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THE HULLS OF C(Y)

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Introduction. Let $C(Y)$ be the set of all continuous real-valued functions on a completely regular space Y. Then $C(Y)$ can be considered as an ℓ-group G_1 or as a semiprime ring G_3, and in each case it admits various X-hulls, which are minimal essential extensions with some property X. We show that G_1^X is essentially the same as G_3^X and investigate the structure of these X-hulls. All of these hulls are contained in the complete ring of quotients $Q(Y)$ of G_3, and, in fact, $Q(Y)$ is the lateral completion of G_1 or of G_3.

In the first two sections we summarize the theory known for abelian ℓ-group and commutative semiprime ring X-hulls. The third section contains a description of the hulls of $C(Y)$, and their relationships with one another. §4 contains characterizations of $C(Y)$ considered as an abstract ℓ-group.

For further information about lattice-ordered groups (ℓ-groups), see **[9]** or **[14]**; for semiprime rings, see **[26]**; for $C(Y)$, see **[24]**.

We will use $\sum T_\lambda(\Pi T_\lambda)$ to represent the restricted (unrestricted) direct product of the groups or rings T_λ; in the case of ℓ-groups, these groups are equipped with the cardinal order.

We wish to acknowledge the valuable advice of Jack Porter about the topological results that appear in this paper. In particular, Theorem 3.9 and Example 3.12 are entirely due to him.

1. The hulls of semiprime rings. Throughout this section let G be a commutative semiprime ring (that is, G is a subdirect product of integral domains) with identity. We summarize some of the X-hull theory of G that is developed in **[18]**, **[19]**, and **[20]**. Actually, this theory also holds for non-commutative semiprime rings.

For $a, b \in G$ define $a \underset{\sim}{\alpha} b$ if $a^2 = ab$. This is a partial order for G (introduced in **[1]**) with smallest element 0 and for $a, b, x \in G$, $a \underset{\sim}{\alpha} b$ implies that $ax \underset{\sim}{\alpha} bx$. Moreover, $a \underset{\sim}{\alpha} b$ if and only if in each representation of $G \subseteqq \Pi T_\lambda$ as a subdirect product of integral domains T_λ, $a_\lambda \neq 0$ implies that $a_\lambda = b_\lambda$.

One says that a is *disjoint* from b or that a is *orthogonal* to b if $ab = 0$ (notation: $a \perp b$). This is equivalent to the fact that a and b have disjoint

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support in each representation of G as a subdirect product of integral domains. Note that $a \underline{\alpha} b$ if and only if $a \perp b - a$, and $a \perp b$ if and only if $a \underline{\alpha}\, a + b$. If X is a subset of G, then $X' = \{g \in G: g \perp x$ for each $x \in X\}$ is the annihilator ideal of X. The set $P(G)$ of all these annihilator ideals is a complete Boolean algebra [**26**, p. 43].

One says that $\{a, b\} \subseteqq G$ is *boundable* if $ab \underline{\alpha} a^2$, and this is the case if and only if a and b agree on their common support in each representation of G as a subdirect product of integral domains. A subset S of G is *boundable* if each pair in S is boundable. G will be called *a P-ring* if $G = g'' \oplus g'$ for each $g \in G$ (*projectable*); an *SP-ring* if $G = X'' \oplus X'$ for each subset $X \subseteqq G$ (*strongly projectable*); an *L-ring* if each pairwise disjoint set has a l.u.b. (*laterally complete*); an *O*-ring if G is both an *L*-ring and an *SP*-ring (*orthocomplete*); a *CC-ring* if each bounded set has a l.u.b. (*conditionally complete*); a *CL-ring* if each bounded disjoint set has a l.u.b. (*conditionally laterally complete*); an *FC-ring* if each finite boundable set has a l.u.b. (*finitely complete*); and a *C-ring* if each boundable set has a l.u.b. (*complete*).

A commutative overring H is an *essential extension* of G if this is the case when H is considered as a G-module. In this case H is also semiprime and the po of G is induced by the po of H. Also, if S is a boundable subset of G, then it is boundable in H.

For a commutative semiprime ring G and $X = P$, SP, L, O, CC, CL, FC or C we have the following theorems.

THEOREM. *If H is an essential extension of G that is an X-ring, then the intersection of all the subrings of H that contain G and are X-rings is a minimal essential extension of G that is an X-ring; it is called an X-hull of G.*

THEOREM. *G admits a unique X-hull G^X. Furthermore, $G \subseteqq G^P \subseteqq G^{SP} \subseteqq (G^{SP})^L = (G^P)^L = G^O$, $G^{CL} \subseteqq G^L \subseteqq G^C \subseteqq G^O$, $G^{CL} \subseteqq G^{CC} \subseteqq G^C$, $G^{FC} \subseteqq G^C$, and $G^O \subseteqq Q(G)$, the complete ring of quotients of G.*

THEOREM. *Suppose G is an FC-ring.*

(a) *G is an L-ring if and only if G is a C-ring.*

(b) *G is a CL-ring if and only if G is a CC-ring.*

Furthermore, if G is a P-ring, so is G^{CL} and $G^{CL} = G^{CC}$.

2. The hulls of ℓ-groups and f-rings. Throughout this section let $G = (G, +, \wedge, \vee, \leqq)$ be an abelian ℓ-group. We summarize some of the X-hull theory of G that is developed in [**16**] and [**20**]. Actually, this theory also holds for representable ℓ-groups.

For $a, b \in G$ define $a \underline{\beta} b$ if $|a| \wedge |b - a| = 0$. This is a partial order for G with smallest element 0. Moreover, $a \underline{\beta} b$ if and only if in each representation of $G \subseteqq \Pi T_\lambda$ as a subdirect product of o-groups T_λ, $a_\lambda \neq 0$ implies that $a_\lambda = b_\lambda$.

One says that a is disjoint from b or that a is orthogonal to b if $b \,\underline{\beta}\, a + b$ (notation $a \perp b$). This is equivalent to the fact that $|a| \wedge |b| = 0$ and so disjointness is the same as ℓ-group disjointness. If X is a subset of G, then $X' = \{g \in G\colon g \perp x$ for each $x \in X\}$ is the polar of X. The set $P(G)$ of all these polars is a complete Boolean algebra [**14**, p.2.4].

