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# MINIMAL GENERATION OF BASIC SEMIALGEBRAIC SETS

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Dedicated to the Memory of Gus Efroymson

**Introduction.** Let $R$ be a real closed field and $V$ an affine algebraic $R$-variety. We assume that $V(R)$ is Zariski-dense in $V$. A basic semialgebraic set $S \subset V(R)$ is a set of the form $S = S(f_1, \ldots, f_m) := \{x \in V(R) \mid f_i(x) > 0,\ i = 1, \ldots, m\}$ for suitable $f_i \in R[V]$. How many $f_i$ are needed for such a representation of $S$? It is shown that there exists a finite upper bound depending only on the dimension $n$ of $V$. This bound is equal to $n$ for $n \leqq 3$. I did not succeed in proving (or disproving) this for $n > 3$. Anyway, the best bound is $\geqq n$. We shall also characterize the basic semialgebraic sets among the open semialgebraic sets.

**1. The real spectrum.** For a quasicompact scheme $S$ we denote by $(X(S), \beta(S))$ the real spectrum [**4**]. This is a restricted topological space $X(S)$ with base $\beta(S)$ [**2**]. For an $R$-variety $V$ one has also the restricted topological space $(V(R), \gamma(V))$, where $\gamma(V)$ is the lattice generated by all sets, which are basic semialgebraic after restriction to open affine subsets of $V$. By the ultrafilter theorem [**2**] one has canonical isomorphisms

$$(\hat{V}(R), \hat{\gamma}(V)) \underset{p}{\overset{f}{\rightleftarrows}} (X(V), \beta(V))$$

where $\wedge$ means canonical ultrafilter completion of a restricted topological space.

Now let $x_1, \ldots, x_l$ be real points of the $R$-Variety $V$, and let $A$ be the semilocal ring $A = \lim_{\rightarrow} \mathcal{O}(U)$, $U$ open in $V$, $x_1, \ldots, x_l \in U$. We set $\hat{V}(x_1, \ldots, x_l) := \{F \in \hat{V}(R) \mid x(F)$ generalizes some $x_i\}$, and provide this with the induced base $\hat{\gamma}(x_1, \ldots, x_l)$. Then the projection $\lambda$: $\mathrm{Spec}(A) \to V$ defines an imbedding

$$X(\lambda)\colon (X(\mathrm{Spec}(A)), \beta(\mathrm{Spec}(A)) \to (X(V), \beta(V)),$$

moreover, one has the commutative diagram

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$$\begin{array}{ccc}(X(\mathrm{Spec}(A)), \beta(\mathrm{Spec}(A)) & \xrightarrow[f \mid X(\mathrm{Spec}(A))]{} & (\hat{V}(x_1, \dots x_l), \hat{\gamma}(x_1, \dots, x_l)) \\ \downarrow{\scriptstyle X(\lambda)} & & \downarrow{\scriptstyle i} \\ (X(V), \beta(V)) & \xrightarrow[f]{} & (\hat{V}(B), \hat{\gamma}(V)),\end{array}$$

where $f \mid X(\mathrm{spec}(A))$ is an isomorphism.

We denote by $(\hat{\mathbf{V}}(x_1, \dots, x_l), \hat{\boldsymbol{\gamma}}(x_1, \dots, x_l)$ the subspace of all closed points in $(\hat{V}(x_1, \dots, x_l), \hat{\gamma}(x_1, \dots, x_l)$. The map $S \to \hat{S}$ defines a lattice-isomorphism : $\gamma(V) \to \hat{\gamma}(V)$, but this no longer holds for the above sublattice. Nevertheless, we have the following Proposition.

PROPOSITION 1. *For* $S_1, S_2 \in \gamma(V)$ *one has* $\hat{S}_1 \cap \hat{\mathbf{V}}(x_1, \dots, x_l) = \hat{S}_2 \cap \hat{\mathbf{V}}(x_1, \dots, x_l)$ *iff in* $\overline{(S_1 \cup S_2) \setminus (S_1 \cap S_2)}^Z$ *there is no generalization of some* $x_i$.

(Here we need the index $Z$ for Zariski, whereas general topological symbols without index $Z$ refer to the strong topology, which is generated by the base of the corresponding restricted topological spaces.)

PROOF. See [3].

**2. Relation to spaces of orderings.** Let $A$ be a commutative ring with unit and $W(A)$ its Wittring. Following Knebusch [8], [9], a homeomorphism $\sigma\colon W[A] \to \mathbf{Z}$ is called a signature of $A$. We provide the set $\mathrm{Sign}(A)$ of all signatures of $A$ with the base $Z(A)$, which is generated by all sets $Z(\varphi, n) := \{\sigma \in \mathrm{Sign}(A) \mid \sigma(\varphi) = n\}$ with $\varphi \in W(A)$ and $n \in \mathbf{Z}$. One has a natural map

$$\pi\colon (X(A), \beta(A)) \to (\mathrm{Sign}(A), Z(A));\ (x, P(x)) \mapsto \sigma$$

$$\text{where } \sigma(\varphi) = \mathrm{sign}_{P(x)}(k(x) \underset{A}{\otimes} \varphi) \text{ and } X(A) = X(\mathrm{Spec}(A)).$$

By Dress [5] $\pi$ is surjective; apparently $\pi$ is constant on connected components, and Mahé [10] has even shown that $\pi$ defines a homeomorphism between $\mathrm{Sign}(A)$ and the space of the connected components of $X(A)$. Now, if $A$ is semilocal and connected, each component of $X(A)$ admits exactly one closed point, hence $\pi$ induces a homeomorphism $\mathbf{X}(A) \to \mathrm{Sign}(A)$. Following Schwartz [13] this can be seen directly: $\sigma \in \mathrm{Sign}(A)$ defines a canonical decompostition $A = Q(\sigma) \cup p(\sigma) \cup -Q(\sigma)$; $p(\sigma)$ is a prime ideal and for $Q(\sigma)$ the relations $Q(\sigma) + Q(\sigma) \subset Q(\sigma)$ and $Q(\sigma) \cdot Q(\sigma) \subset Q(\sigma)$ hold. Moreover, $Q(\sigma) \cup p(\sigma) \in \mathbf{X}(A)$ [6] [8].

