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THE MAXIMAL RING OF QUOTIENTS OF A FINITE VON NEUMANN ALGEBRA

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ABSTRACT. Expository article on the embedding of finite von Neumann algebras into regular rings. The algebra of affiliated closed operators is identified with the maximal ring of right quotients. Applications of the theory of self-injective regular rings to operator algebras (matrix algebras, projection lattices, reduction theory).

1. Introduction. Let A be a von Neumann algebra of finite class [**4**, Ch. III, §8]. In their 1936 paper [**17**], F. J. Murray and J. von Neumann showed that A can be embedded in a regular ring $\mathscr{U}(A)$ of closed, densely defined operators "affiliated" with A (in a sense made precise below). Two decades later, Y. Utumi gave a general construction that embeds A in a regular ring Q, which he called its maximal ring of right quotients [**22**]. Twelve years later, J. E. Roos demonstrated that $\mathscr{U}(A)$ and Q are the same ring [**20**]. This apparently glacial progress is due less to the difficulty of the subject than it is to the nearly perfect insulation separating ring theorists from operator theorists. Since the time of von Neumann, and even since the time of Roos' paper, the theory of regular rings has ripened significantly, the maturity of the subject being evident from K. R. Goodearl's recent monograph [**8**]; the regular rings of operator theory can now be perceived in a very general light, and their algebraic properties proved neatly and efficiently. Moreover, these rings are seen, via the general theory of regular rings, to possess a striking property not foreseen from the perspective of operator theory, namely, self-injectivity. This seems, therefore, to be a propitious time to review the regular rings of operator theory, taking advantage of the economies made possible by the algebraic theory of regular rings. The present article is written from the perspective of operator theory, no knowledge of AW^*-algebras being required for the main results; however, the arguments are sufficiently general to apply to AW^*-algebras, yielding, in particular, a brief new proof that the algebra of $n \times n$ matrices over an AW^*-algebra is also an AW^*-algebra. For a purely algebraic view of the subject, the reader should consult the books of Kaplansky and Goodearl ([**15**], [**8**]) and the paper of D. Handelman [**12**].

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The first part of the paper (§§2–4) is devoted to describing the maximal ring of right quotients of a finite von Neumann algebra in operator-theoretic terms; we stick closely to operators and keep the abstract ring theory to a minimum. The rest of the paper is devoted to applications. The main results are stated in terms of Baer *-rings; they specialize easily to the corresponding classical results on operator algebras, but it is hoped that the simplicity of the ring-theoretic formulation justifies what might otherwise be gratuitous generality.

2. Unbounded operators. The basic objective in this circle of ideas is to embed a finite von Neumann algebra A in a regular ring R. {A ring R is *regular* (or von Neumann regular) if, for each element $x \in R$, there exists an element $y \in R$ such that $x = xyx$.} Since a key feature of regularity is that it provides each element x with a "relative inverse" y, it is to be expected that the creation of such relative inverses for the operators in A will entail the introduction of some unbounded operators (closely related to A, to be sure). The most algebraically tractable unbounded operators are the closed, densely defined operators [**19**, Chapter 8]; these are the linear mappings x, defined on a dense linear subspace of a Hilbert space H, such that the graph of x is a closed linear subspace of $H \oplus H$. For such an operator x the adjoint operator, x^*, is also densely defined, $(x^*)^* = x$ [**19**, p. 304], and one has a canonical factorization $x = wr$ with w a partial isometry and $r = (x^*x)^{1/2}$ a positive, self-adjoint operator [**19**, p. 284, Théorème]. One can then define the Cayley transform $u = (r - i)(r + i)^{-1}$ of r, which is a unitary operator such that $r = i(1 + u)(1 - u)^{-1}$ [**19**, p. 318]; thus, x is completely determined in terms of the pair of bounded operators w, u. (Another expression for r in terms of bounded operators is the spectral decomposition $r = \int \lambda de_\lambda$ [**19**, p. 318, Théorème].)

A closed, densely defined operator x is said to be *affiliated* with the finite von Neumann algebra A if $u'xu'^* = x$ for every unitary operator u' in the commutant A' of A; an intrinsic form of this condition is that in the notation of the preceding paragraph one has $w \in A$ and $u \in A$ (equivalently, $w \in A$ and $e_\lambda \in A$ for all λ). Write $\mathscr{U}(A)$ for the set of all such x. Evidently $A \subset \mathscr{U}(A)$; more precisely, A is the set of bounded elements of $\mathscr{U}(A)$ (cf. [**4**, p. 4, Corollary of Proposition 3]). The only obvious "algebraic" property of $\mathscr{U}(A)$ is adjunction; $x \in \mathscr{U}(A)$ implies $x^* \in \mathscr{U}(A)$. Linear operations and multiplication are introduced as follows (cf. [**17**, p. 229], [**24**, p. 89, (VI)], and [**21**, p. 414, Definition 2.2]). For any pair of operators x, $y \in \mathscr{U}(A)$, one can form the natural sum and composite of x and y (but these need not be closed operators); their closures are elements of $\mathscr{U}(A)$ (the finiteness of A is crucial here), called the *strong sum* and *strong product* of x and y, denoted $x + y$ and xy. With these definitions, $\mathscr{U}(A)$ becomes a ring with involution (briefly, a *-ring), containing A as a *-subring.

Moreover, taking $x = \lambda 1$, λ complex, one sees that $\mathscr{U}(A)$ is a complex algebra, with $(\lambda y)^* = \bar{\lambda} y^*$ for all λ and all $y \in \mathscr{U}(A)$. One calls $\mathscr{U}(A)$ *the algebra of unbounded operators affiliated with A*. This algebra can be characterized in the following way.

