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HOLOMORPHIC FUNCTIONS COMMUTING WITH ABSOLUTE VALUES

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Introduction. It is often possble in complex analysis to derive very strong conclusions about holomorphic functions from apparently weak information. Suppose, for example, that f is holomorphic in a disk centered at 0 in the complex plane, and that f commutes with absolute values in the sense that

$$f(|z|) = |f(z)| \tag{1}$$

One can then conclude that

(2) $f(z) = cz^m$, where $c \geqq 0$, and m is a non-negative integer.

A proof of this exercise usually relies on a power series expansion for f. In this note we extend this result in two directions. First of all, we observe that if Ω is a simply connected domain, not containing 0, and such that (1) makes sense for all z in Ω, then we can conclude that

(3) $f(z) = cz^\alpha$ where $c \geqq 0$, and α is an arbitrary real number.

Secondly, if Ω is a domain in $\mathbf{C}^n$ for which real powers of z are holomorphic, and $|z| = (|z_1|, |z_2|, \ldots, |z_n|)$, we can still conclude that (3) holds, except α is then an arbitrary real multi-index.

Our proof relies on the polar form of the Cauchy-Riemann equations, and integration of some real ordinary differential equations.

Statement and proof of the result. Let Ω be an open domain in $\mathbf{C}^n$. We say Ω is R-like if whenever z lies in Ω, so does $|z|$, Here $|z| = (|z_1|, |z_2|, \ldots, |z_n|)$. We say Ω is L-like if the functions $g(z) = \log(z_j)$ are all holomorphic on Ω. In particular this implies that Ω does not intersect any of the coordinate axes. Furthermore, if Ω is L-like, the function $g(z) = z^\alpha$ is holomorphic for any real multi-index α. Note that both concepts, L-like and R-like, are not preserved under general holomorphic changes of coordinates, because the origin and the notion of absolute value must remain invariant.

We recall that f is holomorphic on Ω if any only if f is continuously dif-

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ferentiable and satisfies the Cauchy-Riemann equations in each variable. These equations take the form

$$r_j \frac{\partial f}{\partial r_j} + i \frac{\partial f}{\partial \theta_j} = 0 \tag{4}$$

if $z_j = r \exp(i\theta_j)$ for $j = 1, 2, \ldots, n$. We will write partial derivatives as subscripts, so $f_{r_j} = \partial f / \partial r_j$.

We are now ready to prove our theorem.

THEOREM. *Let Ω be an L-like, R-like open domain in $\mathbf{C}^n$. Suppose that f is holomorphic on Ω, and that*

$$f(|z|) = |f(z)|. \tag{5}$$

Then there is a non-negative number c, and a real multi-index α, so that $f(z) = cz^\alpha$.

PROOF. If f is identically 0, the result is trivial. Otherwise let V denote the zero set of f, and put $\tilde{\Omega} = \Omega - V$. On $\tilde{\Omega}$, there are continuously differentiable real functions g and h so that

$$f(z) = f(re^{i\theta}) = g(r, \theta) \exp(ih(r, \theta)) \tag{6}$$

where $r = (r_1, \ldots, r_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$ are polar coordinates. We have $r > 0$, and $0 \leqq \theta < 2\pi$.

By application of (5), we see immediately that g is actually independent of θ, so that (6) becomes

$$f(re^{i\theta}) = g(r) \exp(ih(r, \theta)). \tag{7}$$

We apply (4) to (7), and separate real and imaginary parts, This gives us the following system of partial differential equations.

$$r_j g_{r_j} = g h_{\theta_j} \text{ and } r_j g h_{r_j} = 0 \text{ for all } j. \tag{8}$$

on $\tilde{\Omega}$, $g \neq 0$, so we can rewrite (8) as

$$h_{\theta_j} = r_j g_{r_j} / g \text{ where } h \text{ is independent of } r. \tag{9}$$

Notice that the left side of (9) is independent of r, and the right side is independent of θ. Therefore there are real constants α_j so that

$$h_{\theta_j} = \alpha_j = r_j g_{r_j} / g \text{ for all } j. \tag{10}$$

We integrate (10) to obtain

$$g(r) = e^\lambda r^\alpha, \; h(\theta) = \alpha\theta + k, \tag{11}$$

where λ and k are real constants of integration. Here $\alpha\theta$ denotes $\sum \alpha_j \theta_j$ and $r^\alpha = \prod r_j^{\alpha_j}$ To insure that (5) holds we must choose $k = 0$. This finally gives

$$f(z) = c(re^{i\theta})^{\alpha}, \tag{12}$$

where $c \geqq 0$, and r, θ, α denote multi-indices.

This finishes the proof, if we note that the solution (12) is holomorphic in $\tilde{\Omega}$, and agrees with our original f on all of Ω. Since Ω is L-like, f is holomorphic on Ω.

COROLLARY. *Suppose that Ω is a domain in* $\mathbf{C}^n$, *and that* 0 *lies in* Ω. *Then, if Ω is R-like and* (5) *holds, we can conclude that* $f(z) = cz^m$, *where* $c \geqq 0$, *and m is a multi-index of non-negative integers.*

PROOF. We apply the proof of previous theorem to Ω-W, where W denotes the union of the coordinate axes. Ω-W is still R-like, so the proof gives us that $f(z) = cz^{\alpha}$. Since f must extend to be analytic on all of Ω, we must have that α is a multi-index of non-negative integers.

COROLLARY. *If* $n = 1$, *Ω is simply connected and does not contain* 0, *and* (5) *holds, then* $f(z) = cz^{\alpha}$ *some* $c \geqq 0$ *and real number* α.

PROOF. The hypothesis implies that Ω is L-like. The result therefore follows immediately from the theorem.

REFERENCE

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