One says that $\{a, b\} \subseteqq G$ is *boundable* if $a \wedge b \,\underline{\beta}\, b \wedge b$ and $a \wedge b \,\underline{\beta}\, a \wedge a$, and this is the case if and only if a and b agree on their common support in each representation of G as a subdirect product of o-groups. A subset S of G is *boundable* if each pair in S is boundable. We can now define X-group, where $X = P$, SP, L, O, CC, CL, FC and C in a way directly analogous to the definition of X-ring in §1.

Note that if $\{a, b\} \subseteqq G$ is boundable, then $h = a^+ \vee b^+ - (a^- \vee b^-)$ is the l.u.b. of $\{a, b\}$ with respect to $\underline{\beta}$. Thus all ℓ-groups are finitely complete.

If G is an ℓ-subgroup of the abelian ℓ-group H, then H is an *essential extension* of G (or G is *large* in H) if $L \cap G \neq 0$ for each non-zero ℓ-ideal L of H.

For an abelian ℓ-group G and $X = P$, SP, L, O, CC, CL, FC or C we have the following theorems.

THEOREM. *If H is an essential extension of G that is an X-group, then the intersection of all the ℓ-subgroups of H that contain G and are X-groups is a minimal essential extension of G that is an X-group; it is called an X-hull of G.*

THEOREM. *G admits a unique X-hull G^X. Moreover, if G is Archimedean, then so is G^X. Furthermore, $G \subseteqq G^P \subseteqq G^{SP} \subseteqq (G^{SP})^L = (G^P)^L = G^O$, and $G = G^{FC} \subseteqq G^{CL} = G^{CC} \subseteqq G^C = G^L$.*

Note that G^L is the minimal extension of G in which each pairwise disjoint set has a l.u.b. with respect to $\underline{\beta}$; it is also the minimal essential extension of G in which each set of pairwise disjoint elements in $(G^L)^+$ has a l.u.b. with respect to $\leqq$. Also, if G is Archimedean, then $G^L = G^O$ [**8**], and $G \subseteqq G^O \subseteqq G^e$, the *essential closure* of G which is the unique essential extension of G that is essentially closed in the category of Archimedean ℓ-groups (i.e., admits no Archimedean essential extensions). Now G^e is of the form $D(Y)$, the ℓ-group of almost finite continuous functions to the extended reals on the compact extremally disconnected (or Stonean) space Y, which corresponds to the complete Boolean algebra of polars of G [**15**, p. 155].

The conditional lateral completion $G^{CL}_{\leqq}$ of G with respect to $\leqq$ is not the same as G^{CL} and if G is Archimedean, then the Dedekind completion $G^{\wedge}$ is not the same as G^{CC}.

In [20] it is shown that for a boundable set $U \subseteqq G^+$, $\vee_{\leq} U$ exists if and only if $\vee_{\beta} U$ exists, and if this is the case these joins are equal. Now suppose that G is Archimedean; then $G \subseteqq G^{CC} \subseteqq G^{\wedge}$ and $G \subseteqq G^P \subseteqq G^{SP} \subseteqq G^{\wedge}$. In particular, if G is a subdirect product of reals, then so is $G^{\wedge}$ [21, p. 189] and hence so are G^P, G^{SP} and G^{CC}.

However, if G is a laterally complete ℓ-group with nonmeasurable cardinality, then G is a subdirect product of reals if and only if $G \cong \Pi T_\lambda$, with each $T_\lambda \subseteqq \mathbf{R}$ [3, p. 74]. Thus, if G is a subdirect product of reals, then so is G^L if and only if $\sum T_\lambda \subseteqq G \subseteqq G^L = \Pi T_\lambda$. Hence, in general, if G is a subdirect product of reals, then G^L need not be.

Recall that an f-ring G is a lattice-ordered ring such that $x \wedge y = 0$ implies that $dx \wedge y = xd \wedge y = 0$ for all s, y, $d \in G^+$. We shall make some remarks here about the existence of f-cones.

PROPOSITION 2.1. *Suppose that G is a ring with no non-zero nilpotent elements.*

(a) *If Q is an f-cone for G and $Q \subseteqq P$, a ring lattice order for G, then $Q = P$.*

(b) *If $S = \{g^2 \colon g \in G\}$ is an f-cone for G, then it is the unique f-cone.*

PROOF. (a) Suppose by way of contradiction that $P \supset Q$ and pick $g \in P \backslash Q$. Then $g = a - b$, with $a \wedge b = 0$ and $b > 0$, with respect to Q. Thus $g, b \in P$ and so $gb = -b^2 \in P$. But $b^2 \in Q \subseteqq P$, a contradiction.

(b) This follows from the fact that S must be contained in any f-cone.

PROPOSITION 2.2. *If G is a commutative semiprime ring and P is a ring lattice order for G that induces $\underline{\alpha}$, then P is an f-cone.*

PROOF. Now $|a| \wedge |b| = 0$ if and only if $a \perp b$, and so the polars are the annihilator ideals. In particular, each polar is a ring ideal and so P is an f-cone.

If G is an f-ring, then there is a unique multiplication on G^X so that it is an f-ring and G is a subring, for $X = P$, SP, L or O [16, Theorem 4.6].

Now, using the fact that G^O is an f-ring it is easy to show that there exists a unique minimal extension G^{Xf} of the f-ring G that is an X-group and also an f-ring. Moreover, G^{Xf} is isomorphic to the intersection of all X-subgroups of G^O that contain G and are f-subrings of G^O, for $X = P$, SP, L, O, CC, CL, FC or C (see [16, Theorem 3.3]), and G^X with the above ring structure equals G^{Xf} for $X = P$, SP, L or O [16, Theorem 4.6]; if G is Archimedean, this is also true for $X = \wedge$. In §3 we show that this is also true for $X = CL$.