THEOREM 1. *If A is a finite von Neumann algebra, then there exists a complex $*$-algebra R with unity, containing A as a $*$-subalgebra, such that* (1) *R is a regular ring, and* (2) *the relations $x, y, z \in R$, $x^*x + y^*y + z^*z = 1$ imply $x, y, z \in A$. These conditions determine R uniquely up to a $*$-isomorphism that leaves fixed the elements of A.*

PROOF. For existence it is easy to see that $\mathscr{U}(A)$ satisfies the condition (2), and the proof that $\mathscr{U}(A)$ is regular is a straightforward piece of spectral theory (cf. [**24**, p. 89, (VI)], [**6**, p. 567, Theorem 2]). {The theorem can also be given an "intrinsic" proof, valid for any finite AW^*-algebra [**3**, p. 237, Corollary].}

For uniqueness suppose A is a $*$-subalgebra of a $*$-algebra R with unity element 1, satisfying the conditions (1) and (2). The $*$-algebra R contains no new partial isometries; if $x \in R$ and $x^*x = e$, e a projection, then $x \in A$ (put $y = 1 - e$ and $z = 0$ in (2)). In particular, R contains no new projections, and if $x \in R$ is isometric ($x^*x = 1$), then $x \in A$ and therefore x is unitary by the finiteness of A. If $x, y \in R$ and $x^*x + y^*y = 0$, then $x, y \in A$ (put $z = 1$ in (2)) and therefore $x = y = 0$. Thus, the hypotheses of [**3**, p. 235, Proposition 3] are fulfilled; the identity mapping of A extends to a $*$-isomorphism $R \to \mathscr{U}(A)$. {The basic idea is straightforward. If $x \in R$ is self-adjoint, then its Cayley transform $u = (x - i)(x + i)^{-1}$ is a unitary element of A, which in turn defines a self-adjoint element $y = i(1 + u)(1 - u)^{-1}$ of $\mathscr{U}(A)$; the mapping in question sends x to y, and is extended to arbitrary elements of R via the Cartesian decomposition.}

3. Self-injectivity. With notations as in Theorem 1, the relations $x \in R$, $x^*x = 0$ imply $x \in A$ and therefore $x = 0$; thus R is a $*$-regular ring, therefore every principal ideal (right or left) of R is generated by a projection (of A)(cf. [**24**, p. 114, Theorem 4.5], [**6**, p. 567, Theorem 2], and [**3**, p. 229, Proposition 3]). Moreover, the projection lattice of R, being identical with that of A, is complete; thus, R is a complete $*$-regular ring. {In other words, R is a regular Baer $*$-ring, equivalently, a $*$-regular Baer ring [**23**, p. 599, Lemma 1].} Our next objective is the theorem that R is, in fact, self-injective (cf. [**8**, p. 162, Corollary 13.5, and p. 169, Theorem 13.17]).

LEMMA 1. *With notations as in Theorem* 1, *R is unit-regular.*

PROOF. The assertion is that every $x \in R$ can be written $x = xyx$ with y an invertible element of R; it suffices [**8**, p. 86, Theorem 8.12] to show that the regular ring R is directly finite ($yx = 1$ implies $xy = 1$) and has generalized comparability. As noted in the proof of Theorem 1, the relations $x \in R$, $x^*x = 1$ imply $xx^* = 1$, that is, R is finite; since the left and right projections of each element of R are equivalent (e.g., by the canonical factorization described in §2), it follows that R is directly finite [**3**, p. 210, Proposition 1, (5)]. Moreover, R has generalized comparability; for, the principal right ideals of R are generated by projections of A, and the assertion follows from the fact that A has generalized comparability [**3**, p. 80, Corollary 1].

It follows from Lemma 1 that every matrix ring $M_n(R)$ is directly finite (even unit-regular) [**8**, p. 40, Corollary 4.7, and p. 50, Proposition 5.2], hence its subring $M_n(A)$ is also directly finite.

LEMMA 2. *If Z is an abelian von Neumann algebra, then the algebra $M_2(Z)$ of 2×2 matrices over Z is a finite von Neumann algebra, and $\mathscr{U}(M_2(Z))$ may be identified with $M_2(\mathscr{U}(Z))$.*

PROOF. If Z acts on the Hilbert space H, then it is clear that $A = M_2(Z)$ may be identified with a von Neumann algebra of operators on the Hilbert space $H \oplus H$ [**4**, p. 23, Lemme 2], and A is finite by the remark following Lemma 1 (or by a simple trace argument; cf. [**4**, p. 217, Proposition 3], and [**2**, p. 176]). Let $R = M_2(\mathscr{U}(Z))$; since $\mathscr{U}(Z)$ is regular (Theorem 1), so is R [**8**, p. 4, Theorem 1.7]. Equipped with the natural involution ($*$-transposition), R is a $*$-algebra containing A as a $*$-subalgebra. If $x = (x_{ij})$, $y = (y_{ij})$, $z = (z_{ij})$ are elements of R with $x^*x + y^*y + z^*z = 1$, inspection of the diagonal entries in this matrix equation shows that $x, y, z \in A$ [**3**, p. 250, Proposition 3]. Thus R satisfies the conditions (1), (2) of Theorem 1, hence may be identified with $\mathscr{U}(A)$. {We remark that if Z is a commutative AW^*-algebra, then $M_2(Z)$ is an AW^*-algebra [**13**, p. 855, Corollary] (finite by the same trace argument as in the case of von Neumann algebras). More generally, see Theorem 4 below.}

THEOREM 2. *With notation as in Theorem* 1, *R is right and left self-injective.*

PROOF. Write $A = Z \times B$ with Z abelian and B properly nonabelian (no abelian summands) [**3**, p. 93, Theorem 1, (2)]. Evidently $\mathscr{U}(A) = \mathscr{U}(Z) \times \mathscr{U}(B)$ (e.g., by Theorem 1), thus one is reduced to the case that A is abelian or properly nonabelian. {It is pertinent here that every idempotent of R is similar to a projection [**15**, p. 24, Exercise 4], whence the concordance of the terms "abelian" as used in [**3**] and [**8**].}

If A is properly nonabelian, then so is R (it has the same projection lattice as A); since, moreover, the projection lattice of A is a continuous geometry (cf. [**3**, p. 185, Theorem 1], and [**8**, p. 160, Proposition 13.1]), it follows that the regular ring R is right self-injective [**8**, p. 169, Corollary

13.18]. Since R possesses an involution, it is immediate (e.g., by Baer's criterion) that R is also left self-injective.

