_i h - f_{i-2} r} \right),$$

where $x_{-3} = p$, $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$, $\{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Proof. Same as the proof of Theorem 4 and will be omitted.

Assume that $x_{-3} = 3$, $x_{-2} = 7$, $x_{-1} = 11$, $x_0 = 6$ (see Figure 4).

5.3. Third equation $x_{n+1} = x_{n-1} - (x_{n-1}^2)/(x_{n-1} + x_{n-3})$**.** In this section we obtain the solution of the following special case of

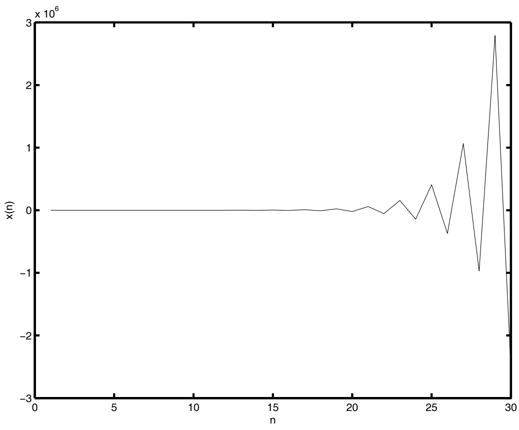

FIGURE 4. Plot of $x(n+1) = x(n-1) + (x(n-1)^* x(n-1))/(x(n-1) - x(n-3))$ vs. n.

equation (1)

$$x_{n+1} = x_{n-1} - \frac{x_{n-1}^2}{x_{n-1} + x_{n-3}}, \tag{10}$$

where the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers.

Theorem 6. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of equation* (10). *Then the solution is unbounded and, for* $n = 0, 1, 2, \dots,$

$$x_{2n-1} = \frac{kp}{(f_n k + f_{n+1} p)}, \qquad x_{2n} = \frac{hp}{(f_n h + f_{n+1} r)},$$

where $x_{-3} = p$, $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$, $\{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, \dots\}$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-2$, $n-3$. That is,

$$x_{2n-5} = \frac{kp}{(f_{n-2} k + f_{n-1} p)}, \qquad x_{2n-4} = \frac{hp}{(f_{n-2} h + f_{n-1} r)},$$
$$x_{2n-3} = \frac{kp}{(f_{n-1} k + f_n p)}, \qquad x_{2n-2} = \frac{hp}{(f_{n-1} h + f_n r)},$$

Now, it follows from equation (10) that

$$\begin{aligned} x_{2n-1} &= x_{2n-3} - \frac{x_{2n-3}^2}{x_{2n-3}+x_{2n-5}} \\ &= \frac{kp}{(f_{n-1}k+f_np)} - \frac{\frac{kp}{(f_{n-1}k+f_np)}\frac{kp}{(f_{n-1}k+f_np)}}{\frac{kp}{(f_{n-1}k+f_np)}+\frac{kp}{(f_{n-2}k+f_{n-1}p)}} \\ &= \frac{kp}{(f_{n-1}k+f_np)} - \frac{\frac{kp}{(f_{n-1}k+f_np)}(f_{n-2}k+f_{n-1}p)}{f_{n-2}k+f_{n-1}p+f_{n-1}k+f_np} \\ &= \frac{kp}{(f_{n-1}k+f_np)}\left(1-\frac{f_{n-2}k+f_{n-1}p}{f_nk+f_{n+1}p}\right) \\ &= \frac{kp}{(f_{n-1}k+f_np)}\left(\frac{f_nk+f_{n+1}-f_{n-2}k-f_{n-1}p}{f_nk+f_{n+1}p}\right) \\ &= \frac{kp}{(f_{n-1}k+f_np)}\left(\frac{f_{n-1}k+f_np}{f_nk+f_{n+1}p}\right). \end{aligned}$$

Therefore,

$$x_{2n-1} = \frac{kp}{(f_nk+f_{n+1}p)}.$$

Also, from equation (10), we see that

$$\begin{aligned} x_{2n} &= x_{2n-2} - \frac{x_{2n-2}^2}{x_{2n-2}+x_{2n-4}} \\ &= \frac{hp}{(f_{n-1}h+f_nr)} - \frac{\frac{hp}{(f_{n-1}h+f_nr)}\frac{hp}{(f_{n-1}h+f_nr)}}{\frac{hp}{(f_{n-1}h+f_nr)}+\frac{hp}{(f_{n-2}h+f_{n-1}r)}} \\ &= \frac{hp}{(f_{n-1}h+f_nr)}\left(1-\frac{f_{n-2}h+f_{n-1}r}{f_nh+f_{n+1}r}\right) \\ &= \frac{hp}{(f_{n-1}h+f_nr)}\left(\frac{f_{n-1}h+f_nr}{f_nh+f_{n+1}r}\right). \end{aligned}$$

Therefore,

$$x_{2n} = \frac{hp}{(f_nh+f_{n+1}r)}.$$

Hence, the proof is completed. □

Figure 5 shows the solution when $x_{-3} = 3$, $x_{-2} = 15$, $x_{-1} = 9$, $x_0 = 17$.