3. The hulls of C(Y). Let Y be a Tychonoff space and let $G = C(Y)$

be the set of all continuous real-valued functions on Y. Consider G as an ℓ-group $G_1(G, +, \leqq)$ with induced po $\underset{\sim}{\beta}$, as an f-ring $C_2(G, +, \cdot, \leqq)$ with induced po $\underset{\sim}{\beta}$, or as a semiprime ring $G_3(G, +, \cdot)$. Since G_1 is archimedean, $G_1^L = G_1^Q$ [8] and so $G_1^Q = G_1^C = G_1^L \supseteqq G_1^{CL} = G_1^{CC}$. Now $|a| \wedge |b| = 0$ in G_1 if and only if $a \perp b$ in G_3 and so it follows that $\underset{\sim}{\beta}$ and $\underset{\sim}{\alpha}$ are the same partial order and the polars in G_1 are the same as the annihilator ideals in G_3.

Since the positive cone of G_2 consists of squares, it follows from Proposition 2.1 that $\leqq$ is the unique f-order on G_3 and also the unique ring lattice order on G_3 that induces $\underset{\sim}{\alpha}$.

Let e be the identity of G_3. Then by Theorem 1.1 in [17] the multiplication of G_3 is the unique multiplication so that G_1 is an f-ring with $e^2 = e$. Also, this is the unique multiplication so that G_1 is an ℓ-ring with identity e. For in this case G_1 is an f-ring by Corollary 3 of Theorem 15 in [10].

Note that an ℓ-cone for G_3 need not be an f-cone. For **R** admits a lattice order that is not total and hence not an f-order [34].

An ℓ-cone for the additive group $(G, +)$ that induces $\underset{\sim}{\beta}$ need not be a ring ℓ-cone. For $(\mathbf{R}, +)$ admits a non-Archimedean total order, and such an order induces $\underset{\sim}{\beta}$ but is not a ring order.

Now, we shall show that G_1^X and G_3^X (and also G_2^{Xf}) are essentially the same for $X = P$, SP and O, in the following sense.

THEOREM 3.1. *There exists a unique multiplication* $\#$ *on* G_1^X *so that it is an f-ring and* $e \# e = e$. *Moreover,* $\#$ *is the unique multiplication so that* G_1^X *is an* ℓ*-ring with identity e and* $(G_1^X, +, \#)$ *is the X-hull of* G_3.

QUESTION. Is $\#$ the unique multiplication so that G_1^X is an ℓ-ring and $e \# e = e$?

THEOREM 3.2. *There exists a unique ring lattice order P on* G_3^X *that induces* $\underset{\sim}{\alpha}$. *Moreover, P is the unique f-order on* G_3^X *and* $(G_3^X, +, P)$ *is the X-hull of* G_1.

Preliminary to the proofs of these theorems, we make the following observations. Since the Boolean algebras of polars and annihilator ideals coincide for $G = C(Y)$, when $X = P$, SP or O, the additive groups $(G_3^X, +)$ and $(G_1^X, +)$ can be constructed using the same direct limits of products of quotients of G by polars (see [16] and [19]). Thus we can and shall assume that $(G_3^X, +) = (G_1^X, +)$, for these X. Furthermore, if A is a polar of G_1 and $a, b \in G$, then $a^2 = ab \pmod{A'}$ if and only if $|a| \wedge |b - a| = 0 \pmod{A'}$, and so from the direct limit construction we have that $(G_1^X, +, \underset{\sim}{\beta}) = (G_3^X, +, \underset{\sim}{\alpha})$, for $X = P$, SP or O.

PROOF OF THEOREM 3.1. G_1^X is Archimedean, and since G_1 is large in G_1^X, e is an order unit in G_1^X. Then by Theorem 1.1 of [17] there exists a unique minimal f-ring with identity e and containing $(G_1^X, +, \leqq)$ as a large ℓ-subgroup. By Theorem 4.6 in **[16]** there exists a unique multiplication $\#$ on G_1^X so that it is an f-ring with identity e and G_2 as a subring. By Theorem 2.2 in **[17]**, $\#$ is the unique multiplication so that $(G_1^X, +, \#, \leqq)$ is an f-ring with $e \# e = e$.

Now suppose that $*$ is a multiplication so that $(G_1^X, +, *, \leqq)$ is an ℓ-ring with identity e. Then since it is Archimedean and e is positive and a weak order unit, it is an f-ring by Corollary 3 of Theorem 15 in **[10]**. Thus $\#$ is the unique multiplication so that $(G_1^X, +, \#, \leqq)$ is an ℓ-ring with identity e.

Now $(G_1^X, +, \#, \leqq)$ is the X_f-hull of G_2 by Theorem 4.6 in **[16]** and so it can be embedded as an f-ring in the f-ring $D(S)$, where S is the Stonean space for G_1. In particular, $(G_1^X, +, \#)$ is semiprime and since $(G_1^X, +, \underline{\beta}) = (G_3^X, +, \underline{\alpha})$ we have, for $a, b \in G_1^X$, $a \, \underline{\alpha} \, b$ if and only if $a \, \underline{\beta} \, b$ if and only if $|a| \wedge |b - a| = 0$ if and only if $a(y) \neq 0$ implies $a(y) = b(y)$ if and only if $a \# a = a \# b$. Therefore $\#$ induces the po $\underline{\alpha}$ on G_3^X and hence by Theorem 7.4 of **[19]**, $(G_1^X, +, \#)$ is the X-hull of G_3.

Proof of theorem 3.2. By Theorem 3.1, $(G_3^X, +, \cdot, \leqq)$ is an f-ring and from the proof of Theorem 3.1 $\leqq$ induces $\underline{\alpha}$. Now each positive element in G_2 is a square and it follows from the direct limit construction that each positive element in $(G_3^X, +, \cdot, \leqq)$ is a square. Thus $\leqq$ is the unique f-order for G_3^X.

If P is a ring lattice order for G_3^X that induces $\underline{\alpha}$, then the polars with respect to P are the annihilator ideals in G_3^X and so P is an f-order. Thus P must be the positive cone for $\leqq$. In particular, $(G_3^X, +, P)$ is the X-hull of G_3.

We shall now show that for $C(Y) = G$, all of the ring and group X-hulls are contained in the ring of quotients $Q(Y)$. In fact, we shall prove that $Q(Y) = G_3^Q = G_1^Q$.