Suppose now that A is abelian. Then $M_2(A)$ is a properly nonabelian, finite von Neumann algebra (Lemma 2), so by the preceding paragraph $\mathscr{U}(M_2(A))$ is self-injective (right and left), that is (Lemma 2) $M_2(\mathscr{U}(A))$ is self-injective, hence so is its "corner" $R = \mathscr{U}(A)$ [**8**, p. 98, Proposition 9.8].

4. Identification of $\mathscr{U}$(A) with the maximal ring of right quotients. With notations as in Theorem 1, let Q be the maximal ring of right quotients of A [**16**, p. 94]; we show in this section that Q may be identified with R (consequently the involution of A is extendible to an involution of Q, so that Q is also a maximal ring of left quotients of A).

LEMMA 3. *The right (and the left) singular ideal of A is zero.*

PROOF. Let $a \in A$, $a \neq 0$, and let $J = \{x \in A : ax = 0\}$; the assertion is that there exists a nonzero right ideal K of A such that $J \cap K = 0$ [**16**, p. 106]. Writing $J = eA$, e a projection, one has $e \neq 1$, and hence $K = (1 - e)A$ meets the requirements.

It follows from Lemma 3 that Q is a regular, right self-injective ring containing A as a subring [**16**, p. 106, Proposition 2 and its corollary].

LEMMA 4. *With notation as in Theorem* 1, *R is a ring of right quotients of A.*

PROOF. Given $x, y \in R$, $x \neq 0$, one seeks $a \in A$ such that $xa \neq 0$ and $ya \in A$ [**16**, p. 99]. It is clear from the identification of R with $\mathscr{U}(A)$ that there exists a sequence of projections $e_n \in A$ with $\sup e_n = 1$ and $ye_n \in A$ for all n (let $y^*y = \int \lambda df_\lambda$ be the spectral decomposition and define $e_n = \int_0^n df_\lambda$); since $x \neq 0$, there must exist n with $xe_n \neq 0$. {For another style of proof, see the proof of Theorem 10.}

It follows from Lemma 4 that the identity mapping of A may be extended to a monomorphism of rings $R \to Q$ [**16**, p. 99, Proposition 8]. In other words, we can suppose that R is a subring of Q.

THEOREM 3. $R = Q$.

PROOF. At any rate, $A \subset R \subset Q$ and Q is a ring of right quotients of A, therefore Q is also a ring of right quotients of R; in particular R_R is an essential submodule of Q_R. Since R_R is injective (Theorem 2), it follows that $R = Q$.

The foregoing arguments are valid with "von Neumann algebra" replaced by AW^*-algebra (and the reference to the Hilbert space H suppressed). This yields a new proof of the following theorem (cf. [**3**, p. 262, Corollary 1]).

Theorem 4. *If A is an AW^*-algebra, then every matrix algebra $M_n(A)$ is also an AW^*-algebra.*

Proof. As in [**3**, p. 262] one quickly reduces to the case that A is finite. Let R be the regular ring paired with A via Theorem 1 (more precisely, the analogue of Theorem 1 for a finite AW^*-algebra). One verifies easily that $M_n(R)$, with its natural involution, is a $*$-algebra containing $M_n(A)$ as a $*$-subalgebra and satisfying the conditions (1), (2) of Theorem 1. In particular, from (2) we see that $M_n(R)$ is a $*$-regular ring all of whose projections belong to $M_n(A)$. Since R is right self-injective (by the AW^* version of Theorem 2), so is $M_n(R)$ [**8**, p. 96, Corollary 9.3]. It follows from a theorem of Utumi that $M_n(R)$ is a Baer ring (cf. [**23**, p. 599, Lemma 1], and [**8**, p. 95, Proposition 9.1, (c)]). Thus $M_n(R)$ is a regular Baer *-ring; since all projections of $M_n(R)$ belong to $M_n(A)$, it is immediate that $M_n(A)$ is also a Baer *-ring, hence an AW^*-algebra (finite, by the remarks following Lemma 1).

It is noteworthy that, modulo some general ring theory, the proof of Theorem 4 is effectively reduced (via the AW^* version of Theorem 2) to the case that $n = 2$ and A is abelian, a very special case of Kaplansky's construction of type I algebras [**13**].

5. Extendibility of the involution. Viewing Theorem 4 as an application of regular ring theory, the crux of the matter is the possibility of extending the involution of a *-algebra to its maximal ring of right quotients. This is not always possible; when it is, there are strong consequences. The applications of regular ring theory, to which the rest of the paper is devoted, are in large part formulated for Baer *-rings whose involution is extendible to the maximal ring of quotients; these results suggest that extendibility of the involution may profitably be taken as an axiomatic point of departure.

Let A be a Baer *-ring, Q its maximal ring of right quotients [**16**, p. 94]. Suppose that the involution of A is extendible to Q (the extension is then unique [**18**, p. 204, Theorem 3.2]). Then Q is a regular Baer *-ring whose projection lattice is identical with that of A [**18**, p. 205, Corollary 3.5]. Since Q is self-injective (both right and left) [**16**, p. 107, Corollary of Proposition 2], its projection lattice is continuous [**8**, p. 162, Corollary 13.5], therefore Q is unit-regular [**8**, p. 170, Corollary 13.23] hence directly finite [**8**, p. 50, Proposition 5.2]. {Alternatively, since Q is a complete *-regular ring, one could cite Kaplansky's results [**14**].} In particular, A is directly finite. For the following remarks, let e, f be a pair of projections of A.

