

STRUCTURE PARAMETERS IN ROTATING COUETTE-POISEUILLE CHANNEL FLOW

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1. Introduction. It is well-known that a number of steady state problems in fluid mechanics involving systems of nonlinear partial differential equations can be reduced to the problem of solving a single operator equation of the form

$$(1.1) \quad v + \lambda Av + \lambda B(v) = 0, \quad v \in H, \lambda \in \mathbf{R}^1,$$

where H is an appropriate (real or complex) Hilbert space. Here λ is a typical "load" parameter, e.g., the Reynolds number A is a linear operator and B is a quadratic operator generated by a bilinear form. In this setting many bifurcation and stability results for problems in fluid mechanics have been obtained, the reader is referred to [1, 10, 11, 21] and the bibliographies therein for a detailed account of such results.

In fact, there may be considerably more structure in a nonlinear stability problem in fluid mechanics than that implied by an operator equation such as (1.1). As shown in a recent series of papers by the authors [12, 13, 19], a "structure" parameter, say γ , also may be present so that equation (1.1) may be actually of the form

$$(*) \quad u - \lambda(L - \gamma M)u - F(u) - \gamma G(u) = 0, \quad u \in H, \lambda \in \mathbf{R}^1, \gamma \in \mathbf{R}^1,$$

where H is again an appropriate Hilbert space, L and M are linear operators, and F and G are generated by bilinear operators. Equations of the form (*) are derived in [12, 13] for Bénard-type convection problems and in [19] for the Taylor problem. It is the purpose of the present paper to describe a setting in which it is possible to determine a complete set of bifurcation equations for an operator equation of the form (*) by using the structure parameter γ as an "amplitude" parameter rather than regarding γ as merely a constant to be incorporated into the

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operators M and G in (*). Complete bifurcation diagrams are obtained by such an approach because, e.g. the “reduced” bifurcation equations contain linear as well as both quadratic and cubic terms. This is in contrast to many other approaches in bifurcation theory based upon splitting methods (e.g., see [10, 11, 18, 21]) in which either linear and quadratic or linear and cubic terms appear in the reduced equations. In this sense the spirit of the approach presented here is somewhat like that of singularity theory as in Golubitsky and Schaeffer [7], however, our approach also has points in common with those in Busse [2] and Chow, Hale and Mallet-Paret [4, 5]. The present paper emphasizes the identification and utilization of structure parameters in fluid mechanics but the approach presented here applies equally well to general classes of bifurcation problems provided that such problems have appropriate “higher order” terms (e.g., see the general class of variational problems studied in [14]).

The following hypotheses are chosen to illustrate our approach with a minimum number of technical difficulties. We consider equation (*) under the following hypotheses in which H is a real Hilbert space with norm $\| \cdot \|$ generated by an inner product (\cdot, \cdot) .

(H1) The linear operator $L : H \rightarrow H$ is selfadjoint and compact, and, if μ_0 denotes the smallest positive characteristics value of L , then the dimension of the null space $N \equiv N(I - \mu_0 L)$ is $n = 1$ and N is spanned by ψ_0 with $\|\psi_0\| = 1$.

(H2) The linear operator $M : H \rightarrow H$ is compact and satisfies $M : N \rightarrow N^\perp$, where N^\perp denotes the orthogonal complement of N in H .

(H3) The critical characteristic value, $\lambda_c = \lambda_c(\gamma)$, of $L_\gamma \equiv L - \gamma M$, i.e., the positive characteristic value of L_γ of least magnitude, is of the form

$$(1.2) \quad \lambda_c(\gamma) = \mu_0 - \mu_0^3 b \gamma^2 + \Lambda(\gamma), \quad |\gamma| < \gamma_1,$$

where Λ is real and analytic and satisfies $\Lambda(\gamma) = 0(\gamma^3)$ as $\gamma \rightarrow 0$. Here $b = (MKM\psi_0, \psi_0)$ where $K : M^\perp \rightarrow N^\perp$ denotes the inverse of the restriction of $(I - \mu_0 L)$ to N^\perp .

(H4) The nonlinear operators $F(w) = \Phi(w, w)$ and $G(w) = \Gamma(w, w)$ are generated by bounded bilinear operators $\Phi : H \times H \rightarrow H$ and $\Gamma : H \times H \rightarrow H$, where, in addition, $\Phi : N \times N \rightarrow N^\perp$.