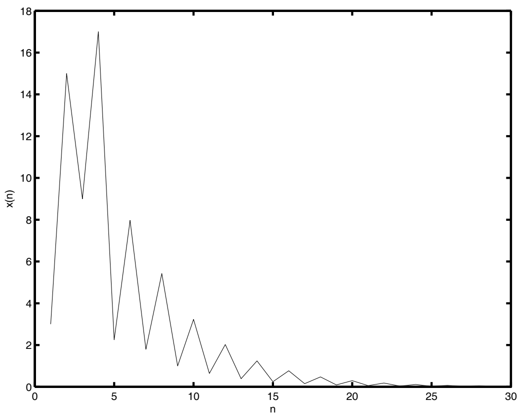

FIGURE 5. Plot of $x(n+1) = x(n-1) - (x(n-1)^* x(n-1))/(x(n-1) + x(n-3))$ vs. n.

5.4. Fourth equation $x_{n+1} = x_{n-1} - (x_{n-1}^2)/(x_{n-1} - x_{n-3})$**.** In this section we study the following special case of equation (1):

$$x_{n+1} = x_{n-1} - \frac{x_{n-1}^2}{x_{n-1} - x_{n-3}}, \tag{11}$$

where the initial conditions x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary non zero real numbers with $x_{-3} \neq x_{-1}$, $x_{-2} \neq x_0$.

Theorem 7. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of equation* (11)*. Then every solution of equation* (11) *is periodic with period* 12*. Moreover,* $\{x_n\}_{n=-3}^{\infty}$ *takes the form*

$$\left\{p, r, k, h, \frac{kp}{p-k}, \frac{hr}{r-h}, -p, -r, -k, -h, \frac{-kp}{p-k}, \frac{-hr}{r-h}, p, r, k, h, \ldots\right\},$$

or

$$\begin{aligned} x_{12n-3} &= p, & x_{12n-2} &= r, & x_{12n-1} &= k, \\ x_{12n} &= h, & x_{12n+1} &= \frac{kp}{p-k}, & x_{12n+2} &= \frac{hr}{r-h}, \\ x_{12n+3} &= -p, & x_{12n+4} &= -r, & x_{12n+5} &= -k, \\ x_{12n+6} &= -h, & x_{12n+7} &= \frac{-kp}{p-k}, & x_{12n+8} &= \frac{-hr}{r-h}, \end{aligned}$$

where $x_{-3} = p$*,* $x_{-2} = r$*,* $x_{-1} = k$*,* $x_0 = h$*.*

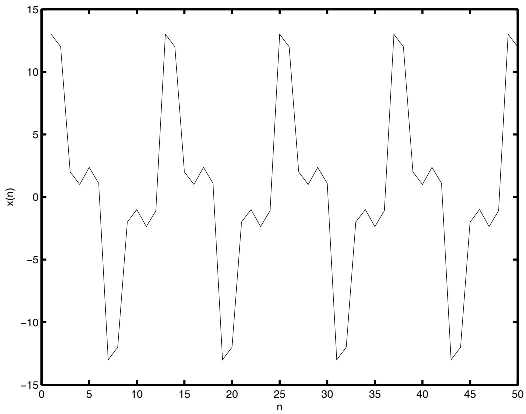

FIGURE 6. Plot of $x(n+1) = x(n-1) - (x(n-1)^{*}x(n-1))/(x(n-1) - x(n-3))$ vs. n.

Proof. Same as the proof of Theorem 6 and will be omitted.

Figure 6 shows the solution when $x_{-3} = 13$, $x_{-2} = 12$, $x_{-1} = 2$, $x_0 = 1$.

6. Conclusion. This paper discussed global stability, boundedness and the solutions of some special cases of equation (1). In Section 2 we proved when $b(c+3d) < (c+d)^2(1-a)$, equation (1) has local stability. In Section 3 we showed that the unique equilibrium of equation (1) is globally asymptotically stable if $c(1-a) \neq b$. In Section 4 we proved that the solution of equation (1) is bounded if $(a + (b/c)) < 1$. In Section 5 we obtained the form of the solution of four special cases of equation (1) and gave numerical examples of each case, drawing them with Matlab 6.5.

REFERENCES

1. R. Agarwal, *Difference equations and inequalities. Theory, methods and applications*, Marcel Dekker Inc., New York, 1992.

2. R.P. Agarwal and E.M. Elsayed, *Periodicity and stability of solutions of higher order rational difference equation*, Adv. Stud. Contemp. Math. **17** (2008), 181–201.

3. M. Aloqeili, *Dynamics of a rational difference equation*, Appl. Math. Comp. **176** (2006), 768–774.

4. C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. Comp. **156** (2004), 587–590.

5. E.M. Elabbasy, H. El-Metwally and E.M. Elsayed, *On the difference equation* $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, Adv. Diff. Equat. **2006**, Article ID 82579, 1–10.

6. ———, *On the difference equations* $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math. **5** (2007), 101–113.

7. ———, *Qualitative behavior of higher order difference equation*, Soochow J. Math. **33** (2007), 861–873.