(i) Suppose e, f are *algebraically equivalent* in Q, that is, there exist elements $x \in eQf$, $y \in fQe$ with $xy = e$ and $yx = f$ (equivalently, eQ and fQ are isomorphic as right Q-modules [**15**, p. 21, Theorem 14]). Since Q is unit-regular, eQ and fQ are perspective in the lattice of principal right ideals of Q [**8**, p. 39, Corollary 4.4], hence e, f are *perspective* in the projection lattice of A (that is, there exists a projection $g \in A$, called a "common complement of e and f", such that $e \cup g = f \cup g = 1$ and $e \cap g = f \cap g = 0$).

(ii) In particular, if $x \in Q$ and if $e = \mathrm{LP}(x), f = \mathrm{RP}(x)$ are the left and right projections of x (that is, the projections such that $xQ = eQ$ and $Qx = Qf$), then e, f are algebraically equivalent in Q, hence perspective in A.

(iii) It follows from (ii) that if e, f are arbitrary projections of A, then $e \cup f - f$ and $e - e \cap f$ are perspective in A [**3**, p. 14, Proposition 7].

(iv) Every idempotent of Q is similar to a projection [**3**, p. 18, Exercise 1]; hence the central idempotents of Q are the central projections of A.

(v) If e, f are arbitrary projections of A, then there exists a central projection h of A such that he is perspective to a projection $f' \leqq hf$ and $(1 - h)f$ is perspective to a projection $e' \leqq (1 - h)e$. This is immediate from [**8**, p. 102, Corollary 9.15] and Remarks (i), (iv).

THEOREM 5. *If A is a Baer *-ring whose involution is extendible to the maximal ring of right quotients, then the following conditions on A are equivalent*: (a) *A satisfies* (P); (b) *A has* GC; (c) *A satisfies* LP $\sim$ RP; *and* (d) *if e, f are perspective projections in A, then $e \sim f$ in A. In such a ring, algebraically equivalent projections are equivalent.*

PROOF. Let us review the definitions. Projections e, f are said to be *equivalent*, written $e \sim f$, if there exists $x \in A$ with $xx^* = e$, $x^*x = f$. If $\mathrm{LP}(a) \sim \mathrm{RP}(a)$ for every $a \in A$, one says that A satisfies LP $\sim$ RP. If $e \cup f - f \sim e - e \cap f$ for every pair of projections e, f, one says that A satisfies the *parallelogram law* (P). One writes $e \precsim f$ if $e \sim e'$ for some projection $e' \leqq f$. One says that A has *generalized comparability* (GC) if, for each pair of projections e, f, there exists a central projection h such that $he \precsim hf$ and $(1 - h)f \precsim (1 - h)e$.

(a) $\Rightarrow$ (d). Let g be a common complement of e and f. Then $e = e - e \cap g \sim e \cup g - g = 1 - g$ and similarly $f \sim 1 - g$, whence $e \sim f$.

(d) $\Rightarrow$ (c). Immediate from Remark (ii) above.

(c) $\Rightarrow$ (a). For projections e, f one has $e \cup f - f = \mathrm{RP}(a)$ and $e - e \cap f = \mathrm{LP}(a)$ with $a = e(1 - f)$ [**3**, p. 14, Proposition 7]. {Incidentally, (c) implies (a) and (b) in any Baer *-ring [**3**, p. 80, Corollary 2].}

(d) $\Rightarrow$ (b). Immediate from Remark (v) above.

(b) $\Rightarrow$ (d). If e, f are perspective in A, hence in the maximal ring of quotients Q, then eQ, fQ are isomorphic as right Q-modules [**8**, p. 46, Corollary 4.23], thus e, f are algebraically equivalent in Q. By (b), there is a central projection h such that in A one has $he \sim e' \leqq hf$ and $(1 - h)f \sim f' \leqq (1 - h)e$ for suitable projections e', f'. Since e, f are algebraically equivalent in Q, so are he, hf; thus hf is algebraically equivalent to its subprojection e', hence $e' = hf$ by direct finiteness of Q. Similarly $f' = (1 - h)e$. Thus $he \sim hf$ and $(1 - h)f \sim (1 - h)e$ in A, whence $e \sim f$ in A.

Finally, assuming these conditions hold, suppose the projections e, f are algebraically equivalent in A, say $xy = e$, $yx = f$ with $x \in eAf$, $y \in fAe$; it is easy to see that $e = \mathrm{LP}(x)$ and $f = \mathrm{RP}(x)$, hence $e \sim f$ in A by condition (c).

We remark that under the conditions of Theorem 5, for every $x \in Q$ one has $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ in A by Theorem 5 and Remark (ii) above.

COROLLARY. *In a regular Baer *-ring, the conditions* (a) – (d) *are equivalent.*

PROOF. If the regular Baer *-ring R is abelian, then perspective (or algebraically equivalent) projections are equal, thus all four conditions hold trivially. On the other hand, suppose R has no abelian summand; since the projection lattice of R is continuous [**14**, p. 535, Theorem 3], it follows that R is (right and left) self-injective [**8**, p. 169, Corollary 13.18], consequently the equivalence of (a) – (d) is immediate from Theorem 5. The general case then follows from structure theory [**3**, p. 93, Theorem 1, (2)].

We have noted above some necessary conditions for the extendibility of the involution of a Baer *-ring A to its maximal ring of right quotients Q (notably, direct finiteness and continuity of the projection lattice). The known necessary and sufficient conditions for extendibility seem to be too shallow to be applicable without further hypotheses. Various effective sufficient sets of conditions have been developed by E. S. Pyle [**18**, p. 205, Theorem 3.6, p. 206, Corollary 3.7], I. Hafner [**9**, p. 158, Theorem 2] and D. Handelman [**12**, p. 8, Theorem 2.3]. In Theorem 6 below, we prove a variation on these results that is easily applied in the operatorial case (therefore providing an alternative approach to the results of §§2–4).