In **[23]** the following construction is given for $Q(Y)$. Let F be the set of all continuous real-valued functions on any dense open subset of Y. Define $f \sim g$ if f and g agree on some dense open subset. Then $Q(Y)$ consists of all the equivalence classes $\tilde{f}$; that is, it is the direct limit of $\{C(V)\colon V \text{ is dense open in } Y\}$. We may define a partial order on $Q(Y)$ by making $\tilde{f}$ positive if $f(y) \geqq 0$ for all y on some dense open subset of Y. Then $\tilde{f}$ is positive if and only if it is a square. It is easily checked that this is an f-cone for $Q(Y)$ and hence is the unique f-order, which extends the f-order on $C(Y)$. In summary then, we have the following proposition.

Proposition 3.3. *The squares form an f-cone for $Q(Y)$ and hence a unique f-cone. Moreover, $Q(Y)$ is Archimedean and an essential extension of the*

f-ring G_2, and so $G_3 \subseteqq G_3^Q \subseteqq Q(Y) \subseteqq G_2^e = D(S)$, where S is the Stonean space of the Boolean algebra of annihilator ideals of G_3.

If R is a commutative semiprime ring, then there exists an embedding

$$R \to \prod\{D_y: y \in Y\} \to \prod\{Q(D_y)\},$$

where the D_y's are integral domains, the first map makes R a subdirect product of the D_y's, and $Q(D_y)$ is the quotient field of D_y. Now R induces a Zariski topology on Y. We say that R is locally inversion closed if for any $f \in R$ and y in the support $F(f)$ of f, there exists a neighborhood $U \subseteqq S(f)$ of y and a $g \in R$ so that $g(x) = 1/f(x)$ for all $x \in U$. Banaschewski **[6]** has shown that if R is locally inversion closed, then its ring of fractions consists of the direct limit $\varinjlim F(V)$, where V ranges over all dense open subsets of Y, and $F(V)$ is the set of all $f \in \prod Q(D_y)$ such that for all $x \in V$, there exists a neighborhood U of x and $g \in R$ such that $g|U = f|U$. We can apply this to $C(Y)$, because Y is Tychonoff and so the topology on Y is the Zariski topology, and because $C(Y)$ is locally inversion closed (see Lemma 3.5 below). Also, note that for $C(Y)$, its ring of fractions is its complete ring of quotients. Thus, we have the following theorem.

THEOREM 3.4. $G_3^Q = G_1^Q = Q(Y)$.

PROOF. The direct limit $\varinjlim F(V)$ is exactly the direct limit which Bleier (implicitly) constructs as the orthocompletion of a representable ℓ-group in **[11]**. The proof is completed by the following lemma.

LEMMA 3.5. *For a Tychonoff space Y, $C(Y)$ is locally inversion closed.*

PROOF. (Stephan Carlson). Let $f \in C(Y)$, and $f(p) \neq 0$. Then choose ε_1, ε_2 so that $0 < \varepsilon_1 < \varepsilon_2 < |f(p)|$. Set $Z_1 = \{y \in Y: |f(y)| \leqq \varepsilon_1\}$ and $Z_2 = \{y \in Y: |f(y)| \geqq \varepsilon_2\}$. Then Z_1 and Z_2 are disjoint zero sets and so there exists $h \in C(Y, [0, 1])$ such that $h(Z_1) \subseteqq \{0\}$ and $h(Z_2) \subseteqq \{1\}$. Let $N = \operatorname{int} Z_2$, which is a neighborhood of p. Define $g: Y \to \mathbf{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in Z_1 \\ h(x)/f(x) & \text{if } x \in Y \backslash \operatorname{int} Z_1. \end{cases}$$

By the pasting lemma **[22**, page 82], $g \in C(Y)$; if $x \in N$, then $g(x) = h(x)/f(x) = 1/f(x)$.

This theorem enables us to characterize those $C(Y)$ which are already orthocomplete.

COROLLARY 3.6. *For $G = C(Y)$ with nonmeasurable cardinality, the following are equivalent*:

(1) $G = G^O$,

(2) $G_1 = G_1^e$, *and*
(3) Y *is discrete.*

PROOF. (3) $\Rightarrow$ (2) is clear, because if Y is discrete, then $G = \Pi\{R_y : y \in Y\}$, which is essentially closed.

(2) $\Rightarrow$ (1) follows since $G_1 \subseteqq G_1^0 \subseteqq G_1^e = G_1$.

(1) $\Rightarrow$ (3). It is shown in **[23]** that $C(Y) = Q(Y)$ if and only if Y is an extremally disconnected p-space; this means Y is discrete, if the cardinality of Y is nonmeasurable (**[23]**, **[24]**). Thus this follows from Theorem 3.4.

In **[13]** Burgess and Raphael introduce an orthogonal completion S of a commutative semiprime ring R. They require that S be laterally complete and that each element of S be the join of a disjoint set from R. In this case R is a large R-submodule of S, and so $R \subseteqq R^O = S$. Such completions need not exist; if one does exist for $C(Y)$, it is just the orthocompletion. Such a completion does exist precisely when G_3 is an ι-dense ring (that is, each annihilator ideal contains an idempotent). Equivalently, G_1 is a subprojectable ℓ-group (that is, for all $g \in G^+$ and polars $0 \neq Q \subseteqq g''$ there exists polar $P \neq 0$ such that $g \in P \oplus P'$). See **[12]** and **[4]**.

We can now identify three hulls of G as convexifications.

(1) G^{SP} *is the smallest* $\underset{\sim}{\alpha}$*-convex* (*or* $\underset{\sim}{\beta}$*-convex*) *subgroup of* G^O *containing* G.

PROOF. Now G_1^{SP} is generated as a group by $\{g[P] : P$ is a polar and $g \in G\}$, where

$$g = g[P] + g[P'] \in P \boxplus P' = G^O.$$

But $h \underset{\sim}{\alpha} g$ if and only if $h = g[P]$ for some polar of G^O.