The assumption in (H1) that $\dim N = 1$ is made for the sake of convenience; more general settings with $\dim N > 1$ may be treated along the lines in [12, 13, 19], however, the actual solution of the bifurcation equations may be, of course, much more difficult in the case $\dim N > 1$. The assumption (1.2) in (H3) is the “natural” expansion for λ_c and also is made for the sake of convenience. It, or a suitable modification, holds in many linear problems in fluid mechanics even if M is not symmetric and $\dim N > 1$ (e.g., see [3, 8, 12, 13, 19]); if M is symmetric, then (H3) may be omitted. The modifications required when H is a complex Hilbert space are somewhat more involved but may be formulated as in [12; §4] in terms of real operators and group representations (see also [20, 21]). Finally, if one wishes to carry out a linear stability analysis of solutions of (*) in the case $\dim N > 1$, one may proceed as in [12; §5] (see also [16]).

REMARK 1.1. In many problems in fluid mechanics the bilinear operator Φ in (H4) will be generated by the nonlinear term $(u \cdot \nabla)u$ in the Navier-Stokes equations. If so, then Φ satisfies the additional condition

$$(1.3) \quad (\Phi(u, v), w) = -(\Phi(u, w), v), \quad u, v, w \in H.$$

The condition (1.3) plays a role in determining the actual form of the bifurcation equations associated with (*) but it is not required in the derivation of the bifurcation equations. Thus, condition (1.3) is not included as part of (H4); however, it is used in the application to rotating Couette-Poiseuille flow in §4 and also in the applications to Bénard-type convection problems in [12, 13] and to the Taylor problem in [19].

The outline of the paper is as follows. In §2 we make use of γ as an amplitude parameter and derive the bifurcation equation associated with (*) under hypotheses (H1) through (H4). In §3 the bifurcation equation is solved under an additional invariance assumption on the remainder term and, for each fixed γ sufficiently small, a complete bifurcation diagram is obtained. In §4 we use the above approach to study rotating Couette-Poiseuille channel flow. We show, in particular, that the superposition of a Poiseuille flow on a rotating Couette channel flow is, in general, destabilizing. This type of result was conjectured in

[17] for non-rotating combined Couette-Poiseuille flow on the basis of numerical calculations for the linearized Navier-Stokes equations.

2. The bifurcation equation. The bifurcation equation associated with (*) is derived here using standard splitting methods except that the structure parameter γ plays the role of an amplitude parameter.

Let $P : H \rightarrow N^\perp$ denote the orthogonal projection of H onto N^\perp and let $S = I - P : H \rightarrow N$ denote the orthogonal projection of H onto N . For γ sufficiently small, we shall seek solutions of (*) of the form

$$(2.1) \quad u = \gamma(\psi + \gamma\Psi), \quad \lambda = \mu_0 - \mu_0\gamma^2(\mu_0^2b - \tau),$$

where $\psi \in N, \Psi \in N^\perp$ and $\tau \in \mathbf{R}^1$ are to be determined. Note that if $\tau = \tau_0 + O(\gamma)$ and if $\lambda_c = \lambda_c(\gamma)$ is defined as in (H3), then $\lambda = \lambda_c + \mu_0\tau_0\gamma^2 + O(\gamma^3)$. Thus, for γ sufficiently small, a solution of (*) of the form (2.1) is subcritical if $\tau_0 < 0$ and supercritical if $\tau_0 > 0$.

Substituting (2.1) into equation (*), using the projection P onto N^\perp and S onto N , and making use of (H2) and (H4), one obtains the following equations in N^\perp and N :

(2.2a)

0 =

$$\begin{aligned} & (I - \mu_0 L)\Psi + \mu_0 M\psi - F(\psi) \\ & + \gamma P(\mu_0 M\Psi - \Phi(\psi, \Psi) - \Phi(\Psi, \psi) - G(\psi)) \\ & + \gamma^2 P(\mu_0(\tau - \mu_0^2 b)M\psi + \mu_0(\mu_0^2 b - \tau)L\Psi - F(\Psi) - \Gamma(\psi, \Psi) - \Gamma(\Psi, \psi)) \\ & + \gamma^3 P(\mu_0(\tau - \mu_0^2 b)M\Psi - G(\Psi)), \end{aligned}$$

(2.2b)

$$\begin{aligned} 0 = & (\mu_0^2 b - \tau)\psi + S(\mu_0 M\Psi - \Phi(\psi, \Psi) - \Phi(\Psi, \psi) - G(\psi)) \\ & - \gamma S(F(\Psi) + \Gamma(\psi, \Psi) + \Gamma(\Psi, \psi)) + \gamma^2 S(\mu_0(\tau - \mu_0^2 b)M\Psi - G(\Psi)). \end{aligned}$$