LEMMA. *Let A be a Baer *-ring whose projection lattice is continuous, and suppose that* (1°) *for every $x \in A$,* $\mathrm{LP}(x)$ *is the supremum of an orthogonal family of projections in xA, and* (2°) *for every right ideal I of A, the set of projections* $\{\mathrm{LP}(x)\colon x \in I\}$ *is increasingly directed. Then, for a right ideal I of A, the following conditions are equivalent*:

(a) *the left annihilator of I is* 0;

(b) *I is an essential right ideal of A*;

(c) *I contains an orthogonal family of projections with supremum* 1;

(d) *the supremum of the set of all projections in I is* 1; *and*

(e) $\sup\{\mathrm{LP}(x)\colon x \in I\} = 1$.

PROOF. Let $g = \sup\{\mathrm{LP}(x)\colon x \in I\}$; as in any Baer *-ring, the left annihilator of I is $A(1 - g)$ [**3**, p. 21, Proposition 2], whence (a) $\Leftrightarrow$ (e). Let h

be the supremum of all projections in I. Obviously $h \leqq g$, hence (d) $\Rightarrow$ (e). Also (c) $\Rightarrow$ (d) is trivial. Thus, in any Baer *-ring, one has (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Leftrightarrow$ (a). {It is also easy to see that (b) $\Rightarrow$ (a) in any Rickart ring (a ring in which the right annihilator of any element is generated by an idempotent), but this is not needed for the present proof.} Note that the hypothesis (1°) implies that A has "sufficiently many projections" (that is, every nonzero one-sided ideal contains a nonzero projection).

(b) $\Rightarrow$ (c). Let (e_α) be a maximal orthogonal family of nonzero projections in I (Zorn's lemma) and let $e = \sup e_\alpha$; it suffices to show that $e = 1$. If, on the contrary, $e \neq 1$, then $(1 - e)A \neq 0$; since I is essential, $I \cap (1 - e)A \neq 0$. Let $f \in I \cap (1 - e)A$ be a nonzero projection; then f is orthogonal to e, hence to every e_α, contradicting maximality. {We remark that, by a result of Hafner [**9**, p. 158, Lemma 5], (b) $\Leftrightarrow$ (c) in any Baer *-ring with continuous projection lattice and sufficiently many projections.}

(a) $\Rightarrow$ (b). (cf. [**18**, p. 205, Theorem 3.6]) In view of (a) $\Leftrightarrow$ (e) and (2°), one has $\mathrm{LP}(x) \uparrow 1$ as $x \in I$. Let J be a nonzero right ideal of A; we are to show that $I \cap J \neq 0$. Let $f \in J$ be a nonzero projection. By continuity, $f \cap \mathrm{LP}(x) \uparrow f \neq 0$, hence there exists $x \in I$ with $f \cap \mathrm{LP}(x) \neq 0$. Citing (1°) and passing to finite sums, one obtains an increasingly directed family (g_α) of projections in xA such that $g_\alpha \uparrow \mathrm{LP}(x)$; then $f \cap g_\alpha \uparrow f \cap \mathrm{LP}(x) \neq 0$, hence there exists an index α with $f \cap g_\alpha \neq 0$. Since $f \cap g_\alpha \in J \cap xA \subset J \cap I$, the implication is proved. This completes the proof of the lemma.

In order that the involution of a Baer *-ring be extendible to the maximal ring of right quotients, it is necessary and sufficient that the implication "(a) $\Rightarrow$ (b)" of the lemma be valid [**18**, p. 204, Theorem 3.2]. (A ring in which this implication is valid is said to satisfy Utumi's condition.)

THEOREM 6. *Let A be a finite Baer *-ring satisfying the parallelogram law* (P), *with generalized comparability* (GC), *and satisfying the conditions* (1°), (2°) *of the lemma. Then the involution of A is uniquely extendible to the maximal ring of right quotients, and A satisfies* LP $\sim$ RP.

PROOF. (cf. [**18**, p. 206, Corollary 3.7]) The projection lattice of A is continuous [**3**, p. 185, Theorem 1], thus the hypotheses of the lemma are fulfilled. In particular (a) $\Rightarrow$ (b), thus the involution is extendible by the remark following the lemma; uniqueness is noted in [**18**, p. 204, Theorem 3.2], and A satisfies LP $\sim$ RP by Theorem 5.

Condition (1°) of the Lemma holds in any AW^*-algebra, by spectral theory [**3**, p. 130, proof of Theorem 3]; condition (2°) also holds in any AW^*-algebra (indeed, in any Rickart $*$-ring in which $xx^* + yy^* = 0$ implies $x = y = 0$ [**3**, p. 225, Lemma]). In a $*$-regular ring A, both conditions hold trivially; $xA = \mathrm{LP}(x)A$ for all $x \in A$, and $(e \cup f)A = eA + fA$ for all projections e, f.

6. Projection ortho-isomorphisms. If A is a $*$-ring, let us write $\tilde{A}$ for the partially ordered set of projections of A, where $e \leqq f$ means $e = ef$. For $*$-rings A and B, a mapping $\theta\colon \tilde{A} \to \tilde{B}$ is called an *ortho-isomorphism* if (i) θ is an isomorphism for the order structures, and (ii) $\theta(1 - e) = 1 - \theta(e)$ for all $e \in \tilde{A}$. It is easy to see that a bijective mapping $\theta\colon \tilde{A} \to \tilde{B}$ is an ortho-isomorphism if and only if, for $e, f \in \tilde{A}$, the relations $ef = 0$ and $\theta(e)\theta(f) = 0$ are equivalent [**5**, p. 75, Lemma 1].

If A, B are *-rings and $\phi\colon A \to B$ is the direct sum of a $*$-isomorphism and a $*$-anti-isomorphism, then ϕ induces an ortho-isomorphism $\tilde{A} \to \tilde{B}$. H. A. Dye has shown, conversely, that every projection ortho-isomorphism between von Neumann algebras arises in this way, provided that direct summands of type I_2 are excluded [**5**, p. 83, Corollary]. Dye's arguments are valid for AW^*-algebras, and, as also shown by J. Feldman [**6**], the case of finite algebras can be treated by von Neumann's theory of regular rings; our aim in this section is to give an exposition of the finite case, patterned after Feldman's discussion of the type II_1 case.