(2) $G^{CL}_{\lneqq}$, *the conditional lateral completion of* G *with respect to* $\lneqq$, *is the smallest* $\lneqq$*-convex subgroup of* G^O *containing* G.

PROOF. Let H be the smallest $\lneqq$-convex subgroup of G^O containing G. Then H is conditionally laterally complete, and so $G^{CL}_{\lneqq} \subseteqq H$. Let $0 < g \in G^O$ and $g \lneqq h \in G^{CL}_{\lneqq}$. Now $g = \bigvee g_\alpha[P_\alpha]$, where $g_\alpha \in G$. But conditional lateral completeness implies strong projectability for Archimedean ℓ-groups **[31]**, and so $g \in G^{CL}_{\lneqq}$.

(3) $G^\wedge$ *is the smallest* $\lneqq$*-convex subgroup of* G^e *containing* G.

PROOF. Since G is divisible, this follows from Lemma 2.3 of **[21]**.

NOTE. In fact, if Y is a weak cb-space, then $C(Y)^\wedge \cong C(eY)$, where eY is the absolute or minimal projective extension of Y **[28]**.

If G equals any one of these three hulls, it equals the others and in this case Y is extremally disconnected. This famous theorem appears as Theorems 4.3 and 43.11 of [27], excepting the assertion concerning $G = G_{\leqq}^{CL}$, which is obvious, since $G_1^{SP} \subseteq G_{\leqq}^{LC}$ [31].

A similar theorem (appearing as Theorems 43.2 and 43.8 in [27]) asserts that $G = G^P$ if and only if Y is basically disconnected or equivalently, if G is σ-conditionally complete with respect to $\leqq$.

The first statement of a theorem like these was by Stone [32]. Nakano's proofs of these or similar results appeared in [29]. Stone's proofs are in [33].

We shall now discuss the remaining various completions and conditional completions of G_1 and G_3.

In [5, p. 251] it is shown that G_1^0 is generated as a group by joins of disjoint subsets of positive elements of G_1. But G_3^L is a subgroup of $(G^O, +)$ which contains the joins of disjoint subsets of G_3 and hence the group generated by them. Thus $G_3^L = G_3^C = G^O$, since we know that $G_3^L \subseteq G_3^C \subseteq G_3^O$.

Finally, we turn to the conditional completions G_1^{CC}, G_1^{CL}, G_3^{CC}, and G_3^{CL}.

PROPOSITION 3.7. *G_1^{CL} is a subring of $G_1^{\wedge}$ and so $G_1^{CL} = G_1^{CC} \supseteqq G_3^{CC} \supseteqq G_3^{CL}$ as groups and hence they are all subdirect products of reals.*

PROOF. If $\{a_\mu\}$ and $\{b_\nu\}$ are disjoint subsets of G_1 that are bounded by a and b respectively in G_1, then $\{a_\mu b_\nu\}$ is a disjoint subset of G_1 bounded by ab, and $(\vee a_\mu)(\vee b_\nu) = \vee a_\mu b_\nu$ in $G_1^{\wedge}$. Hence the set T of all joins in $G_1^{\wedge}$ of bounded disjoint subsets of G_1 is a multiplicative semigroup. Therefore, the subgroup $[T]$ of $G_1^{\wedge}$ generated by T is a subring of $G_1^{\wedge}$. Moreover, the ℓ-subgroup of $G_1^{\wedge}$ generated by T is a subring of $G_1^{\wedge}$ [25, p. 542]. It follows that G_1^{CL} is a subring of $G_1^{\wedge}$.

We can now extend Theorem 4.6 in [16] to include $X = CL$.

PROPOSITION 3.8. *Let H be any f-ring with H^{CL} the CL-hull of the ℓ-group $(H, +)$ and H^{CLf} the CL-full of the f-ring H. There exists a unique multiplivation on H^{CL} so that it is an f-ring with H as an f-subring. Moreover, H^{CL} with this multiplivation is the CL-hull H^{CLf}. If e is the identity for H, then it is also the identity for H^{CLf}.*

PROOF. The proof of Proposition 3.7 shows that H^{CL} is in fact a subring of H^O, for any f-ring H. Now suppose we have any multiplication on H^{CL} so that it is an f-ring with H as a subring. Then since H^O is the orthocompletion of H^{CL}, there exists by the analogous result for O a multiplication on H^O so that it is an f-ring and H^{CL} is a subring. But this is an f-ring multiplication on H^O so that H is a subring and so unique. Thus, the multiplication on H^{CL} is unique.

As a consequence of Proposition 3.8, in order to extend Theorems 3.1 and 3.2 to all of the hulls of G_1 and G_3 we need to show that $G_1^{CC} = G_3^{CC} =$

$G_3^{\underline{C}L}$ and that each positive element in $G_1^{\underline{C}C}$ is a square. This we have been unable to do.

QUESTIONS. Is $[T] = G_1^{\underline{C}L}$? If so, then $G_1^{\underline{C}L} = [T] \subseteqq G_3^{\underline{C}L}$. Is $[T]$ an ℓ-group? If so, then it follows that $G_3^{\underline{C}L}$ is an ℓ-group and hence $G_1^{\underline{C}L} = G_3^{\underline{C}L}$. Is $[T]$ a CL-ring? If so, then $[T] = G_3^{\underline{C}L}$. Note that it is known that $G_1^{\underline{C}}$ is generated as a group by joins of disjoint subsets of G_1^+ [5].

A topological space Y is said to be locally connected at a point $p \in Y$ if each neighborhood of p contains a connected neighborhood of p; it is extremally disconnected at p if for each pair of disjoint open sets U and V, $p \notin \operatorname{Cl} U \cap \operatorname{Cl} V$.