PROPOSITION 2. *Let $A$ be semilocal and connected. The map* $\mathrm{Sign}(A) \to \mathbf{X}(A)$; $\sigma \mapsto Q(\sigma) \cup p(\sigma)$ *inverts* $\pi$; $\pi\colon \mathbf{X}(A) \to \mathrm{Sign}(A)$ *is a homeomorphism.*

Note that $\pi$ need not be an isomorphism of restricted topological spaces.

Now for $q^*(A) = \{a \in A^* \mid \sigma\langle a \rangle = 1 \text{ for all } \sigma \in \mathrm{Sign}(A)\}$ and $G(A) = A^*/q^*(A)$ the pair $(\mathrm{Sign}(A), G(A))$ is a space of orderings in the sense of

Marshall [**11**], [**12**]. For the proof see 6.4 in [**7**], 2.5a in [**9**]. Now, by propositions 1 and 2 the theory of the spaces of orderings is made applicable for geometrical problems. In particular, we use the following proposition.

PROPOSITION 3. *Let* $(X, G)$ *be a space of orderings and* $B \subset X$ *a clopen subset.*

a) *There exist elements* $g_1, \ldots, g_n \in G$, *such that* $B = B(g_1, \ldots, g_n)$ *iff for all fans* $Y \subset X$ *with* $|Y| = 4$ *one has* $|Y \cap B| \neq 3$.

b) *If, moreover, for each finite fan* $Y \subset X$ *one has* $2^k|B \cap Y| \equiv 0 \bmod |Y|$, *there exist* $g_1, \ldots, g_k \in G$ *with* $B = B(g_1, \ldots, g_k)$.

Here $B(g_1, \ldots, g_k) = \{\sigma \in X \mid \sigma(g_i) = 1 \text{ for } i = 1, \ldots, k\}$. Without b) this is [**12**, 3.16], and b) can be proved correspondingly using 5.5 in [**11**].

**3. Generation and characterization of basic semialgebraic sets.** Let $V$ be an affine algebraic $R$-variety, $R$ real closed, such that $V(R)$ is $Z$-dense in $V$, $n = \dim V$. Among the open semialgebraic sets $S \subset V(R)$ a basic one has the following additional properties:

(A) $S \cap \overline{\partial S}^Z = \emptyset$.

For $U < V$, $U$ real, integral and closed, one has

(F) $|Y \cap p(\hat{S})| \neq 3$ for all fans $Y \subset X(R(U))$ with $|Y| = 3$.

Here $p$ is defined as in §1. If moreover $S$ is of the form $S = S(a_1, \ldots, a_k)$, then for the above $U < V$ one has

($F_k$) $2^k|Y \cap p(\hat{S})| \equiv 0 \bmod |Y|$ for all finite fans $Y \subset X(R(U))$.

PROPOSITION 4. *Let* $S \subset V(R)$ *be open semialgebraic.*

a) *If* (A) *holds for* $S$ *and also* ($F_k$) *for all* $U < V$ *as above. then* $S$ *is basic.*

b) *If, moreover, for all* $m \leqq n = \dim V$ *there exists a number* $k(m) \in \mathbf{N}$ *such that* ($F_{k(m)}$) *holds for all* $U < V$ *with* $\dim U \leqq m$ *then there exists a sequence* $1 < i_1 < \cdots < i_r = n$ *with* $i_{j+1} - i_j \geqq 2$ *such that* $S$ *is of the form* $S = S(b_1, \ldots, b_k)$ *for* $k \leqq \prod_{j=1}^{r} k(i_j)$.

For the proof one applies Prop. 3 on a suitable semilocalization of $V$. So by Prop. 2 and Prop. 1 one gets a representation of $S$ of the form $S = S(b_1, \ldots, b_{k(n)})$ up to a set of lower dimension. This aberration can be represented by further elements $c_1, \ldots, c_{k(i)}$. Unfortunately, the number of elements we need to drop the dimension of the defect increases multiplicatively in our proof. See [**3**] for the details.

COROLLARY. *Let* $S \subset V(R)$ *be basic semialgebraic. Then* $S$ *is of the form* $S = S(b_1, \ldots, b_m)$ *with* $m \leqq \prod_{i=1}^{[(n+1)/2]} (2i - (1/2)(1 - (-1)^n)$.

This follows from the fact that stability index of $F$ = transcendence degree of $F$ for function fields over $R$ [**1**].

COROLLARY. *For $n = 1$ every open semialgebraic set $S \subset V(R)$ is of the form $S = S(b)$.*

This is more of less well known [**14**].

COROLLARY. *Suppose that $R$ is the field* **R** *and $V(R)$ complete. Let $S \subset V(R)$ be semialgebraic and open. If for each pair $x, y$ of points in $V(R)$ there exists an open set $0 \subset V(R)$ with $x, y \in 0$ and a basic semialgebraic set $S' \subset V(R)$ such that $S \cap 0 = S' \cap 0$ then $S$ is basic. If, moreover, $S'$ is always of the form $S(b')$, then $S$ is of the form $S(b)$ too.*

PROOF. See [**3**].

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