Since $K \equiv ((I - \mu_0 L)|_{N^\perp})^{-1}$ is bounded on N^\perp , given $\rho > 0$, there exists $\gamma_0 > 0$ such that, if $(\psi, \tau) \in N \times \mathbf{R}^1$ with $|\tau| < \rho$ and $\|\psi\| < \rho$, then by the implicit function, theorem (2.2a) can be solved uniquely for $\Psi = \Psi(\psi, \tau, \gamma)$ in N^\perp provided that $|\gamma| < \gamma_0$. In fact, Ψ is analytic and of the form

$$(2.3) \quad \Psi = -\mu_0 K M \psi + K F(\psi) + \gamma \Psi_1,$$

where $\Psi_1 = \Psi_1(\psi, \tau, \gamma) \in N^\perp$ is bounded with bound depending only on ρ . It is important here that ρ can be arbitrarily large provided that γ_0 is chosen sufficiently small. Substituting Ψ into (2.2b), taking the inner product with ψ_0 , and making use of the definition of b in (H3), one obtains, for ψ of the form $\psi = \beta\psi_0$ with $\beta \in \mathbf{R}^1$, the *bifurcation equation associated with (*)*:

(2.4)

$$\begin{aligned} 0 = & -\tau\beta + \beta^2[\mu_0(MKF(\psi_0), \psi_0) + \mu_0(\Phi(\psi_0, KM\psi_0), \psi_0) \\ & + \mu_0(\Phi(KM\psi_0, \psi_0), \psi_0) - (G(\psi_0), \psi_0)] \\ & - \beta^3((\Phi(\psi_0, KF(\psi_0)), \psi_0 + (\Phi(KF(\psi_0), \psi_0), \psi_0)) + r(\beta, \tau, \gamma) \\ \equiv & -\tau\beta + a\beta^2 + c\beta^3 + r(\beta, \tau, \gamma) \equiv D(\beta, \tau, \gamma). \end{aligned}$$

Here, for $|\tau| < \rho$, $|\beta| < \rho$ and $|\gamma| < \gamma_0$, the remainder term r is given by

$$\begin{aligned} r(\beta, \tau, \gamma) = & (\gamma S\{\mu_0 M\Psi_1 - \Phi(\psi, \Psi_1) - \Phi(\Psi_1, \psi) - F(\Psi) \\ (2.5) \quad & - \Gamma(\psi, \Psi) - \Gamma(\Psi, \psi) \\ & + \gamma(\mu_0(\tau - \mu_0^2 b)M\Psi - G(\Psi))\}, \psi_0), \end{aligned}$$

where $\psi = \beta\psi_0$ and $\Psi_1 = \Psi_1(\beta\psi_0, \tau, \gamma)$. Moreover, r is analytic in (β, τ, γ) and, for some r_0 depending only on ρ , satisfies

$$(2.6) \quad |r(\beta, \tau, \gamma)| \leq r_0|\gamma|, \quad |\beta| < \rho, |\tau| < \rho, |\gamma| < \gamma_0.$$

Note that if $(\beta^*, \tau^*, \gamma)$ is a solution of (2.4) with $|\tau^*| < \rho$, $|\beta^*| < \rho$, and for fixed γ satisfying $|\gamma| < \gamma_0$, then $(v^*, \lambda^*) \in H \times \mathbf{R}^1$ given by

$$(2.7) \quad v^* = \gamma(\beta^*\psi_0 + \gamma\Psi(\beta^*\psi_0, \tau^*, \gamma)), \quad \lambda^* = \mu - \mu_0\gamma^2(\mu_0^2 b - \tau^*)$$

is a solution of equation (*).

In the next section we solve the bifurcation equation (2.4) under an additional invariance assumption on the remainder term r .

3. Solutions of equation (*). In many problems of the type considered here the remainder term r in (2.5) may satisfy additional properties. For example, because of invariance and symmetry considerations, r may have the form

$$(3.1) \quad r(\beta, \tau, \gamma) = \beta s(\beta, \tau, \gamma),$$

where s is analytic and s and its partials with respect to β and τ are uniformly of order $O(\gamma)$ as $\gamma \rightarrow 0$; one has, e.g.,

$$(3.2) \quad |s(\beta, \tau, \gamma)| \leq s_0 |\gamma|, \quad |\beta| < \rho, |\tau| < \rho, |\gamma| < \gamma_0,$$

for some s_0 depending only on ρ . In such problems the bifurcation equation (2.4) may be factored and replaced by

$$(3.3) \quad 0 = -\tau + a\beta + c\beta^2 + s(\beta, \tau, \gamma) \equiv E(\beta, \tau, \gamma).$$

It is this type of “factoring” that is also the key to the solution of more difficult problems with $\dim N > 1$ (e.g., see [13, 15, 19]). Since E is a mapping of a neighborhood of $(0, 0, 0) \in \mathbf{R}^3$ into \mathbf{R}^1 , it is natural to seek solutions of (3.3) for γ near $\gamma = 0$, hence solutions (v^*, λ^*) of $(*)$ in the form (2.7), by means of the implicit function theorem. We have, e.g., the following result.