THEOREM 3.9. (Jack Porter). *If Y is either locally connected or extremally disconnected at each of its points, then $C(Y) = C(Y)_3^{\underline{C}C}$. If Y is locally connected, then the join of an $\underline{\alpha}$-bounded set is pointwise.*

PROOF. Suppose that T is a subset of $C(Y)$ which is $\underline{\alpha}$-bounded by t. Without loss of generality, all elements of T are positive. Let $\operatorname{coz} T = \bigcup \{\operatorname{coz}(s) : s \in T\}$. Define

$$h(x) = \begin{cases} t(x), & x \in \operatorname{Cl}(\operatorname{coz} T) \\ 0, & \text{otherwise.} \end{cases}$$

First, we show that h is continuous by showing that $h(x) = 0$ for all $x \in \operatorname{Cl}(Y \backslash \operatorname{Cl}(\operatorname{coz} T))$. Now, if $x \in \operatorname{Cl}(\operatorname{coz} T) \cap \operatorname{Cl}(Y \backslash \operatorname{Cl}(\operatorname{coz} T))$, then Y is not extremally disconnected at x. So, Y is locally connected at x. Since $x \in \operatorname{Cl}(Y \backslash \operatorname{Cl}(\operatorname{coz} T)) \subseteqq T \backslash \operatorname{coz} T$, then $s(x) = 0$ for all $s \in T$. Assume, by way of contradiction, that $h(x) = t(x) > 1/n$ for some $n \in \mathbf{N}$. Then there is a connected neighborhood W of x such that $t(y) > 1/n$ for all $y \in W$. Since $W \cap \operatorname{coz} T \neq \varnothing$, there is an $s \in T$ and a $y \in W$ such that $s(y) > 0$. Because $s(x) = 0$ and W is connected, there exists $z \in W$ such that $0 < s(z) < 1/n$. Because $s(z) > 0$, $t(z) = s(z) < 1/n$, which is a contradiction. Thus, h is continuous. Now, we show that $T \,\underline{\alpha}\, h$. Let $s \in T$ and $s(x) > 0$. Then $x \in \operatorname{coz} T$, and so $s(x) = t(x) = h(x)$. Finally, suppose that $T \underline{\alpha} g$. If $x \in \operatorname{coz} T$, then $g(x) = t(x) = h(x)$. Let $x \in \operatorname{Cl}(\operatorname{coz} T) \backslash \operatorname{coz} T$. Choose a net $\{x_\alpha\} \subseteqq \operatorname{coz} T$ such that $\{x_\alpha\} \to x$. For each α, there is an $s_\alpha \in T$ such that $s_\alpha(x_\alpha) > 0$. Now $s_\alpha(x_\alpha) = h(x_\alpha) = t(x_\alpha) = g(x_\alpha)$. Because g and h are continuous, $h(x) = g(x)$. This shows that $h \,\underline{\alpha}\, g$.

COROLLARY. [12]. *If Y is locally connected, then $C(Y) = C(Y)_3^{\underline{C}C}$.*

The converse of Theorem 3.9 is false; see example 3.12. However, we do have the following theorem.

THEOREM 3.10. *Y is locally connected if and only if $C(Y) = C(Y)_3^{\underline{C}C}$, and each join of a pairwise disjoint set is pointwise.*

PROOF. ($\Rightarrow$) is just Theorem 3.10.

($\Leftarrow$) Call $h \underline{\alpha}\, g$ an atom of g if $0 \neq k \underline{\alpha}\, h$ implies that $k = h$. Notice that if $h \underline{\alpha}\, g$, then coz h is clopen in coz h. Conversely, if C is clopen in coz g, then $h|C \in C(Y)$ and $h|C \underline{\alpha}\, g$. We first show that each g in $C(Y)$ has atoms. For suppose that g does not. Choose $z \in \operatorname{coz} g$. Then we may pick a maximal disjoint set $\mathfrak{M}$ of sets clopen in coz g, such that $x \notin \bigcup \mathfrak{M}$. Now $h = \vee \{g|M \colon M \in \mathfrak{M}\} \in C(Y)$, because $C(Y)$ is conditionally complete. If $h \neq g$, then $g - h = a + b$, where $a \neq 0 \neq b$ and a and b are disjoint, because $g - h$ is not an atom of g. But we may assume that $x \in \operatorname{coz} a$, and so the existence of coz b contradicts the maximality of $\mathfrak{M}$. Thus $h = g$. But joins are pointwise and so $\operatorname{coz} g = \bigcup \mathfrak{M}$, a contradiction Thus, g has atoms. Now suppose that $x \in Y$ and let coz g be a basic neighborhood of x, where $g \in C(Y)$. Now $g = \vee \{g_\alpha \colon g_\alpha$ is an atom of $g\}$, and $\operatorname{coz} g = \bigcup \operatorname{coz} g_\alpha$, because joins are pointwise. Therefore $x \in \operatorname{coz} g_\alpha$, for some α. But since g_α is an atom, coz g_α is connected, and so Y is locally connected.

Finally, we note that $C(Y)_3 = C(Y)_3^{FC}$.

PROPOSITION 3.11. *$C(Y)_3$ and $C(Y)_3^X$ are FC-rings, for $X = P$, SP and O.*

PROOF. Let $\{a, b\}$ be a boundable set in one of these rings. Each such ring is also an ℓ-group, and so $a = a^+ - a^-$ and $b = b^+ - b^-$. Then $h = (a^+ \vee b^+) - (a^- \vee b^-)$ is the l.u.b. of $\{a, b\}$ with respect to $\underline{\alpha}$.

In summary, we have the following containment relations for the possibly distinct ring and group hulls of $G = C(Y)$.

$$\begin{tikzcd}
 & & & & G^O \arrow[dr, dash] & \\
 & G^P \arrow[r, dash] & G^{SP} \arrow[r, dash] & G^{CL} \arrow[ur, dash] \arrow[ddr, dash] & & G_1^e \arrow[ddl, dash] \\
G \arrow[ur, dash] \arrow[ddr, dash] & & & & & \\
 & & & & G_1 & \\
 & G_3^{CL} \arrow[r, dash] & G_3^{CC} \arrow[r, dash] & G_1^{CL} \arrow[ur, dash] & &
\end{tikzcd}$$

EXAMPLE 3.12 (Jack Porter). Let $\mathbf{N} \subset Z \subseteqq \beta\mathbf{N}$, and Y be the cone over Z (that is $Y = I \times Z/\{0\} \times Z$, the quotient space of $I \times Z$ with $\{0\} \times Z$ identified to a point). The $C(Y) = C(Y)_3^{CC}$, but Y is neither locally connected nor extremally disconnected at any point of the form $\pi(s, z)$ where $\pi \colon I \times Z \to Y$ is the quotient map, $0 < s \leqq$ x, and $z \in Z \backslash \mathbf{N}$.