8. E.M. Elabbasy and E.M. Elsayed, *On the global attractivity of difference equation of higher order*, Carpathian J. Math. **24** (2008), 45–53.

9. ———, *Global attractivity and periodic nature of a difference equation*, World Appl. Sci. J. **12** (2011), 39–47.

10. H. El-Metwally, *Global behavior of an economic model*, Chaos, Solit. Fract. **33** (2007), 994–1005.

11. H. El-Metwally, E.A. Grove and G. Ladas, *A global convergence result with applications to periodic solutions*, J. Math. Anal. Appl. **245** (2000), 161–170.

12. H. El-Metwally, E.A. Grove, G. Ladas and H.D. Voulov, *On the global attractivity and the periodic character of some difference equations*, J. Diff. Equat. Appl. **7** (2001), 1–14.

13. H. El-Metwally, E.A. Grove, G. Ladas, M. Raidan and R. Levins, *On the difference equation* $y_{n+1} = A + By_{n-1}e^{-y_n}$, Nonlin. Anal. **47** (2001), 4623–4634.

14. E.M. Elsayed, *Dynamics of a recursive sequence of higher order*, Commun. Appl. Nonlin. Anal. **16** (2009), 37–50.

15. ———, *On the solution of recursive sequence of order two*, Fasc. Math. **40** (2008), 5–13.

16. ———, *On the difference equation* $x_{n+1} = \frac{x_{n-5}}{-1+x_{n-2}x_{n-5}}$, Inter. J. Contem. Math. Sci. **3** (2008), 1657–1664.

17. ———, *Qualitative behavior of difference equation of order three*, Acta Sci. Math. (Szeged) **75** 113–129.

18. ———, *On the global attractivity and the solution of recursive sequence*, Stud. Sci. Math. Hungar. **47** (2010), 401–418.

19. A. Gelişken, C. Cinar and I. Yalcinkaya, *On the periodicity of a difference equation with maximum*, Discr. Dynam. Nat. Soc. **2008**, Article ID 820629.

20. E.A. Grove and G. Ladas, *Periodicities in nonlinear difference equations*, Chapman & Hall, CRC Press, New York, 2005.

21. A.E. Hamza and S.G. Barbary, *Attractivity of the recursive sequence* $x_{n+1} = (\alpha - \beta x_n)F(x_{n-1}, \dots, x_{n-k})$, Math. Comp. Model. **48** (2008), 1744–1749.

22. V.L. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, 1993.

23. M.R.S. Kulenovic and G. Ladas, *Dynamics of second order rational difference equations with open problems and conjectures*, Chapman & Hall, CRC Press, New York, 2001.

24. M. Saleh and S. Abu-Baha, *Dynamics of a higher order rational difference equation*, Appl. Math. Comp. **181** (2006), 84–102.

25. M. Saleh and M. Aloqeili, *On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$*, Appl. Math. Comp. **176** (2006), 359–363.

26. ———, *On the difference equation $x_{n+1} = A + \frac{x_n}{x_{n-k}}$*, Appl. Math. Comp. **171** (2005), 862–869.

27. H. Sedaghat, *Nonlinear difference equations, Theory with applications to social science models*, Kluwer Academic Publishers, Dordrect, 2003.

28. D. Simsek, C. Cınar, R. Karatas and I. Yalcinkaya, *On the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$*, Int. J. Pure Appl. Math. **28** (2006), 117–124.

29. A. Sobiło and M. Migda, *Bounded solutions of third order nonlinear difference equations*, Rocky Mountain J. Math. **36** (2006), 23–34.

30. I. Yalçınkaya, C. Cinar and M. Atalay, *On the solutions of systems of difference equations*, in *Advances in difference equations*, Vol. 2008, Article ID 143943, 9 pages, doi: 10.1155/2008/ 143943.

31. X. Yang, W. Su, B. Chen, G.M. Megson and D.J. Evans, *On the recursive sequence $x_{n+1} = \frac{ax_{n-1}+bx_{n-2}}{c+dx_{n-1}x_{n-2}}$*, Appl. Math. Comp. **162** (2005), 1485–1497.

32. E.M.E. Zayed and M.A. El-Moneam, *On the rational recursive sequence $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$*, Commun. Appl. Nonlin. Anal. **12** (2005), 15–28.

33. ———, *On the rational recursive sequence $x_{n+1} = ax_n - \frac{bx_n}{cx_n-dx_{n-k}}$*, Commun. Appl. Nonlin. Anal. **15** (2008), 47–57.

Dept. Math., Faculty of Science & Art in Rabigh, King Abdelaziz University, Rabigh 21911, Saudi Arabia and Dept. Math., Faculty of Science, Mansoura University, Mansoura 35516, Egypt
Email address: helmetwally@mans.edu.eg, helmetwally2001@yahoo.com

King AbdulAziz University, Faculty of Science, Mathematics Department, P.O. Box 80203, Jeddah 21589, Saudi Arabia and Dept. Math., Faculty of Science, Mansoura University, Mansoura 35516, Egypt
Email address: emelsayed@mans.edu.eg, emmelsayed@yahoo.com

Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
Email address: emelabbasy@mans.edu.eg