THEOREM 3.1. *Given $\rho > 0$ there exists $\gamma_0 > 0$ such that, for $|\gamma| < \gamma_0$, equation (3.3) has a solution*

$$(3.4) \quad \tau = \tau(\beta, \gamma) \equiv a\beta + c\beta^2 + \tau_1(\beta, \gamma)$$

that is bounded, analytic and unique in

$$(3.5) \quad B = \{(\beta, \tau, \gamma) : |\tau - a\beta - c\beta^2| < k|\gamma|, |\beta| < \rho, |\gamma| < \gamma_0\},$$

where the constant k depends only on ρ and τ_1 is $O(\gamma)$ as $\gamma \rightarrow 0$, uniformly for $|\beta| < \rho$. For each fixed γ satisfying $|\gamma| < \gamma_0$, the corresponding nontrivial solution branch $(v^(\beta), \lambda^*(\beta))$ of $(*)$, $|\beta| < \rho$, has the form (2.7) with $\beta^* = \beta$ and τ^* given by (3.4).*

PROOF. Let β_0 satisfying $|\beta_0| < \rho$ be given and set $\tau_0 = a\beta_0 + c\beta_0^2$. It follows from (3.2) that $(\beta_0, \tau_0, 0)$ is a solution of (3.3) at which $\frac{\partial E}{\partial \tau} = -1$. Thus, by the implicit function theorem, (3.3) has a solution $\tau = \tau(\beta, \gamma)$ that is bounded, analytic and unique in a neighborhood of $(\beta, \tau, \gamma) = (\beta_0, \tau_0, 0)$ with $\tau(\beta_0, 0) = \tau_0$. Since

$$(3.6) \quad |\tau - a\beta - c\beta^2| = |s(\beta, \tau, \gamma)| < s_0 |\gamma| < s_0 \gamma_0,$$

a finite number of such neighborhoods cover B , provided that γ_0 is sufficiently small. For fixed γ satisfying $|\gamma| < \gamma_0$, the form of the corresponding solution branch $(v^*(\beta), \lambda^*(\beta)) \in H \times \mathbf{R}^1$ is an immediate consequence of the above construction.

REMARK 3.1. One can show also that, for each fixed γ satisfying $|\gamma| < \gamma_0$, if $c > 0$, then the branch of solutions, Q , of (3.3) determined by (3.4) has a unique turning point in B at $\beta = \beta_T(\gamma)$, where β_T is of the form

$$(3.7) \quad \beta_T(\gamma) = -(a/2c) + \beta_1(\gamma)$$

with $\beta_1(\gamma) = O(\gamma)$ as $\gamma \rightarrow 0$. The fact that $\frac{\partial \tau}{\partial \beta}(\beta, \gamma) = 0$ in (3.4) at a unique point in B follows from another application of the implicit function theorem; one considers the equation

$$(3.8) \quad 0 = \frac{\partial}{\partial \beta} \tau(\beta, \gamma) = a + 2c\beta + \frac{\partial}{\partial \beta} \tau_1(\beta, \gamma)$$

and uses the fact that, at $\gamma = 0$, $\frac{\partial \tau}{\partial \beta} = 0$ only at $\beta = -a/2c$.

It can be shown (e.g., see [12, §5; 16, 21]) that, for fixed γ , the linearized stability at points (v^*, λ^*) along the solution branch of (*) in Theorem 3.1 is determined by the sign of $\frac{\partial D}{\partial \beta}$ along the branch Q . Since, along Q , (2.4) holds and

$$(3.9) \quad \frac{\partial D}{\partial \beta} = \beta \left(a + 2c\beta + \frac{\partial s}{\partial \beta} \right),$$

one sees that, for γ_0 sufficiently small, the stability of $(v^*(\beta), \lambda^*(\beta))$ is indeterminate only at $\beta = 0$ and at the unique turning point on Q given by (3.7). Thus, we have the following corollary to Theorem 3.1. For the statement of the corollary we may assume that $a \leq 0$; if $a > 0$, one replaces β by $-\beta$ in the given intervals.

COROLLARY 3.1. *If $a \leq 0$ and $c > 0$, then, for γ_0 sufficiently small, the solution branch $(v^*(\beta), \lambda^*(\beta))$ of (*) obtained in Theorem 3.1 is stable for $-\rho < \beta < 0$ and $\beta(\gamma) < \beta < \rho$ and unstable for $0 < \beta < \beta_T(\gamma)$.*

We wish to emphasize that, for each fixed γ sufficiently small, the solution branches of the bifurcation equation (2.4) obtained in the above discussion are “global” in the (β, τ) plane, however, the corresponding solution branch (v^*, λ^*) of $(*)$ may, of course, be “small” because of the amplitude factor γ in (2.7).