4. Lattice-ordered group characterizations of C(Y). We first characterize $C(Y)$ for Y a Stonean space.

THEOREM 4.1. *For an ℓ-group G, the following are equivalent:*
(1) *$G \cong C(Y)$, where Y is a Stonean space;*
(2) *G is a complete vector lattice with a strong order unit;*
(3) *G is complete, divisible and has a strong order unit; and*

(4) *G is Archimedean with a strong order unit e, and e is not a strong order unit for any proper essential extension of G.*

In particular, two such ℓ-groups are isomorphic if and only if their Boolean algebras of polars are isomorphic.

PROOF. (1) $\Rightarrow$ (2). This follows from the Nakano-Stone theorem.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Let H be an essential extension of G with e as a strong order unit. We first show that H is Archimedean. If not, then $0 < h_1 \ll h_2$ in H, and since G is dense in H, $0 < g \leqq h_1 \ll h_2 < me$ for some $g \in G$ and positive integer m. But then $g \ll me$, which contradicts the fact that G is Archimedean. Then

$$G \underbrace{\quad\subseteq\quad}_{\text{dense}} H \underbrace{\quad\subseteq\quad}_{\text{dense}} H^{\wedge}$$

and so $G = G^{\wedge}$ is an ℓ-ideal of H [**21**; p. 184]; thus $G \supseteqq H(e) = H$.

(4) $\Rightarrow$ (1). By Bernau's embedding theorem [7, p. 617], there exists an ℓ-isomorphism τ of G onto a large ℓ-subgroup of $C(Y)$, so that $e\tau = \bar{1}$, where Y is the Stonean space of G and $\bar{1}(x) = 1$ for all $x \in Y$. Since $\bar{1}$ is a strong order unit for $C(Y)$, it follows that $G\tau = C(Y)$.

Note that $C(Y)$ is the ℓ-ideal of $D(Y)$ generated by $\bar{1}$. If G is Archimedean and $R(G) = 0$ (where $R(G)$ is the radical of G; see [**14**; p. 5.3]), then the associated Stonean space Y has a dense discrete set S and so $D(Y) = \Pi\{R_s : s \in S\}$. Thus $C(Y)$ consists of all the bounded functions in ΠR_s.

COROLLARY I. *If $R(G) = 0$, then each of* (2), (3) *and* (4) *is equivalent to*

(1′) $\qquad G \cong \{f \in \Pi_S \mathbf{R}_s : f \textit{ is bounded}\}$, *for some set S.*

COROLLARY II. *A divisible ℓ-group G is complete if and only if for each $g \in G$, $G(g) \cong C(Y)$, for some Stonean space Y.*

PROOF. This follows from the theorem and the fact that G is complete if and only if each $G(g)$ is complete.

COROLLARY III. *For an archimedean ℓ-group G, the following are equivalent*:

(a) $(G^d)^{\wedge} \cong C(Y)$, *for a Stonean space Y; and*

(b) *G has a strong order unit.*

PROOF. G has a strong order unit if and only if $(G^d)^{\wedge}$ does. Thus (a) $\Rightarrow$ (b) follows. For (b) $\Rightarrow$ (a), use (3) of Theorem 4.1.

COROLLARY IV. *For an ℓ-group G, the following are equivalent*:

(a) *G is complete, divisible and has weak order unit e; and*

(b) *G is (ℓ-isomorphic to) ℓ-ideal of $D(Y)$ that contains $C(Y)$, for some Stonean space Y.*

PROOF. (a) $\Rightarrow$ (b). Now $G(e) \cong C(Y)$, where Y is the Stonean space for G. Thus we may assume that $e = \bar{1}$ and $G(e) = C(Y) \subseteqq G \subseteqq D(Y)$. But since G is complete, it must be an ℓ-ideal of $D(Y)$.

(b) $\Rightarrow$ (a). Each ℓ-ideal of $D(Y)$ is complete and divisible.

THEOREM 4.2. *For an ℓ-group G, the following are equivalent:*
(1) *$G \cong D(Y)$, for some Stonean space Y;*
(2) *G is Archimedean and admits no Archimedean essential extensions;*
(3) *G is a complete, laterally complete vector lattice;*
(4) *G is divisible, complete and laterally complete;*
(5) *G is divisible, complete and each disjoint set is bounded;*
(6) *G is divisible, complete and has the splitting property;*
(7) *$G = ((H^d)^\wedge)^L$ for some Archimedean ℓ-subgroup H;*
(8) *$G = ((H^d)^L)^\wedge$ for some Archimedean ℓ-subgroup H; and*
(9) *If G is an ℓ-subgroup of an Archimedean ℓ-group, H and G are large in G'', then $H = G \boxplus G'$.*

In particular, such ℓ-groups are ℓ-isomorphic if and only if their Boolean algebras of polars are isomorphic.

PROOF. A complete ℓ-group is a vector lattice if and only if it is divisible; so (3) $\Leftrightarrow$ (4).

Bernau [7, page 617] remarks that (1) $\Leftrightarrow$ (4). Bernau [**8**] shows that $(H^\wedge)^L = (H^L)^\wedge$ for each Archimedean ℓ-group and so (7) $\Leftrightarrow$ (8).

Pinsker [**30**] shows that (2) $\Leftrightarrow$ (3).

Clearly (4) $\Leftrightarrow$ (5).

See Conrad [**15**] for a proof that (1), (2), (4), (6) and (7) are equivalent.

(6) $\Rightarrow$ (9). G is divisible and complete and large in G'' implies that G is an ℓ-ideal of G'', and hence of H. Thus, by the splitting property, $H = G \boxplus G'$.

(9) $\Rightarrow$ (1). We may assume that G is large in $D(Y)$, and since $G'' = D(Y)$, we have that $D(Y) = G \boxplus G' = G$.

We will now obtain an ℓ-group characterization of $C(Y)$ for any Tychonoff space. We need two lemmas.