For the following series of remarks, A and B are assumed to be Baer $*$-rings with generalized comparability (GC) and satisfying the parallelogram law (P), and $\theta\colon \tilde{A} \to \tilde{B}$ is an ortho-isomorphism. The remarks culminate in the observation that A and B have the same type-structure.

1. For each $e \in \tilde{A}$, θ induces an ortho-isomorphism $(eAe)^{\sim} \to (\theta(e)B\theta(e))^{\sim}$.

2. For $e, f \in \tilde{A}$ one has $\theta(e \cup f) = \theta(e) \cup \theta(f)$ and $\theta(e \cap f) = \theta(e) \cap \theta(f)$; hence e, f are perspective (that is, have a common complement) if and only if $\theta(e)$, $\theta(f)$ are perspective. If $ef = 0$, then $\theta(e + f) = \theta(e) + \theta(f)$.

3. For $e, f \in \tilde{A}$, one has $ef = fe$ if and only if $\theta(e)\theta(f) = \theta(f)\theta(e)$, and in this case $\theta(ef) = \theta(e)\theta(f)$ [cf. **10**, p. 58].

4. $h \in \tilde{A}$ is a central projection in A if and only if it has a unique complement [**3**, p. 39, Exercise 11]; hence h is a central projection of A if and only if $\theta(h)$ is a central projection of B.

5. If $e, f \in \tilde{A}$ are perspective, then they are unitarily equivalent, hence $e \sim f$ [**3**, p. 109, Exercise 12, (iv)].

6. If $e, f \in \tilde{A}$, $ef = 0$ and $e \sim f$, then e, f are perspective [**3**, p. 109, Exercise 12, (vi)].

7. In order that A be infinite (i.e., not finite) it is necessary and sufficient that there exist in A an infinite sequence of pairwise orthogonal, pairwise perspective nonzero projections (cf. 5, 6 and [**3**, p. 101, Proposition 1]).

8. A is finite if and only if B is finite. (Immediate from 7.) {If A and B

satisfy LP $\sim$ RP (cf. [3, p. 80, Corollary 2]), then A is directly finite if and only if B is directly finite [3, p. 210, Proposition 1].}

9. $e \in \tilde{A}$ is a finite projection if and only if $\theta(e)$ is a finite projection. (Immediate from 1 and 8.)

10. For finite projections $e, f \in \tilde{A}$, $e \sim f$ if and only if e, f are perspective [3, p. 109, Exercise 12, (viii)].

11. For finite projections $e, f \in \tilde{A}$, one has $e \sim f$ if and only if $\theta(e) \sim \theta(f)$. (Immediate from 2, 9 and 10.)

12. $e \in \tilde{A}$ is faithful (that is, the only central projection of A orthogonal to e is 0) if and only if $\theta(e)$ is faithful in B. (Immediate from 3 and 4.)

13. $e \in \tilde{A}$ is an abelian projection in A (that is, all projections of eAe are central in eAe) if and only if $\theta(e)$ is an abelian projection in B. (Immediate from 1 and 4.)

14. A is type I (that is, has a faithful abelian projection) if and only if B is type I. (Immediate from 12 and 13.)

15. A is semifinite (that is, has a faithful finite projection) if and only if B is semifinite. (Immediate from 9 and 12.)

16. A is continuous (that is, contains no nonzero abelian projections) if and only if B is continuous. (Immediate from 13.)

17. A is purely infinite (that is, contains no nonzero finite projections) if and only if B is purely infinite. (Immediate from 9.)

18. A is properly infinite (that is, has no nonzero finite central projections) if and only if B is properly infinite. (Immediate from 4 and 9.)

19. Projections $e_1, \ldots, e_n$ in A are pairwise orthogonal and equivalent if and only if $\theta(e_1), \ldots, \theta(e_n)$ are pairwise orthogonal and equivalent. (Immediate from 2, 5 and 6.)

20. For an integer n, A is of type I_n (that is, $1 \in A$ is the sum of n orthogonal, equivalent abelian projections [3, p. 112, Remark 1]) if and only if B is of type I_n. (Immediate from 13 and 19.)

21. For any cardinal $\aleph$, A is of type $\mathrm{I}_\aleph$ [3, p. 116, Definition 3] if and only if B is of type $\mathrm{I}_\aleph$. (The proof is similar to 20.)

22. Summarizing, A and B have the same type-structure. {For example, if $A = h_1A \oplus h_2A \oplus h_3A$ is the decomposition of A into types I, II, III [3, p. 94, Theorem 2], then $B = \theta(h_1)B \oplus \theta(h_2)B \oplus \theta(h_3)B$ is the analogous decomposition of B. If A is finite and $h, h_1, h_2, \ldots$ are orthogonal central projections with supremum 1, such that hA is of type II_1 and h_nA is of type I_n [3, p. 115, Theorem 3], then $\theta(h), \theta(h_1), \theta(h_2), \ldots$ have the analogous properties relative to B.}

LEMMA. *Let R and S be regular Baer *-rings satisfying one* (*hence all*) *of the conditions* (a) – (d) *of Theorem* 5, *and suppose that R and S have no abelian or type* I_2 *summands. If $\theta\colon \tilde{R} \to \tilde{S}$ is an ortho-isomorphism of the* *projection lattices, then there exists a $*$-isomorphism $\phi: R \to S$ that extends θ.*

PROOF. (cf. [**6**, p. 567, Theorem 3]) As observed in the proof of the corollary of Theorem 5, R and S are self-injective. {We remark that in Theorem 7 below, the appeal to Kaplansky's theorem in the cited corollary is avoided. As to the need for excluding abelian summands, note that all involutive division rings (in particular, all fields equipped with the identity involution) have the same projection lattice; the pathology of type I_2 is sketched in [24, p. 103] and [**5**, p. 83].} By Remark 22 above, R and S have the same type-structure; moreover, if (h_n) is a central partition of 1 in R (hence $\theta(h_n)$ is a central partition of 1 in S), one has $R = \Pi h_n R$ and $S = \Pi\theta(h_n)S$ [**8**, p. 99, Proposition 9.10]. Thus, dropping down to a direct summand, we can suppose that for some $n \geqq 3$, R (hence S) contains n pairwise orthogonal, equivalent projections $e_1, \ldots, e_n$ with sum 1 (Remark 22 above).