The above results are formulated only to illustrate the approach when $\dim N = 1$. Complete solutions of the bifurcation equations and the corresponding stability results may be, of course, much more difficult to obtain in problems where $\dim N > 1$ because in such problems, e.g., there may be points of secondary bifurcation at which the null space of the appropriate linearized operator has dimension greater than one (see the discussion of hexagonal solutions in [13]).

4. Rotating Couette-Poiseuille channel flow. In this section we study rotating Couette-Poiseuille channel flow and show that, in general, the superposition of a Poiseuille flow on a rotating Couette channel flow is destabilizing. In the case of non-rotating Couette channel flow this result was conjectured in [17] on the basis of numerical calculations for the linearized Navier-Stokes equations. In the nonlinear analysis presented here, it is crucial that there are Coriolis effects present in the problem so that the swirl-like parameter S defined in (4.2) is positive.

We consider viscous incompressible flow in a rotating infinite-channel of width l . The non-dimensional Navier-Stokes equations in a rectangular coordinate system rotating about the z -axis with constant angular speed, Ω , are given by (e.g., see [9, p. 163])

$$(4.1) \quad \mathbf{R}_e^{-1} \Delta \underline{v} - \nabla \underline{p}' - (\underline{v} \cdot \nabla) \underline{v} + 2S \begin{bmatrix} v_2 \\ -v_1 \\ 0 \end{bmatrix} = 0, \quad (x, y, z) \in C$$

$$\nabla \cdot \underline{v} = 0,$$

where $C = \{(x, y, z) : -\infty < x < \infty, -1/2 < y < 1/2, -\infty < z < \infty\}$, and $\underline{v} = (v_1, v_2, v_3)$. Here the Coriolis acceleration terms are determined by the square brackets, the rectangular coordinates are scaled by l , the velocity components are scaled by U_c , where U_c denotes the maximum velocity of pure Couette flow at $y = 1/2$ in the direction

of the x -axis with no Poiseuille flow present (e.g., see [9, p. 179]), S is the swirl-like parameter

$$(4.2) \quad S = l\Omega/U_c,$$

and R_e is Reynolds number

$$(4.3) \quad R_e = lU_c/\nu,$$

where ν is the kinematic viscosity. We assume throughout that S is fixed and $0 < S < \frac{1}{2}$.

For the basic unperturbed flow we take a pressure, P , and a combination Couette-Poiseuille velocity distribution $\underline{U} = (U_1(y), 0, 0)$, where

$$(4.4) \quad U_1(y) = \left(y + \frac{1}{2}\right) + \frac{\delta}{4}(1 - 4y^2), \quad -\frac{1}{2} \leq y \leq \frac{1}{2},$$

$$(4.5) \quad \delta = U_p/U_c.$$

Here $\frac{1}{4}U_p$ denotes the maximum velocity of pure Poiseuille flow in the direction of the x -axis with no Couette flow present (e.g., see [9, p.66]).

We shall seek solutions of (4.1) that are perturbations of \underline{U} and P , and are independent of x . Setting $\underline{v} = \underline{U} + \underline{w}$, $p' = P + q$ in (4.1) and assuming $\underline{w} = \underline{w}(y, z)$ and $q = q(y, z)$, one obtains the disturbance equations

$$R_e^{-1} \Delta \underline{w} - \nabla q + 2S \begin{bmatrix} w_2 \\ -w_1 \\ 0 \end{bmatrix} - \underline{w} \cdot \nabla \underline{U} - \underline{w} \cdot \nabla \underline{w} = 0,$$

$$\nabla \cdot \underline{w} = 0, \quad -\frac{1}{2} < y < \frac{1}{2}, \quad -\infty < z < \infty,$$

where $\nabla = (0, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. The boundary conditions here are $\underline{w} = 0$ on $y = \pm \frac{1}{2}$. To introduce the appropriate parameters we set

$$(4.6) \quad \lambda = (2S(1 - 2S))^{1/2} R_e, \quad 0 < S < \frac{1}{2},$$

$$(4.7) \quad w_1 = \lambda^{-1}(2S - 1)u_1, \quad w_2 = \lambda^{-1}(2S(1 - 2S))^{1/2}u_2 \\ w_3 = \lambda^{-1}(2S(1 - 2S))^{1/2}u_3, \quad q = \lambda^{-1}R_e^{-1}(2S(1 - 2S))^{1/2}p,$$

and define the structure parameter, γ , as

$$(4.8) \quad \gamma = \frac{\delta}{1 - 2S} = \frac{U_p}{U_c(1 - 2S)}, \quad 0 < S < \frac{1}{2}.$$