LEMMA 1. *Let G be an archimedean ℓ-group with weak order unit e. Let $\mathfrak{M}$ be the collection of all maximal primes of G such that $e \notin M$, for all $M \in \mathfrak{M}$. Suppose that $\bigcap \mathfrak{M} = 0$. Then G may be embedded as a large ℓ-subgroup of $C(\mathfrak{M})$, where $\mathfrak{M}$ has been equipped with the Zariski topology induced by G; in this case $\mathfrak{M}$ is real compact and Tychonoff.*

PROOF. We may embed $G \to \Pi\{G/M\colon M \in \mathfrak{M}\} \subseteqq \Pi\mathbf{R}$ so that $e \to (1, 1, 1, \ldots)$. It is straightforward to check that $G \subseteqq C(\mathfrak{M})$, that G is large, and that $\mathfrak{M}$ is Tychonoff. It remains to show that $\mathfrak{M}$ is realcompact. If P is a real ideal of $C(\mathfrak{M})$ which is not fixed, then $e \notin P$, and so $P \cap G \in \mathfrak{M}$. Thus we may define a map $\pi: \upsilon\mathfrak{M} \to \mathfrak{M}$, where $\upsilon\mathfrak{M}$ is the real compactification of $\mathfrak{M}$. This map is continuous, because $\pi^{-1}(\mathrm{coz}_{\mathfrak{M}}(g)) = \mathrm{coz}_{\upsilon\mathfrak{M}}(g^{\upsilon})$, where $g \to g^{\upsilon}$ is the natural isomorphism between $C(\mathfrak{M})$ and $C(\upsilon\mathfrak{M})$. But then $\mathfrak{M}$ is a retract of $\upsilon\mathfrak{M}$ and hence closed in $\upsilon\mathfrak{M}$, and so $\mathfrak{M} = \upsilon\mathfrak{M}$.

LEMMA 2. *Let M be a maximal prime of the ℓ-group $C(Y)$, where Y is real-compact and Tychonoff. Then M is a real ring ideal.*

PROOF. We first show that $\bar{1} \notin M$. For if $\bar{1} \in M$, choose $f \notin M$ such that $f \geqq \bar{1}$. Then $f^2 \notin M$. For $n \in \mathbf{N}$, $0 \leqq (f - \bar{n})^2 = f^2 - 2\bar{n}f + \bar{n}^2$, and so $2\bar{n}f \leqq f^2 + \bar{n}^2$. Thus $2n(M + f) \leqq M + f^2$. But this cannot be, since $C(Y)/M$ is Archimedean.

If M is fixed at a point of Y, we are done. If not, we may embed $C(Y) \to C(X)$, where $X = Y \cup \{M\}$ is equipped with the Zariski topology, $\bar{1} \to \bar{1}$. But Y is dense in X and has the subspace topology, and so this embedding is an isomorphism. Since M corresponds to a (fixed) real ring ideal of $C(X)$, it is a ring ideal of $C(Y)$.

The following definition will enable us to state our characterization of $C(Y)$. Let G be an Archimedean ℓ-group with weak order unit e. Then $H \supseteqq G$ is an e-extension if

(i) H is Archimedean,

(ii) G is large in H,

(iii) $M \to M \cap G$ is a one-to-one correspondence between the maximal primes of H and of G, and

(iv) if P is a polar of H and $P \nsubseteq M$, a maximal prime of H, then there exists $g \in P \cap G^+$ such that $g \notin M$.

THEOREM 4.3. *The following are equivalent for an ℓ-group G:*

(a) *$G \cong C(Y)$, Y a Tychonoff space; and*

(b) *G is an Archimedean ℓ-group with a weak order unit e such that $e \notin P$, for all $P \in Y$, the set of maximal primes of G. Furthermore, $\bigcap Y = 0$, and G admits no proper e-extensions.*

PROOF. (a) $\Rightarrow$ (b). If we assume that Y is real compact and $e = \bar{1}$, we need only show that G admits no proper e-extensions. Suppose that H is an e-extension. Then by (1) we have $G = C(Y_\sigma) \subseteqq H \to C(Y_\tau)$, where σ is the topology on Y induced by G, and τ is the topology on Y induced by H. Note that $\sigma \subseteqq \tau$. We may assume that $e = \bar{1} \to \bar{1}$. Let $U \in \tau$ be regularly open. Then $U = \mathrm{coz}\ P$, where P is a polar of H. If $x \in U$, x corresponds to M, a maximal prime of H. But $x \in U$ if and only if $P \nsubseteq M$. Since H is an e-extension, there exists $g \in P \cap C(Y_\sigma)$ such that $g \notin M$. Thus $x \in \mathrm{coz}\ g \subseteqq \mathrm{coz}\ P = U$, and so $U \in \sigma$. Thus $\sigma = \tau$ and so $C(Y_\sigma) = H = C(Y_\tau)$.

(b) $\Rightarrow$ (a). By (1) we may assume that $G \subseteqq C(Y)$, with $e \to \bar{1}$, and Y real compact Tychonoff. Since each maximal prime of $C(Y)$ is a real ideal and so fixed, we have (i), (ii) and (iii) satisfied, and we need only check (iv) in order to show that $C(Y)$ is an e-extension of G. Let P be a polar of $C(Y)$ and suppose $P \nsubseteq M$, a maximal prime of $C(Y)$ which is fixed at y. Then $y \in \operatorname{coz} P = U \operatorname{coz} g_\gamma$, with each $g_\gamma \in G^+$, since the topology on Y is induced by G. Then $x \in \operatorname{coz} g_\gamma$, some γ, and so $g_\gamma \in P \cap G^+$ and $g_\gamma \notin M \cap G$. But since G admits no e-extensions, this means that $G = C(Y)$.

COROLLARY. *The following are equivalent for an ℓ-group G:*

(a) *$G \cong C(Y)$, Y compact Hausdorff; and*

(b) *G is an Archimedean ℓ-group with strong order unit e, which admits no e-extensions.*

REMARK. This theorem is similar in flavor to the characterization of $C(Y)$ in [2] as a real algebra.

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