Writing $L(R)$ and $L(S)$ for the lattices of principal right ideals, one obtains a lattice isomorphism $\bar{\theta}: L(R) \to L(S)$ by composing the mappings $eR \mapsto e \mapsto \theta(e) \mapsto \theta(e)S$ ($e \in \tilde{R}$). The right ideals $e_1R, \ldots, e_nR$ are independent submodules of R_R; moreover, the e_i are pairwise perspective in $\tilde{R}$ (Remark 6 above), hence the e_iR are pairwise perspective in $L(R)$; thus $L(R)$ (and similarly $L(S)$) is of "order n" in von Neumann's sense [**24**, p. 93, Definition 3.2]. Since $n \geqq 3$, it follows from a theorem of von Neumann [**24**, p. 108, Theorem 4.2] that there exists a ring isomorphism ϕ: $R \to S$ such that $\phi(eR) = \bar{\theta}(eR)$, that is, (i) $\phi(e)S = \theta(e)S$ for all $e \in \tilde{R}$. Then also $\phi(1 - e)S = \theta(1 - e)S$, that is, (ii) $(1 - \phi(e))S = (1 - \theta(e))S$ for all $e \in \tilde{R}$. For all $e \in \tilde{R}$ one has $\phi(e) = \theta(e)$; for, $\phi(e)\theta(e) = \theta(e)$ by (i), whereas by (ii) one has $\phi(e)[1 - \theta(e)] = 0$, that is, $\phi(e)\theta(e) = \phi(e)$.

Finally, we assert that ϕ is a $*$-isomorphism. For any $x \in R$, we are to show that $\phi(x^*) = \phi(x)^*$, that is, $x = \phi^{-1}(\phi(x^*)^*)$. Define $\psi: R \to R$ by the formula $\psi(x) = \phi^{-1}(\phi(x^*)^*)$. Clearly ψ is a ring automorphism of R, and for all $e \in \tilde{R}$ one has $\psi(e) = e$ (because $\phi(e) = \theta(e)$ is self-adjoint), therefore $\psi(eR) = \psi(e)\psi(R) = eR$. Thus, ψ induces the identity mapping on $L(R)$; since R has order $n \geqq 2$, it follows from a theorem of von Neumann that ψ is the identity mapping of R [**24**, p. 104, Theorem 4.1].

THEOREM 7. *Let A and B be finite Baer $*$-rings satisfying the hypotheses of Theorem* 6, *and let $Q(A)$ and $Q(B)$ be their maximal rings of quotients. Assume, moreover, that A and B have no abelian or type I_2 summands. Then every projection ortho-isomorphism $\tilde{A} \to \tilde{B}$ extends to an isomorphism of $*$-rings $Q(A) \to Q(B)$.*

PROOF. Since A and $Q(A)$ have the same projection lattice (see §5), they have the same type-structure; similarly for B and $Q(B)$. The theorem is then immediate from the lemma. {Moreover, since $Q(A)$ and $Q(B)$ are self-injective, in the proof of the lemma one need not cite the corollary of Theorem 5.}

LEMMA. *Let A and B be finite AW^*-algebras, $Q(A)$ and $Q(B)$ their maximal rings of right quotients. If ϕ: $Q(A) \to Q(B)$ is a $*$-isomorphism, then $\phi(A) = B$.*

PROOF. Let $a \in A$. Choose a positive integer n such that $a^*a \leqq n1$, that is, $n1 - a^*a = c^*c$ for some $c \in A$. Then $0 \leqq \phi(c)^* \phi(c) = \phi(c^*c) = n1 - \phi(a)^*\phi(a)$, thus $\phi(a)^*\phi(a) \leqq n1$, whence $\phi(a) \in B$ [3, p. 245, Theorem 1].

THEOREM 8. ([5], [6]) *Let A and B be finite AW^*-algebras with no summand of type I_2. Then every projection ortho-isomorphism $\tilde{A} \to \tilde{B}$ extends to an isomorphism of $*$-rings $A \to B$.*

PROOF. In view of Theorem 7 and the lemma, one is reduced to the case that A and B are abelian. Then by M. H. Stone's representation theory of Boolean algebras, the ortho-isomorphism of the projection lattices induces an ortho-isomorphism of the lattices of closed-open subsets of the spectra of the algebras, which in turn induces a homeomorphism of the spectra [11, p. 85, Theorem 8], hence a $*$-isomorphism of A and B.

The same proofs show that if A and B are finite Baer $*$-rings satisfying the conditions 1°–6° of [3, pp. 248–249] and having no abelian or type I_2 summands, then every projection ortho-isomorphism $\tilde{A} \to \tilde{B}$ extends to an isomorphism of $*$-rings $A \to B$.

7. Reduction theory. Let A be a Baer $*$-ring whose involution is extendible to the maximal ring of right quotients Q (cf. Theorem 6). As observed in §5, the self-injectivity of Q implies that its projection lattice is continuous and that Q is unit-regular, hence directly finite; in particular, A is directly finite.