The disturbance equations for $\underline{u} = (u_1, u_2, u_3)$ and p become

$$(4.9a) \quad \Delta u_1 + \lambda(1 - 2\gamma y)u_2 - \left(u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z}\right) = 0$$

$$(4.9b) \quad \Delta u_2 - \frac{\partial p}{\partial y} + \lambda u_1 - \left(u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z}\right) = 0$$

$$(4.9c) \quad \Delta u_3 - \frac{\partial p}{\partial z} - \left(u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z}\right) = 0$$

$$(4.9d) \quad \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0, \quad -\frac{1}{2} < y < \frac{1}{2}, -\infty < z < \infty,$$

with the boundary conditions

$$(4.10) \quad u_1 = u_2 = u_3 = 0 \quad \text{on} \quad y = \pm \frac{1}{2}.$$

The equations in (4.9) are closely related to those for the generalized Bénard problem studied in [12, 13]. In carrying out the analysis below for rotating Couette-Poiseuille flow we shall make repeated use of this relationship.

We next introduce an appropriate Hilbert space setting in which to seek solutions of (4.9) that are periodic in z . Given a positive number α (to be specified below in (4.22)) we set

$$(4.11) \quad R = \left\{ (y, z) : -\frac{1}{2} < y < \frac{1}{2} \text{ and } -\frac{\pi}{\alpha} < z < \frac{\pi}{\alpha} \right\}.$$

The Hilbert space, H , used through out this section is defined as the closure of the set $\{v = (v_1, v_2, v_3) : v \text{ is smooth, periodic in } z \text{ with period } \frac{2\pi}{\alpha}, \text{ and vanishing in a neighborhood of } |y| = \frac{1}{2} \text{ with } \nabla \cdot v = 0\}$ in the norm, $\|\cdot\|$, associated with the inner product

$$(4.12) \quad (v, w) = \int_R \sum_{j=1}^3 \nabla v_j \cdot \nabla w_j.$$

Here and in the sequel, whenever possible, the vector notation \underline{v} is suppressed when dealing with elements of H .

To formulate the problem as an operator equation in H , we take the scalar product of (4.9a) through (4.9c) with $w \in H$ and use (4.10) and integration by parts to obtain

$$(4.13) \quad (u, w) - \lambda(L_\lambda u, w) - (F(u), w) = 0.$$

Here, for each $\gamma \in \mathbf{R}^1$, the linear operator $L_\gamma : H \rightarrow H$ and the quadratic operator $F : H \rightarrow H$ are given by

$$(4.14) \quad L_\gamma = L - \gamma M,$$

$$(4.15) \quad F(u) = \Phi(u, u), \quad u \in H,$$

where the operators $L : H \rightarrow H$ and $M : H \rightarrow H$ are defined (weakly) by

$$(4.16) \quad (Lv, w) = \int_R (v_2 w_1 + v_1 w_2),$$

$$(4.17) \quad (Mv, w) = 2 \int_R y v_2 w_1, \quad v, w \in H,$$

and the bilinear operator $\Phi : H \times H \rightarrow H$ is defined (weakly) by

$$(4.18) \quad (\Phi(u, v), w) = - \int_R ((u \cdot \nabla)v) \cdot w = - \int_R \left(u_2 \frac{\partial v}{\partial y} + u_3 \frac{\partial v}{\partial z} \right) \cdot w.$$

Since w in (4.13) is an arbitrary element of H , one obtains an operator equation of the form (*) in §1, namely

$$(\dagger) \quad u - \lambda L_\gamma u - F(u) = 0, \quad u \in H, \lambda \in \mathbf{R}^1, \gamma \in \mathbf{R}^1.$$

Standard regularity methods (e.g., see [10, 11]) can now be used to show that the problems of finding classical solutions of (4.9), (4.10) and solutions of (\dagger) in H are equivalent.

We require the following facts about the linear problem associated with (\dagger) when $\gamma = 0$, namely the problem

$$(4.19) \quad u - \mu Lu = 0, \quad u \in H, \mu \in \mathbf{R}^1.$$

The linear problem (4.19) is equivalent to the classical problem, for smooth u and p periodic with period $2\pi/\alpha$ in z , obtained by setting $\gamma = 0$ and omitting the nonlinear terms in (4.9). It is sufficient to consider the solutions of (4.19) given by (see also [12, (2.11) ff] and [10; §3])

$$(4.20a) \quad u = (\phi_1(y) \cos \sigma z, \phi_2(y) \cos \sigma z, \phi_3(y) \sin \sigma z),$$

$$(4.20b) \quad p = -\sigma^{-1} \cos \sigma z D^2 \phi_3$$

$$(4.20c) \quad \phi_3 = -\sigma^{-1} \phi'_2,$$

where $D^2 = \frac{d^2}{dy^2} - \sigma^2$, a prime, denotes $\frac{d}{dy}$, and ϕ_1 and ϕ_2 satisfy

$$(4.21a) \quad D^4 \phi_2 - \mu \sigma^2 \phi_1 = 0,$$

$$(4.21b) \quad D^2 \phi_1 + \mu \phi_2 = 0,$$

$$(4.21c) \quad \phi_1 = \phi'_2 = \phi_2 = 0 \text{ at } y = \pm \frac{1}{2}.$$

(The solutions of (4.19) with $\cos \sigma z$ and $\sin \sigma z$ interchanged lead to flows that differ only by a translation from those obtained here). One can show for $\sigma > 0$ (e.g., see [8]) that the eigenvalue problem (4.21) has a countable number of positive, simple eigenvalues, $0 < \mu_1(\sigma) < \mu_2(\sigma) < \dots$, depending continuously on σ . Moreover, $\mu_1(\sigma) \rightarrow \infty$ as

either $\sigma \rightarrow 0^+$ or $\sigma \rightarrow \infty$. Consequently, $\mu_1(\sigma)$ assumes an absolute minimum at some $\sigma_0 > 0$. We assume that σ_0 is unique so that $\mu_1(\sigma) > \mu_1(\sigma_0)$ if $\sigma \neq \sigma_0$. (This property is suggested by numerical calculations (e.g., see [3, §15 (b)] and [6, §10]) and is usually assumed in such problems.) For some given integer $p_0 \geq 1$ we now choose α so that

$$(4.22) \quad \sigma_0 = p_0 \alpha$$

and use this α to define the basic Hilbert space H of this section.

We may now determine a complete solution of the linear problem (4.19). Since $\cos \sigma z$ and $\sin \sigma z$ in (4.20) must have period $2\pi/\alpha$ in z , it follows that the only wave numbers, σ , corresponding to eigenvalues having the required period in z are those for which $\sigma^2 = p^2 \alpha^2$ for some integer p . For each p ($p = 1, 2, \dots$) the eigenvalue problem (4.21) has an infinite sequence of real, nontrivial solutions

$$(\mu, \phi_1, \phi_2) = (\mu_{pq}, \phi_1^{pq}, \phi_2^{pq}), \quad p = 1, 2, \dots, \quad q = \pm 1, \pm 2, \dots$$

Since $(-\mu, -\phi_1, \phi_2)$ is a solution of (4.21) whenever (μ, ϕ_1, ϕ_2) is a solution of (4.21), we may order the indices so that

$$(4.23) \quad \phi_1^{p(-q)} = -\phi_1^{pq}, \quad \phi_2^{p(-q)} = \phi_2^{pq}, \quad \mu_{p(-q)} = -\mu_{pq}, \quad \text{and}$$

$$0 < \mu_{p1} < \mu_{p2} < \dots$$

Using this notation, we see from our assumption (4.22) on α that

$$(4.24) \quad \mu_0 \equiv \min_p \mu_{p1} = \mu_{p_0 1}$$

so that $\mu_{pq} > \mu_0$ if $(p, q) \neq (p_0, 1), q \geq 1$.

The above discussion of the underlying problem (4.21) shows that the full eigenvalue problem (4.19) in H has the solutions

$$(4.25) \quad \lambda = \mu_{pq} \text{ and } u = \psi^{pq}, \quad p = 1, 2, \dots, q = \pm 1, \pm 2, \dots,$$

where

$$(4.26) \quad \psi^{pq} = (\phi_1^{pq} \cos p\alpha z, \phi_2^{pq} \cos p\alpha z, \phi_3^{pq} \sin p\alpha z)$$

with ϕ_3^{pq} determined by (4.20c). It follows as in [12; Appendix] that the eigenfunctions $\{\psi^{pq}\}$ may be assumed orthonormal in H , after rescaling by constants depending upon p and q , i.e.,

$$(4.27) \quad (\psi^{pq}, \psi^{rs}) = \delta_{pr} \delta_{qs},$$

where δ_{ik} is the usual Kronecker delta symbol.

The following lemma summarizes some of the basic facts for the linearized problem (4.19). The compactness properties are essentially known (e.g., see [10, 11]) while the characterization (4.28) follows easily from (4.17).

LEMMA 4.1. (i) *The linear operator $L : H \rightarrow H$ defined in (4.16) is self-adjoint and compact and its characteristic values and eigenfunctions are given by (4.25). The eigenfunctions $\{\psi^{pq}\}$ satisfy (4.27) and are complete in H .*

(ii) *The linear operator $M : H \rightarrow H$ is compact and its adjoint, M^* is characterized by*

$$(4.28) \quad (M^*v, w) = 2 \int_R y v_1 w_2, \quad v, w \in H.$$

We show next that hypotheses (H1) through (H4) of §1 are satisfied so that we can make use of the structure parameter approach developed in §2. To minimize the calculations, we shall make repeated use of the results obtained in [12, 13]. This may be done in most cases simply by replacing ϕ_3 and ϕ_4 in [12] by ϕ_2 and ϕ_1 of the present paper, respectively.