Let J be a maximal ideal of Q. Then Q/J is a self-injective regular ring [8, p. 107, Theorem 9.32], hence is a regular Baer ring [8, p. 95, Proposition 9.1, (c)]; Q/J is also unit-regular, hence directly finite. Moreover, if $x \in Q$ and $e = \mathrm{LP}(x)$, then $xQ = eQ$; thus $x \in J$ if and only if $\mathrm{LP}(x) \in J$. It follows that J is a $*$-ideal of Q, and since $\mathrm{LP}(xx^*) = \mathrm{LP}(x)$, one sees that the natural involution of Q/J is proper ($uu^* = 0$ implies $u = 0$), thus Q/J is $*$-regular. Thus, Q/J is a $*$-regular Baer ring, equivalently, a regular Baer $*$-ring. Moreover, Q/J is simple, hence factorial (i.e., indecomposable). Briefly, Q/J is a regular Baer $*$-factor. Let $I = A \cap J$, which is a $*$-ideal of A.

LEMMA. *A/I is a directly finite Baer $*$-factor.*

PROOF. One has a natural $*$-monomorphism ϕ: $A/I \to Q/J$. Every projection of Q/J has the form $e + J$ with e a projection of Q (that is, of A) [**3**, p. 142, Proposition 1], hence every projection of A/I has the form $e + I$, e a projection of A. Thus, the range of ϕ is a $*$-subring of Q/J containing all projections, hence it is a Baer $*$-ring; so, therefore, is its $*$-isomorph A/I. Moreover, A/I is directly finite (because Q/J is directly finite). Finally, if u is a central projection of A/I, then $\phi(u)$ is a central projection of Q/J. {To prove this note that the projection $e = \phi(u)$ commutes with all projections of $\phi(A/I)$, that is, of $R = Q/J$. To see that e is central in R, it suffices to show that $eR(1 - e) = 0$; to this end, it suffices to show that e commutes with every idempotent f in R [**15**, p. 17, Exercise 5]. Write $fR = gR$, g a projection. Then $efR = egR = geR \subset gR = fR$, thus $ef = fef$; applying this to f^* instead of f, one sees that $fe = fef$, thus $fe = ef$. I am indebted to the referee for this brief argument, and for the observation that it applies in any Rickart $*$-ring.} Since Q/J is factorial, $\phi(u) = 0$ or 1; hence $u = 0$ or 1.

The intersection of all maximal ideals J of Q is 0 [**8**, p. 105, Corollary 9.27], hence so is the intersection of the corresponding ideals $I = A \cap J$ of A. This "reduction theory" for A can be summarized as follows.

THEOREM 9. *If A is a Baer $*$-ring whose involution is extendible to the maximal ring of quotients, then A is the subdirect product of directly finite Baer $*$-factors.*

If, in particular, A is a finite AW^*-algebra, then the closures $M = \bar{I}$ of the ideals I described above are precisely the maximal ideals of A, the intersection of the ideals M is also 0, and the A/M are simple, finite AW^*-factors [**3**, §45]. If A is a finite von Neumann algebra, then by a theorem of Feldman [7] the A/M can also be represented as von Neumann algebras (cf. [**3**, p. 280, note for Exercise 2 of §45]); thus, a finite von Neumann algebra is the subdirect product of factorial finite von Neumann algebras.

Finally, we observe that for a finite AW^*-algebra A, the reduction of A is "compatible" with that of its ring of quotients Q, in the following sense.

THEOREM 10. *If A is a finite AW^*-algebra, Q is its maximal ring of right quotients* (cf. §4), *J is a maximal ideal of Q, and $I = A \cap J$, then Q/J is the maximal ring of right quotients of A/I.*

PROOF. Observe first that Q is also a classical ring of (right and left) quotients of A [**16**, p. 108]; the crux of the matter is that each $x \in Q$ can be written in the form $x = ab^{-1}$ with a, b in A—for example $a = x(1 + x^*x)^{-1}$ and $b = (1 + x^*x)^{-1}$ [**3**, p. 246, Corollary 2].

Let $\phi: A/I \to Q/J$ be the natural $*$-monomorphism, and write $S = \phi(A/I)$, $R = Q/J$; the problem is to show that R is the maximal ring of right quotients of S. We first note that R is a ring of right quotients of S

[**16**, p, 99]. {For, suppose $u, v \in R$, $u \neq 0$; we seek $w \in S$ with $uw \neq 0$ and $vw \in S$. Say $u = x + J$, $v = y + J$. Write $y = ab^{-1}$ with a, b in A; then $xb \notin J$ (because $x \notin J$), whereas $yb = a \in A$, thus $w = b + J$ meets the requirements.} It follows that if $Q(S)$ is the maximal ring of right quotients of S, then the identity mapping of S extends to a ring monomorphism $R \to Q(S)$ [**16**, p. 99, Proposition 8]; thus, one can suppose that $S \subset R \subset Q(S)$. Since $R = Q/J$ is right self-injective, one concludes, as in the proof of Theorem 3, that $R = Q(S)$.

If, in the notation of Theorem 10, $M = \bar{I}$, then the maximal ring of right quotients of A/M is in general not $*$-isomorphic to Q/J [**1**, p. 508, (5)]; this means, in view of the lemma to Theorem 7, that the projection lattice of A/M is in general not ortho-isomorphic to the projection lattice of Q/J (i.e., of A/I).

ADDENDUM. The corollary of Theorem 5 was proved by J. L. Burke for the case of a regular Baer $*$-factor of order $k \geqq 4$ [Canad. Math. Bull. **19** (1976), 21–38, Theorem 1.3]. The general case has been treated by D. Handelman (unpublished). I am grateful to the referee for correcting an error in my original proof of the Lemma to Theorem 6, and for suggesting a number of improvements in the exposition.

A Baer $*$-ring satisfying the parallelogram law (P) automatically satisfies generalized comparability (GC) [S. Maeda and S. S. Holland, *Equivalence of projections in Baer $*$-rings*, J. Algebra **39** (1976), 150–159]. The statement of Theorem 6 can thus be simplified; the hypothesis of GC is redundant. The same is therefore true in the remarks of §6 and the statement of Theorem 7.

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