Since μ_0 , defined in (4.24) is a simple eigenvalue of (4.21) and $\mu_0 < \mu_{pq}$ for $(p, q) \neq (p_0, 1)$, μ_0 is simple and also the smallest positive characteristic value of L in H . The associated null space, N , of $I - \mu_0 L$ is spanned by $\psi_0 \equiv \psi^{p_0 1}$ and N^\perp is spanned by $\{\psi^{pq} : (p, q) \neq (p_0, 1)\}$. Thus, making use of part (i) of Lemma 4.1 above, we see that (H1) is satisfied. If M is defined as in (4.17), then (H2) can be verified by using part (ii) of Lemma 4.1 above and part (ii) of Lemma 3.1 in [12]. The form of $\lambda_c = \lambda_c(\gamma)$ in (H3) may be derived as in Lemma 3.2 of [12]. In fact, the characteristic values of

$L_\gamma \equiv L - \gamma M$ are determined by the problem obtained from (4.21) by replacing (4.21a) by

$$(4.29) \quad D^4 \phi_2 - \mu \sigma^2 (1 - 2\gamma y) \phi_1 = 0.$$

Thus, λ_c is simple and is also real as an eigenvalue of problem (4.21) with (4.21a) replaced by (4.29) and σ set equal to σ_0 (e.g., see [8]). Finally, setting $G(w) \equiv 0$ in (H4) and recalling the definition of Φ in (4.18), one sees that F in (4.15) is generated by the bounded bilinear operator Φ (see also [10, 11]). Since it follows as in part (iv) of Lemma 3.1 in [12] that $\Phi : H \times N \rightarrow N^\perp$, we see that (H4) is also satisfied.

To determine the coefficients a and c in the bifurcation equation (2.4), we note first of all that Φ satisfies, in addition, the condition (1.3) in Remark 1.1 (see part (iii) of Lemma 3.1 in [12]). Thus, we may make use of parts (v) and (vi) in Lemma 3.1 of [12] to show that $a = 0$ always and $c > 0$ in essentially all cases. Moreover, by making use of the invariance of equations (4.9) under the translation $z \rightarrow z + \pi/\alpha$, one can show that, for each τ and γ , the remainder term r in (2.5) is odd in β (see the last part of Appendix B in [19] and also the proof of Lemma 3.2 in [13]). Thus, r has the special form given in (3.1), where, in addition, s is even in β .

In view of the above discussion one can now use Theorem 3.1 and Corollary 3.1 with $0 < \gamma < \gamma_0$ to determine nontrivial solutions of the operator equation (†) and, hence, roll-like solutions of the disturbance equations (4.9) satisfying the boundary conditions (4.10). Thus, for each fixed γ satisfying $0 < \gamma < \gamma_0$, where γ_0 is sufficiently small, rotating Couette-Poiseuille flow is stable up to $\lambda_c(\gamma)$ at which point it loses stability to a supercritical, stable roll-type solution given by (2.7) with $\beta^* = \beta, \tau^*$ given by (3.4) with $a = 0$, and $\psi_0 = \psi^{p_0 1}$ given by (4.26).

As we now show, the above result implies that the superposition of a Poiseuille flow on a rotating Couette channel flow is, in general, destabilizing. Recall that $\lambda_c(\gamma)$ is given by an expression such as (1.2) with $b = (MKM\psi_0, \psi_0)$. If $b > 0$, then, for γ sufficiently small, $\lambda_c(\gamma) < \lambda_c(0) = \mu_0$, where μ_0 is the critical eigenvalue of the linearized problem for rotating Couette flow; to see that $b > 0$ here, one can make use of the results in [3; §71(d)] (see also [6, p.98]) for narrow-gap Taylor problems with the parameter $(1 - \mu)/(1 + \mu)$ in [3] replaced by γ in

(4.8). Thus, for a given value of the swirl S , $0 < S < \frac{1}{2}$, the addition of a sufficiently small component of Poiseuille flow to a basic rotating Couette channel flow always leads to roll-type solutions at values of λ that are greater than $\lambda_c(\gamma)$ but less than the critical eigenvalue, μ_0 , at which rotating Couette flow loses stability. This type of result was conjectured in [17] for non-rotating combined Couette-Poiseuille flow on the basis of numerical calculations for the linearized Navier-Stokes equation.

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